A LEAST-SQUARES APPROACH FOR THE DESIGN OF NONRECURSIVE DIFFERENTIATORS OF ARBITRARY ORDER

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Abstract

A method which can be used to design lowpass nonrecursive linear-phase digital differentiators is described. The method involves formulating an error function based on the absolute mean-square difference between the amplitude responses of the practical and ideal differentiators as a quadratic function. The filter coefficients are obtained by solving a system of linear equations. This method leads to a lower mean-square error and is computationally more efficient than the eigenfilter method and the method based on the Remez exchange algorithm.

1 Introduction

Digital differentiators are used to obtain samples of the derivatives of a bandlimited continuous time signal from the samples of the continuous time signal. In this paper, we focus on systems that yield the higher-order derivatives (greater than one) of the signal since the first-order case has been dealt with sufficiently. These systems have been used for the calculation of geometric moments [1] and in biological signal processing [2]. Higher-order differentiators can be realized as a nonrecursive digital filter with an amplitude response that approximates the ideal frequency characteristic which varies as a power of frequency with frequency.

Higher-order differentiators have been designed by extending the approaches that are used to design first-order differentiators [3]-[4]. In [3], the McClellan-Parks method based on the Remez exchange algorithm [5] has been extended to incorporate the parameters involved in the design of higher-order differentiators. The eigenfilter method in [6] has been extended to the design of higher-order differentiators in [4] by formulating an error function in terms of a quadratic form. The error function involves the square of the difference between the desired amplitude response and the actual amplitude response of the designed nonrecursive filter. In this method, the desired amplitude response is equal to the amplitude response of the designed filter at any arbitrary reference frequency, as opposed to being equal to the ideal amplitude characteristic. The filter coefficients are found by computing the eigenvector that corresponds to the smallest eigenvalue of a real positive-definite symmetric matrix.

In this paper, the least-squares approach described in [7] is extended to the design of higher-order differentiators. The method allows the explicit inclusion of the ideal amplitude response in the error function and hence, leads to a more meaningful formulation than the eigenfilter method. Also, it does not necessitate the use of a reference frequency. The filter coefficients are obtained by solving a system of linear equations, thereby leading to a lower computational complexity than the eigenfilter design.

2 Higher-order Differentiators

A nonrecursive digital filter with N taps can be represented by its impulse response $h(n)$ for $0 \le n \le N-1$. For the case of a linear-phase filter having a symmetric impulse response, we have $h(n) = h(N-1-n)$. Consequently, $H(e^{j\omega}) = M(\omega)\, e^{-j\omega(N-1)/2}$ where

$$M(\omega) = \begin{cases} \displaystyle\sum_{n=0}^{(N-1)/2} a(n)\cos n\omega & N \text{ odd} \\ \displaystyle\sum_{n=1}^{N/2} a(n)\cos(n-1/2)\omega & N \text{ even} \end{cases} \tag{1}$$

If N is odd, $a(0) = h[(N-1)/2]$ and $a(n) = 2h[(N-1)/2-n]$. If N is even, $a(n) = 2h(N/2-n)$.

For a linear-phase filter having an antisymmetric impulse response, we have $h(n) = -h(N-1-n)$. Therefore, $H(e^{j\omega}) = jM(\omega)\, e^{-j\omega(N-1)/2}$ where

$$M(\omega) = \begin{cases} \displaystyle\sum_{n=1}^{(N-1)/2} b(n)\sin n\omega & N \text{ odd} \\ \displaystyle\sum_{n=1}^{N/2} b(n)\sin(n-1/2)\omega & N \text{ even} \end{cases} \tag{2}$$

If N is odd, $b(n) = 2h[(N-1)/2-n]$. If N is even, $b(n) = 2h(N/2-n)$.

An ideal kth-order differentiator has a frequency response

$$H_I(e^{j\omega}) = D(\omega)\, e^{jk\pi/2} \tag{3}$$

where $D(\omega) = (\omega/2\pi)^k$ for $0 \le \omega \le \omega_p \le \pi$. The upper passband edge frequency is ω_p. For an even-order differentiator (k is even), it can be seen that $H_I(e^{j\omega})$ is a real-valued function. Therefore, a nonrecursive filter with a symmetrical impulse response can be used to design even-order differentiators provided the coefficients $a(n)$ are determined such that the amplitude response $M(\omega)$ given in Eqn. (1) approximates $D(\omega)$. It should be noted that a full-band differentiator ($\omega_p = \pi$) can be designed only when N is odd. When N is even, it is required that $\omega_p < \pi$.

For an odd-order differentiator (k is odd), it can be seen that $H_I(e^{j\omega})$ is purely imaginary. We note that a nonrecursive filter with an antisymmetrical impulse response can be used to design odd-order differentiators provided the coefficients $b(n)$ are determined such that the amplitude response $M(\omega)$ given in Eqn. (2) approximates $D(\omega)$. For this case, a full-band differentiator can be designed only when N is even.

3 Error Function Minimization

The mean-square difference between $D(\omega)$ and $M(\omega)$ with respect to the differentiator passband can be expressed as

$$E_{\text{mse}} = \frac{1}{\pi}\int_0^{\omega_p} (D(\omega) - M(\omega))^2 \, d\omega \tag{4}$$

By minimizing the error function E_{mse} with respect to the filter coefficients, the required differentiator can be designed. For the case of even-order differentiators, we have $M(\omega) = \mathbf{a}^T\mathbf{c}(\omega)$ where

$$\mathbf{a} = \begin{cases} [a(0)\ \ a(1)\ \ \cdots\ \ a((N-1)/2)]^T & N \text{ odd} \\ [a(1)\ \ a(2)\ \ \cdots\ \ a(N/2)]^T & N \text{ even} \end{cases} \tag{5}$$

and

$$\mathbf{c}(\omega) = \begin{cases} [1\ \ \cos\omega\ \ \cdots\ \ \cos(\frac{N-1}{2}\omega)]^T & N \text{ odd} \\ [\cos\frac{1}{2}\omega\ \ \cos\frac{3}{2}\omega\ \ \cdots\ \ \cos(\frac{N-1}{2}\omega)]^T & N \text{ even} \end{cases} \tag{6}$$

Equation (4) can be written as

$$E_{\text{mse}} = \frac{1}{\pi}\int_0^{\omega_p} [D^2(\omega) - 2D(\omega)\mathbf{a}^T\mathbf{c}(\omega) + \mathbf{a}^T\mathbf{c}(\omega)\mathbf{c}^T(\omega)\mathbf{a}]\, d\omega \tag{7}$$

In minimizing E_{mse}, we set $\frac{\partial E_{\text{mse}}}{\partial a(i)} = 0$ to obtain a system of linear equations $\mathbf{Qa} = \mathbf{d}$ where

$$\mathbf{Q} = \int_0^{\omega_p} \mathbf{c}(\omega)\mathbf{c}^T(\omega)\, d\omega \tag{8}$$

and

$$\mathbf{d} = \int_0^{\omega_p} D(\omega)\mathbf{c}(\omega)\, d\omega \tag{9}$$

It can be noted from (8) that $\mathbf{Q}$ is a positive-definite (unless $\omega_p = 0$) real symmetric matrix and thus, a unique solution is guaranteed. Consequently, the system of linear equations can be solved by a computationally efficient method, like the Cholesky decomposition that avoids matrix inversion. The entries of $\mathbf{Q}$ and $\mathbf{d}$ can be computed by evaluating the respective integrals in closed form.

Similarly, for odd k, $M(\omega) = \mathbf{b}^T\mathbf{s}(\omega)$ where

$$\mathbf{b} = \begin{cases} [b(1)\ \ b(2)\ \ \cdots\ \ b((N-1)/2)]^T & N \text{ odd} \\ [b(1)\ \ b(2)\ \ \cdots\ \ b(N/2)]^T & N \text{ even} \end{cases} \tag{10}$$

and

$$\mathbf{s}(\omega) = \begin{cases} [\sin\omega\ \ \sin 2\omega\ \ \cdots\ \ \sin(\frac{N-1}{2}\omega)]^T & N \text{ odd} \\ [\sin\frac{1}{2}\omega\ \ \sin\frac{3}{2}\omega\ \ \cdots\ \ \sin(\frac{N-1}{2}\omega)]^T & N \text{ even} \end{cases} \tag{11}$$

The resulting system of equations is given by $\mathbf{Qb} = \mathbf{d}$ where

$$\mathbf{Q} = \int_0^{\omega_p} \mathbf{s}(\omega)\mathbf{s}^T(\omega)\, d\omega \tag{12}$$

and

$$\mathbf{d} = \int_0^{\omega_p} D(\omega)\mathbf{s}(\omega)\, d\omega \tag{13}$$

As in the case of even-order differentiators, $\mathbf{Q}$ is a positive-definite real symmetric matrix. The entries of $\mathbf{Q}$ and $\mathbf{d}$ can be calculated by evaluating the respective integrals in closed form.

4 Design Examples

Two design examples are provided to demonstrate the design of both odd-order and even-order differentiators. Figure 1 shows the amplitude response of a second-order full-band differentiator with 25 taps and the variation of the error $D(\omega) - M(\omega)$ with respect to ω. In Fig. 2, we demonstrate the design of a 27-tap third-order differentiator with a passband edge $\omega_p = 0.88\pi$.

5 Performance Results

In this section, we compare our design method with the minimax technique [3] and the eigenfilter approach from three points of view, namely, the number of floating point operations (flops), the passband mean-square error E_{mse}, and the passband peak error E_{peak}, where

$$E_{\text{peak}} = \max_{0 \le \omega \le \omega_p} |D(\omega) - M(\omega)| \tag{14}$$

A comparison of the three methods with respect to the number of flops is shown in Table 1, with respect to E_{mse} is shown in Table 2 and with respect to E_{peak} is shown in Table 3 for the examples in the previous section. It must be mentioned that the entries in Table 1 have been normalized relative to the number of flops for our method. The reference frequencies for all the designs using the eigenfilter method have been chosen to be equal to half the corresponding passband edge frequencies as in [4].

5.1 Error Measure

Our method formulates a better error measure than the eigenfilter method in that we explicitly minimize the mean-square error between the ideal and the practical amplitude responses. In contrast, the eigenfilter method does not take the ideal response into account but rather the frequency response of the practical filter at an arbitrary frequency. In fact, the filter that is designed depends upon the reference frequency. The performance comparisons show that our method leads to a lower mean-square error than the eigenfilter method although the differences are small.

The error criteria for our method and the minimax method are different in the sense that in the former we find the filter coefficients that minimizes E_{mse} whereas the latter determines the filter coefficients that minimizes E_{peak}. As a consequence, our method leads to a lower E_{mse} while the minimax method guarantees a lower E_{peak}.

5.2 Computational Complexity

For our method, the filter parameters are obtained by a system of linear equations involving a positive-definite matrix $\mathbf{Q}$. It is well known that a real symmetric positive-definite matrix can be decomposed as [8] $\mathbf{Q} = \mathbf{LL}^T$ where $\mathbf{L}$ is a real lower triangular matrix. Consequently, the system of linear equations can be written as $\mathbf{LL}^T\mathbf{a} = \mathbf{d}$. By letting $\mathbf{v} = \mathbf{L}^T\mathbf{a}$, we get $\mathbf{d} = \mathbf{Lv}$. Given $\mathbf{L}$ and $\mathbf{d}$, we can obtain $\mathbf{v}$ by recursively solving a set of linear equations. Let l_{ij} be the element in the ith row and jth column of $\mathbf{L}$. It can be shown that

$$v(n) = \frac{1}{l_{nn}}\left\{d(n) - \sum_{j=0}^{n-1} l_{nj}v(j)\right\} \tag{15}$$

for $n = 0, 1, \cdots, N_E$, where N_E is the dimension of the system of equations. Since $\mathbf{Q}$ is positive-definite, the l_{nn} in the above equation are nonzero. We first solve for $v(0)$ and then recursively obtain $v(n)$. A total of $N_E(N_E-1)/2$ multiplications, N_E divisions, and $N_E(N_E-1)/2$ additions are required to compute $\mathbf{v}$. Similarly, we can find the vector $\mathbf{a}$ for a given $\mathbf{v}$ and $\mathbf{L}$ by solving

$$a(n) = \frac{1}{l_{nn}}\left\{v(n) - \sum_{j=0}^{n-1} l_{nj}a(j)\right\} \tag{16}$$

The total time required to obtain the solution is

$$T_o = (T_a + T_m)N_E(N_E - 1) + 2N_E T_d \tag{17}$$

where T_a, T_m, T_d, are, respectively, the time required for one real addition, multiplication, and division.

In the eigenfilter approach, the error (for even-order differentiators) is formulated as $E_{\text{mse}} = \mathbf{a}^T\mathbf{Pa}$ where

$$\mathbf{P} = \frac{1}{\pi}\int_0^{\omega_p} [R\mathbf{c}(\omega_0) - \mathbf{c}(\omega)][R\mathbf{c}(\omega_0) - \mathbf{c}(\omega)]^T\, d\omega \tag{18}$$

and $R = (\omega/\omega_0)^k$ is the normalizing factor such that the actual frequency response at the reference frequency ω_0 is approximately equal to the desired value [4]. A similar definition for $\mathbf{P}$ exists for odd-order differentiators. It can be noted that $\mathbf{P}$ is a real symmetric positive-definite matrix whose entries can be calculated by evaluating the above integral in closed form. The coefficients of the differentiators are obtained as the eigenvector that corresponds to the smallest eigenvalue of $\mathbf{P}$.

In order to compute the smallest eigenvalue and its corresponding eigenvector, generally an iterative inverse power method is used [9]. At the $(k+1)$th iteration, a vector $\mathbf{x}_{k+1}$ is computed from the previous iterate $\mathbf{x}_k$ as

$$\mathbf{y}_{k+1} = \mathbf{P}^{-1}\mathbf{x}_k \tag{19}$$

$$\mathbf{x}_{k+1} = \mathbf{y}_{k+1} / \|\, \mathbf{y}_{k+1} \,\| \tag{20}$$

where $\|\, \mathbf{y}_{k+1} \,\|$ denotes the L_2 norm of $\mathbf{y}_{k+1}$. If $\|\, \mathbf{x}_{k+1} - \mathbf{x}_k \,\| \le \epsilon$ (ϵ is usually 10^{-6}), then $\mathbf{x}_{k+1}$ is a good approximation of the eigenvector corresponding to the smallest eigenvalue. We can rewrite Eqn. (19) as $\mathbf{x}_k = \mathbf{P}\mathbf{y}_{k+1}$. Using the technique described above for solving a system of linear equations, we can obtain $\mathbf{y}_{k+1}$ and subsequently $\mathbf{x}_{k+1}$.

It can now be seen that the eigenfilter method requires solving a system of linear equations several times before obtaining the eigenvector corresponding to the smallest eigenvalue. On the contrary, our approach requires solving a system of linear equations only once. Thus if T_e is the time taken for the eigenfilter method, then from (17) $T_e = MT_o$, where M is the number of iterations required in the eigenfilter method. The value of M increases as the ratio λ_2/λ_1, where λ_1 is the smallest eigenvalue and λ_2 is the next smallest eigenvalue, decreases. If the ratio is too small, it may not even be possible to evaluate the smallest eigenvalue and its corresponding eigenvector using the inverse power method (the inverse power method may not converge).

From a computational point of view, our method is considerably more efficient than the minimax method. The minimax method uses the iterative Remez exchange algorithm which takes up the major computational burden. It must be mentioned that in designing the differentiators using the minimax method, the procedure forwarded in [3] has been used. More efficient techniques to design nonrecursive filters using the Remez exchange algorithm have been advanced in [10]. However, at this stage we have not extended the method advanced in [10] to the design of higher-order differentiators.

6 Alternative Error Function

The error function E_{mse} is based on the integration of the square of the difference between $D(\omega)$ and $M(\omega)$. This integral can be approximated as a Riemann sum to yield an alternative error function E_{sum}

$$E_{sum} = \frac{1}{\pi}\sum_{k=1}^{r} f(\omega_k')\Delta(\omega_k) \tag{21}$$

where $f(\omega) = (D(\omega) - M(\omega))^2$, $\Delta(\omega_k) = \omega_k - \omega_{k-1}$ and $\omega_{k-1} \leq \omega_k' < \omega_k$. Note that $\omega_0 = 0$ and $\omega_r = \omega_p$ (the upper passband edge frequency). Finding the filter coefficients that minimize E_{sum}, again leads to a system of equations. The difference between minimizing E_{sum} and E_{mse} is that the entries of $\mathbf{Q}$ and $\mathbf{d}$ are now evaluated as Riemann sums as opposed to integrals in closed form. The motivation of attempting this approach is to see if a savings in computation results with a corresponding loss in performance. Obviously, the loss in performance diminishes as the number of points involved in the summation is increased.

Filters were designed by minimizing E_{sum} with $r = 21$, $\omega_k = \frac{k\omega_p}{r}$, $\Delta(\omega_k) = \frac{\omega_p}{r}$ and $\omega_k' = \omega_{k-1}$. Although the differences in the mean-square error obtained by minimizing E_{sum} and E_{mse} were found to be negligible, the number of flops required for minimizing E_{sum} increased by a factor of 4.

7 Summary and Conclusions

In this paper, a method to design nonrecursive linear-phase higher-order digital differentiators has been presented. The characteristics of this method are that it (a) explicitly minimizes the absolute mean-square error between the ideal and actual frequency responses, (b) offers a closed form solution for the filter coefficients, and (c) is implemented in a noniterative and computationally simple manner. Design examples are provided. The mean-square error achieved by our method is lower than that achieved by the eigenfilter and the minimax methods. In addition, our method is computationally more efficient than both the eigenfilter and minimax methods.

8 References

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Examples	Floating point operations (flops) (Normalized relative to our method)		
	Our method	Eigenfilter method	Minimax method
1	1	4.22	5.69
2	1	4.75	7.48

Table 1: Comparison of the three methods with respect to the number of floating point operations.

Examples	Mean-square error E_{mse}		
	Our method	Eigenfilter method	Minimax method
1	8.732e-07	8.799e-07	6.872e-06
2	1.817e-03	1.818e-03	1.817e-03

Table 2: Comparison of the three methods with respect to Emse.

Examples	Peak error E_{peak}		
	Our method	Eigenfilter method	Minimax method
1	8.101e-03	8.105e-03	3.724e-03
2	1.022e-03	1.010e-03	2.967e-04

Table 3: Comparison of the three methods with respect to E_{peak}.

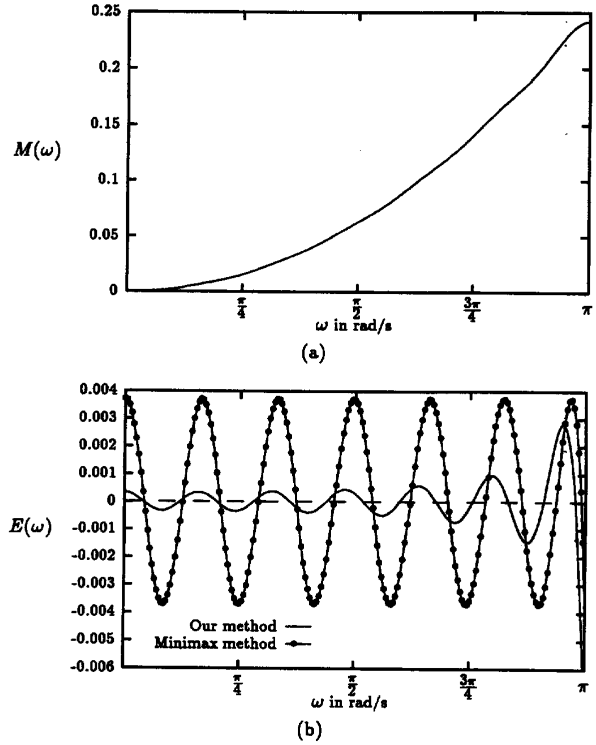

Figure 1: (a) Amplitude response of a second-order differentiator with $N = 25$. (b) Variation of the error function with respect to ω.

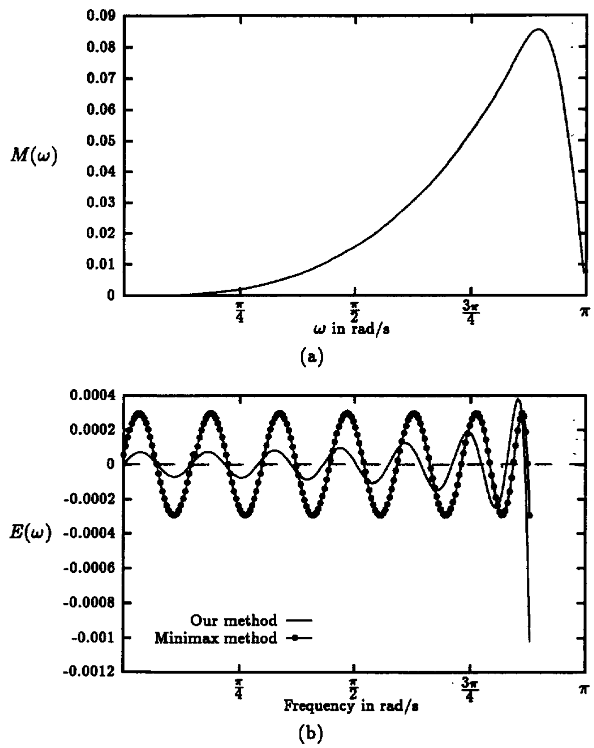

Figure 2: (a) Amplitude response of a third-order differentiator with $N = 27$. (b) Variation of the error function with respect to ω.