A LEAST-SQUARES DESIGN OF NONRECURSIVE FILTERS SATISFYING PRESCRIBED MAGNITUDE AND PHASE SPECIFICATIONS

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Abstract

A method is described which can be used to design nonrecursive filters satisfying prescribed magnitude and phase specifications. The method is based on formulating the absolute mean-square error between the frequency response of the practical filter and the desired response as a quadratic function. The coefficients of the filters are obtained by solving a set of linear equations. Examples for designing lowpass filters, differentiators, and allpass phase equalizers are presented. It is shown that our method leads to a lower mean-square error and is computationally more efficient than the eigenfilter method.

1 Introduction

The design of linear-phase finite impulse response (FIR) filters is generally carried out using the McClellan-Parks (MP) algorithm [1] and least-squares methods [2]-[4]. A linear-phase FIR filter has symmetry/antisymmetry constraints imposed on its impulse response. Consequently, for a given filter length, the group delay assumes a constant value for all frequencies. Furthermore, a large length FIR filter is needed to satisfy a narrow transition-band specification thereby leading to a rather high group delay. On the other hand, a minimum phase FIR filter can be designed to achieve a lower group delay but does not provide a constant group delay for all frequencies including those in the passband. Therefore, to achieve an arbitrary constant group delay filter, neither a linear phase nor a minimum phase filter can be used. Other nonlinear phase characteristics, such as mandated by allpass phase equalizers, can only be achieved by filters satisfying arbitrary magnitude and phase specifications.

In [5]-[7], several methods have been proposed to simultaneously approximate magnitude and phase specifications. In [5], a complex Chebyshev approximation is first converted into a real approximation problem. The solution to an overdetermined set of linear equations obtained by linear programming techniques yields the filter coefficients. A linear programming approach is also used in [6] to design FIR allpass phase equalizers. This method requires a large memory space and considerable computing time. In [7], the eigenfilter method for the design of linear phase filters is extended to the design of F[illegible] filters whose frequency response approximates a complex-valued fu[illegible] 335 tion in a least-squares sense. It is shown that the eigenfilter method is computationally efficient and yields filters that are comparable in performance with those obtained in [5]-[6].

It has been shown that the least-squares method in [4] for the design of linear phase filters is computationally more efficient and leads to a lower mean-square error than the eigenfilter method. In this paper, we extend this method to the design of FIR filters satisfying prescribed magnitude and phase specifications. Our method leads to a lower mean-square error and is computationally more efficient than the eigenfilter method of [7].

2 FIR design procedure

The frequency response of an FIR digital filter with N taps specified by a real-valued impulse response $h(n)$ is given by

$$H(e^{j\omega}) = \mathbf{h}^T\mathbf{c}(\omega) - j\mathbf{h}^T\mathbf{s}(\omega) \qquad (1)$$

where

$$\mathbf{h} = [h(0)\ h(1)\ h(2)\ \cdots\ h(N-1)]^T$$

$$\mathbf{c}(\omega) = [1\ \cos(\omega)\ \cos(2\omega)\ \cdots\ \cos((N-1)\omega)]^T$$
$$\mathbf{s}(\omega) = [0\ \sin(\omega)\ \sin(2\omega)\ \cdots\ \sin((N-1)\omega)]^T$$

The desired frequency response $D(\omega)$ having an amplitude response $M(\omega)$ and a phase response $\rho(\omega)$ is given by

$$D(\omega) = \begin{cases} M(\omega)\, e^{-j\rho(\omega)} = M(\omega)\ \cos(\rho(\omega)) \\ \qquad\qquad -jM(\omega)\ \sin(\rho(\omega)) & \omega \in P \\ 0 & \omega \in S \end{cases} \qquad (2)$$

where P is the passband, S is the stopband and $\tau_\rho(\omega) = \frac{d\rho(\omega)}{d\omega}$ is the desired group delay response. Comparing Eqns. (1) and (2) it can be seen that a nonrecursive filter satisfying prescribed magnitude and phase specifications can be designed provided the coefficients $h(n)$ are determined appropriately.

The mean-square error between $D(\omega)$ and $H(e^{j\omega})$ can be expressed as

$$E_{\text{mse}} = \frac{\alpha}{\pi}\int_P |D(\omega) - H(e^{j\omega})|^2\, d\omega + \frac{\beta}{\pi}\int_S |H(e^{j\omega})|^2\, d\omega \qquad (3)$$

By minimizing the error function E_{mse} with respect to the filter coefficients, the required filter can be designed. In minimizing E_{mse}, we set $\frac{\partial E_{\text{mse}}}{\partial \mathbf{h}} = 0$ to obtain a system of linear equations $(\alpha\mathbf{Q} + \beta\mathbf{R})\mathbf{h} = \alpha\mathbf{d}$, where

$$\mathbf{Q} = \int_P (\mathbf{c}(\omega)\mathbf{c}^T(\omega) + \mathbf{s}(\omega)\mathbf{s}^T(\omega))\, d\omega \qquad (4)$$
$$\mathbf{R} = \int_S (\mathbf{c}(\omega)\mathbf{c}^T(\omega) + \mathbf{s}(\omega)\mathbf{s}^T(\omega))\, d\omega \qquad (5)$$
$$\mathbf{d} = \int_P M(\omega)\,(\cos(\rho(\omega))\mathbf{c}(\omega) + \sin(\rho(\omega))\mathbf{s}(\omega))\, d\omega \qquad (6)$$

It can be noted that $\mathbf{Q}$ and $\mathbf{R}$ are positive-definite, real, and symmetric matrices. Consequently, the system of linear equations can be solved by a computationally efficient method, like the Cholesky decomposition, that avoids matrix inversion [8].

336 [illegible]n examples

Example 1

The design of a lowpass filter with the following specifications is considered.

$$D(\omega) = \begin{cases} \cos(\gamma\omega) - j\sin(\gamma\omega) & \omega \in P = [0,\ 0.12\pi] \\ 0 & \omega \in S = [0.24\pi,\ \pi] \end{cases}$$

The desired group delay in the passband is $\gamma = 12$. The magnitude and group delay response of a 31-tap lowpass filter designed using $\alpha = 1$ and $\beta = 5$ are shown in Figs. 1(a) and 1(b), respectively.

Example 2

The ideal response of a differentiator with a constant group delay in the passband is given by

$$D(\omega) = \begin{cases} j\omega\, e^{-j\gamma\omega} = & \omega\sin(\gamma\omega) \\ & + j\omega\cos(\gamma\omega) \quad \omega \in P = [0,\ \omega_p] \\ 0 & \omega \in S = [\omega_s,\ \pi] \end{cases}$$

where ω_p and ω_s are the passband and stopband edges, respectively and γ is the constant group delay.

For this example, we consider the design of a 31-tap full-band differentiator (i.e., $\omega_p = \omega_s = \pi$) having a constant group delay $\gamma = 11.5$. A nonintegral group delay has been chosen in order to avoid phase discontinuity at the folding frequency. The magnitude and group delay response of the differentiator are shown in Figs. 2(a) and 2(b), respectively.

4 Design of FIR allpass phase equalizers

For an allpass phase equalizer, the desired characteristic can be expressed as

$$D(\omega) = e^{-j\rho(\omega)} = \cos(\rho(\omega)) - j\sin(\rho(\omega)) \quad \omega \in P = [0,\ \pi] \tag{7}$$

where $\rho(\omega)$ is the desired phase response. Following the development in Section 2, E_{mse} is minimized and a system of linear equations $\mathbf{Qh} = \mathbf{d}$ is obtained. Since $\mathbf{Q} = \pi\mathbf{I}$ ($\mathbf{I}$ is an $N \times N$ identity matrix), the filter coefficients are obtained simply as $\mathbf{h} = \mathbf{d}/\pi$ thereby reducing the computational effort significantly. In addition, if $\rho(\omega)$ is symmetric or antisymmetric with respect to $\pi/2$, the computational complexity is further reduced since only half the number of coefficients need to be determined. Below we shall consider the design of two allpass phase equalizers advanced in [6] using our method.

4.1 Symmetric phase characteristics

The desired phase characteristic is given by

$$\rho(\omega) = \frac{N-1}{2}\omega - \hat{\rho}(\omega) \tag{8}$$

where the first term on the right-hand side is the linear phase term and $\hat{\rho}(\omega)$ is a function of ω symmetric about $\pi/2$. It must be mentioned that the number of filter taps N is odd. It follows that $h((N-1)/2 - n)) = h((N-1)/2 + n)$ when n is even and $h((N-1)/2 - n)) = -h((N-1)/2 + n)$ when n is odd [6]. The allpass phase equalizer can be characterized by

$$e^{j\frac{N-1}{2}\omega}H(e^{j\omega}) = \mathbf{a}^T\hat{\mathbf{c}}(\omega) + j\mathbf{b}^T\hat{\mathbf{s}}(\omega) \tag{9}$$

where

$$\begin{aligned}
\mathbf{a} &= [a(0)\ a(1)\ \dots\ a(U)]^T \\
\mathbf{b} &= [b(1)\ b(2)\ \dots\ b(V)]^T \\
\hat{\mathbf{c}}(\omega) &= [1\ \cos(2\omega)\ \dots\ \cos(2U\omega)]^T \\
\hat{\mathbf{s}}(\omega) &= [\sin(\omega)\ \sin(3\omega)\ \dots\ \sin((2V-1)\omega)]^T
\end{aligned}$$

Also, $U = \lfloor\frac{N-1}{4}\rfloor$, $V = \lfloor\frac{N+1}{4}\rfloor$, $a(0) = h((N-1)/2)$, $a(n) = 2h((N-1)/2 - 2n)$ for $n = 1, 2, \dots, U$ and $b(n) = 2h((N-1)/2 - 2n + 1)$ for $n = 1, 2, \dots, V$.

Since the coefficients associated with the real and imaginary parts of eqn. (9) are of different lengths, they are computed separately. The mean-square error associated with the real part can be written as

$$E_R = \int_0^{\pi} \left(\cos(\hat{\rho}(\omega)) - \mathbf{a}^T\hat{\mathbf{c}}(\omega)\right)^2 d\omega \tag{10}$$

By setting $\frac{\partial E_R}{\partial \mathbf{a}} = 0$, we get $\hat{\mathbf{Q}}_a\mathbf{a} = \hat{\mathbf{d}}_a$ where the elements of $\hat{\mathbf{Q}}_a$ are given by

$$\hat{Q}_a(n,m) = \begin{cases} \pi & n = m = 0 \\ \frac{\pi}{2} & n = m \neq 0 \\ 0 & n \neq m \end{cases} \tag{11}$$

for $0 \leq n, m \leq U$. The elements of $\hat{\mathbf{d}}_a$ are given by

$$\hat{d}_a(n) = \int_0^{\pi} \cos(\hat{\rho}(\omega))\cos(2n\omega) \quad n = 0, 1, \dots, U \tag{12}$$

From eqns. (11) and (12), $a(0) = \hat{\mathbf{d}}_a(n)/\pi$ and $a(n) = 2\hat{\mathbf{d}}_a(n)/\pi$ for $n = 1, 2, \dots, U$.

Similarly, the mean-square error associated with the imaginary part can be written as

$$E_I = \int_0^{\pi} \left(\sin(\hat{\rho}(\omega)) - \mathbf{b}^T\hat{\mathbf{s}}(\omega)\right)^2 d\omega \tag{13}$$

Again, by setting $\frac{\partial E_I}{\partial \mathbf{b}} = 0$, we get $\hat{\mathbf{Q}}_b\mathbf{b} = \hat{\mathbf{d}}_b$ where the elements of $\hat{\mathbf{Q}}_b$ are given by

$$\hat{Q}_b(n,m) = \begin{cases} \frac{\pi}{2} & n = m \\ 0 & n \neq m \end{cases} \tag{14}$$

for $1 \leq n, m \leq V$. The elements of $\hat{\mathbf{d}}_b$ are given by

$$\hat{d}_b(n) = \int_0^{\pi} \sin(\hat{\rho}(\omega))\sin((2n-1)\omega) \quad n = 1, 2, \dots, V \tag{15}$$

From eqns. (14) and (15), $b(n) = 2\hat{\mathbf{d}}_a(n)/\pi$ for $n = 1, 2, \dots, V$.

Example 3 *Chirp allpass phase equalizer*

The desired group delay of the chirp allpass equalizer is given by

$$\tau_\rho(\omega) = \frac{N-1}{2} + \frac{16}{\pi}\left(\omega - \frac{\pi}{2}\right) \tag{16}$$

The amplitude and group delay response of a 61-tap phase equalizer are shown in Figs. 3(a) and 3(b), respectively.

4.2 Antisymmetric phase characteristics

When $\hat{\rho}(\omega)$ is antisymmetric with respect to $\pi/2$, then the coefficients satisfy $h((N-1)/2 - n) = h((N-1)/2 + n) = 0$ for odd n where, again, N is odd [6]. As before, $\mathbf{h} = \mathbf{d}/\pi$. The dimension of $\mathbf{h}$ comes down due to the zero-valued coefficients.

Example 4 *Sine-delay allpass phase equalizer*

The desired characteristic of the sine-delay allpass equalizer is given by

$$\tau_\rho(\omega) = \frac{N-1}{2} - 2\pi\sin\omega \tag{17}$$

The amplitude and group delay response of a 61-tap phase equalizer are shown in Figs. 4(a) and 4(b), respectively.

5 Performance Results

In this section, we compare our design method with the eigenfilter approach from four points of view, namely, the number of floating point operations (flops), the mean-square error E_{mse}, the peak error given by

$$E_M = \max_{\omega \in P \cup S} |D(\omega) - H(e^{j\omega})|, \tag{18}$$

and the peak passband group-delay error given by

$$E_\tau = \max_{\omega \in P} |\tau_\rho(\omega) - \tau(\omega)| \tag{19}$$

A comparison of the two methods with respect to E_{mse}, E_M and E_τ is shown in Table 1 and that with respect to the number of flops is shown in Table 2 for the examples in the previous section. It must be mentioned that the entries in Table 2 have been normalized relative to the number of flops in our method. The reference frequencies for all the designs using the eigenfilter method have been chosen as in [7].

5.1 Error Measure

Our method formulates a better error measure than the eigenfilter method in that we explicitly minimize the mean-square error between the ideal response and the frequency response of the obtained filter. In contrast, the eigenfilter method does not take the ideal response into account. Rather, it uses a scaled version of the desired frequency response where the scaling factor is $H(e^{j\omega_0})/D(\omega_0)$ and ω_0 is an arbitrary reference frequency. As a consequence, our method yields a lower mean-square error. However, the differences in the value of E_{mse} obtained for both methods are small.

5.2 Computational Complexity

For our method, the filter parameters are obtained by a system of linear equations involving a positive-definite matrix $\mathbf{G} = (\alpha\mathbf{Q} + \beta\mathbf{R})$. It is well known that a real symmetric positive-definite matrix can be decomposed as $\mathbf{G} = \mathbf{L}\mathbf{L}^T$ where $\mathbf{L}$ is a real lower triangular matrix [8]. Consequently, the system of linear equations can be written as $\mathbf{L}\mathbf{L}^T\mathbf{a} = \mathbf{d}$. By letting $\mathbf{v} = \mathbf{L}^T\mathbf{a}$, we get $\mathbf{d} = \mathbf{L}\mathbf{v}$. Given $\mathbf{L}$ and $\mathbf{d}$, we can obtain $\mathbf{v}$ by recursively solving a set of linear equations. Let l_{ij} be the element in the ith row and jth column of $\mathbf{L}$. It can be shown that

$$v(n) = \frac{1}{l_{nn}}\left\{d(n) - \sum_{j=0}^{n-1} l_{nj}v(j)\right\} \tag{20}$$

for $n = 0, 1, \cdots, N_E - 1$, where N_E is the dimension of the system of equations. Since $\mathbf{G}$ is positive-definite, the l_{nn} in the above equation are nonzero. We first solve for $v(0)$ and then recursively obtain $v(n)$. A total of $N_E(N_E-1)/2$ multiplications, N_E divisions, and $N_E(N_E-1)/2$ additions are required to compute $\mathbf{v}$. Similarly, we can find the vector $\mathbf{a}$ for a given $\mathbf{v}$ and $\mathbf{L}$ by solving

$$a(n) = \frac{1}{l_{nn}}\left\{v(n) - \sum_{j=0}^{n-1} l_{nj}a(j)\right\} \tag{21}$$

The total time required to obtain the solution is

$$T_o = (T_a + T_m)N_E(N_E-1) + 2N_ET_d \tag{22}$$

where T_a, T_m and T_d are, respectively, the time required for one real addition, multiplication and division.

In the eigenfilter approach [7], the mean-square error is of the form $\mathbf{h}^T\mathbf{P}\mathbf{h}$ where $\mathbf{P}$ is a real, symmetric and positive-definite matrix. The coefficients of the filters are obtained as the eigenvector that corresponds to the smallest eigenvalue of $\mathbf{P}$. In order to compute the smallest eigenvalue and its corresponding eigenvector, generally an iterative inverse power method is used. At the $(k+1)$th iteration, a vector $\mathbf{x}_{k+1}$ is computed from the previous iterate $\mathbf{x}_k$ as

$$\mathbf{y}_{k+1} = \mathbf{P}^{-1}\mathbf{x}_k \tag{23}$$

$$\mathbf{x}_{k+1} = \mathbf{y}_{k+1}/\,||\,\mathbf{y}_{k+1}\,|| \tag{24}$$

where $||\,\mathbf{y}_{k+1}\,||$ denotes the L_2 norm of $\mathbf{y}_{k+1}$. If $||\,\mathbf{x}_{k+1} - \mathbf{x}_k\,|| \leq \epsilon$ (typically ϵ is about 10^{-6}), then $\mathbf{x}_{k+1}$ is a good approximation of the eigenvector corresponding to the smallest eigenvalue. We can rewrite Eqn. (23) as $\mathbf{x}_k = \mathbf{P}\mathbf{y}_{k+1}$. Using the technique described above for solving a system of linear equations, we can obtain $\mathbf{y}_{k+1}$ and subsequently $\mathbf{x}_{k+1}$.

It can now be seen that the eigenfilter method requires solving a system of linear equations several times before obtaining the eigenvector corresponding to the smallest eigenvalue. On the contrary, our approach requires solving a system of linear equations only once. Let T_e be the time taken for the eigenfilter method. If M is the number of iterations required in the eigenfilter method, the total time for solving the system of equations is MT_o. For each iteration, $\mathbf{y}_{k+1}$ is normalized by its L_2 norm. The time taken for this is $N_ET_d + T_{norm}$ where T_{norm} is the time required to compute the norm. Therefore,

$$T_e = M(T_o + N_ET_d + T_{norm}) \tag{25}$$

The value of M increases as the ratio λ_2/λ_1, where λ_1 is the smallest eigenvalue and λ_2 is the next smallest eigenvalue, decreases. If the ratio is too small, it may not even be possible to evaluate the smallest eigenvalue and its corresponding eigenvector using the inverse power method (the inverse power method may not converge).

The other aspect that influences the computational complexity is in finding the entries of $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{d}$ for our method and those of $\mathbf{P}$ for the eigenfilter approach. It can be seen from eqns. (4) and (5) that expressions for $\mathbf{Q}$ and $\mathbf{R}$ remain the same for any design. It is only the expression for $\mathbf{d}$ that is different for different designs. In general, the number of multiplications, additions and trigonometric function evaluations is more for the eigenfilter method. In particular, the computational complexity for fullband differentiators and allpass phase equalizers designed using the eigenfilter approach is much more than that designed using our method since numerical integration is involved in obtaining the entries of the associated matrices and vectors. It can be noted from Table 2 that the flops required for the eigenfilter method for these designs are significantly high.

6 Conclusions

In this paper, a method to design FIR filters satisfying arbitrary magnitude and phase specifications has been presented. In this method, the absolute mean-square error between the ideal and actual frequency response is explicitly minimized. This leads to a closed form solution for the filter coefficients in terms of a system of linear equations. The filter coefficients are found in a noniterative and computationally simple manner. The mean-square error is lower than that obtained by the eigenfilter method. Furthermore, the computational complexity achieved by our method is significantly lower than that achieved by the eigenfilter approach.

7 References

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List of captions

Figure 1: Frequency response of a 31-tap lowpass filter with $\omega_p = 0.12\pi$, $\omega_s = 0.24\pi$, and $\gamma = 12$. (a) Magnitude response. (b) Group delay response in the passband.

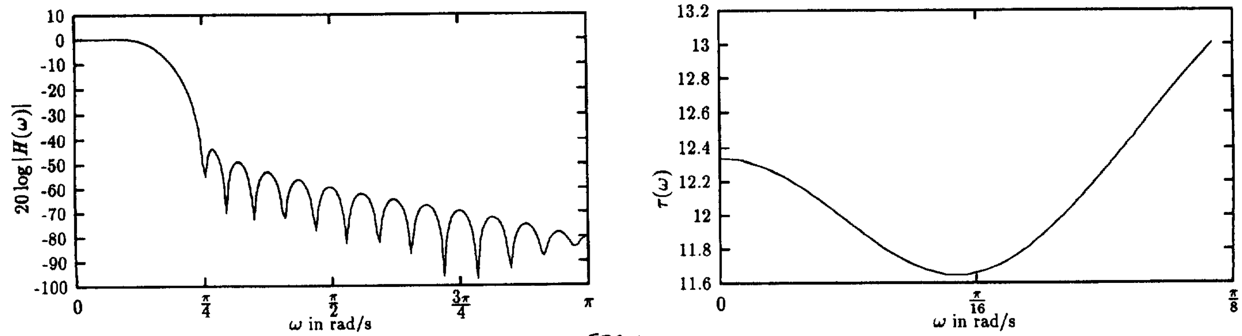
Figure 2: Frequency response of a 31-tap fullband differentiator with $\gamma = 11.5$. (a) Magnitude response. (b) Group delay response.

Figure 3: Frequency response of a 61-tap chirp allpass phase equalizer. (a) Magnitude response. (b) Group delay response.

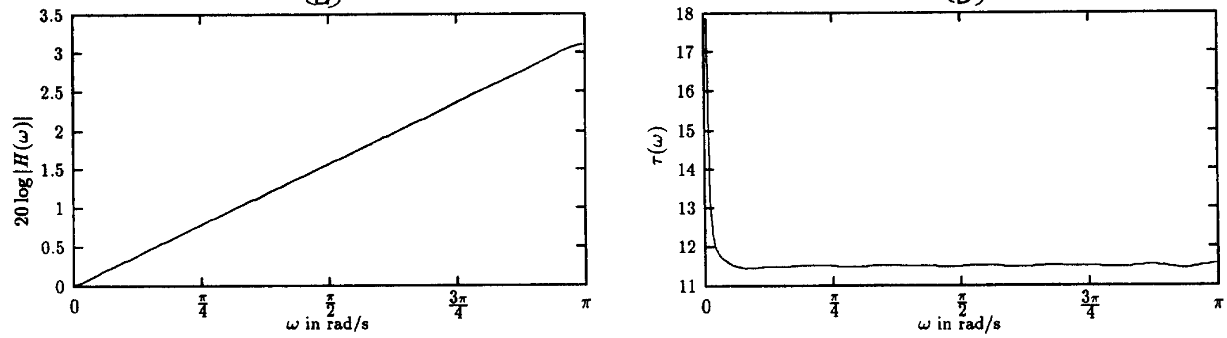
Figure 4: Frequency response of a 61-tap sine-delay allpass phase equalizer. (a) Magnitude response. (b) Group delay response.

Table 1: Comparison of the two methods with respect to the mean-square, peak magnitude and peak group-delay errors.

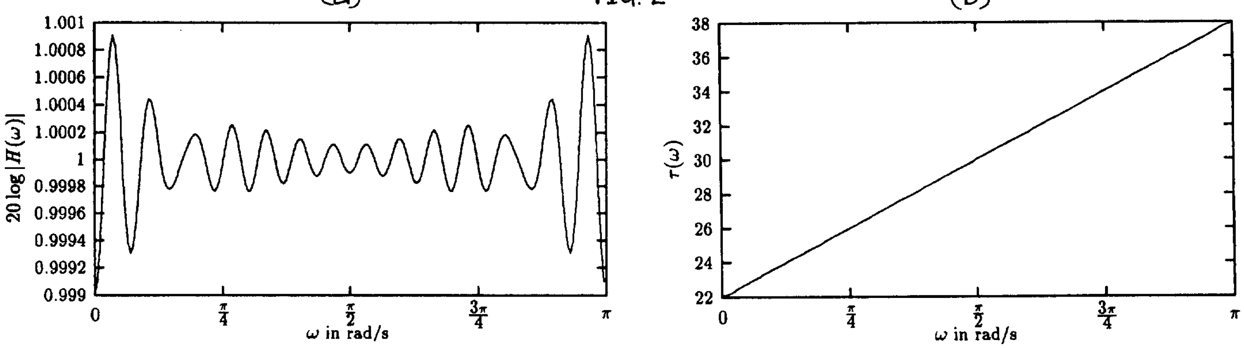
Table 2: Comparison of our method with the eigenfilter approach with respect to the number of floating point operations.



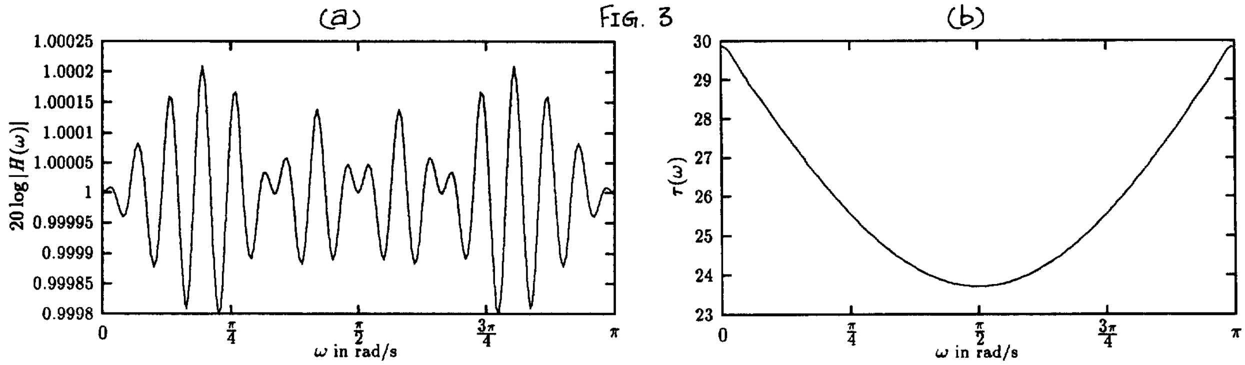
(a) FIG. 1 (b)



(a) FIG. 2 (b)



(a) FIG. 3 (b)



(a) FIG. 4 (b)

Examples	E_{mse}		E_M		E_τ	
	Our method	Eigenfilter method	Our method	Eigenfilter method	Our method	Eigenfilter method
1	6.414e-05	6.525e-05	6.706e-02	6.503e-02	1.007e-00	1.029e-00
2	2.439e-05	2.443e-05	4.325e-02	4.339e-02	4.587e-02	4.609e-02
3	1.803e-07	1.806e-07	1.769e-03	1.773e-03	1.172e-01	1.174e-01
4	2.934e-07	2.935e-07	1.583e-03	1.584e-03	1.290e-01	1.290e-01

TABLE 1

Examples	Flops	
	Our method	Eigenfilter method
1	1	3.64
2	1	234.61
3	1	19.57
4	1	330.11

TABLE 2