A UNIFIED FRAMEWORK FOR THE LEAST-SQUARES DESIGN OF LINEAR-PHASE NONRECURSIVE FILTERS

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Abstract

A method is described which can be used to design a wide class of nonrecursive linear-phase filters including those with arbitrary magnitude specifications and with time domain constraints. The approach is based on formulating the weighted mean-square error between the amplitude responses of the practical and ideal filters as a quadratic function. The filter coefficients are obtained by solving a set of linear equations. This method leads to a lower weighted mean-square error and is computationally more efficient than both the eigenfilter method and the method based on the Remez exchange algorithm. However, the main advantage of our method lies in its computational efficiency. Design examples of many different types of filters are provided.

Introduction

The design of nonrecursive filters with linear phase characteristics has been dealt with extensively over the past three decades. Over these years, the McClellan-Parks (MP) algorithm [1], used for the design of filters that are optimal in the minimax sense, and the least-squares methods have received considerable attention. The least-squares approach, first proposed in [2] for the design of lowpass filters, involves the solution of a linear system of equations to obtain the filter coefficients. This approach was later revisited in [3] resulting in the concept of eigenfilters. In the eigenfilter method, an error function between the desired and the practical amplitude responses is formulated in a quadratic form. The desired amplitude response is equal to the amplitude response of the designed filter at an arbitrary reference frequency. The coefficients of the filter are obtained as the eigenvector corresponding to the smallest eigenvalue of a real symmetric positive-definite matrix. This method has been used to design frequency selective filters including those involving time-domain constraints, differentiators and Hilbert transformers [4]-[5].

The method advanced in [2] has been used to design first-order differentiators in [6] and higher-order differentiators in [7]. In this paper, this method is generalized to accommodate various types of linear-phase nonrecursive filters including those with time domain constraints. The motivation of our approach is to formulate an error function that directly minimizes the weighted mean-square error by explicitly including the ideal amplitude characteristic and obtaining the filter coefficients with low computational complexity. This results in a more meaningful formulation than the eigenfilter method. Our design approach is for a class of least-square nonrecursive filters that includes the usual frequency selective filters (lowpass, highpass, bandpass and bandstop), differentiators, Hilbert transformers, maximally flat filters, interpolating filters, minimum energy filters and those with arbitrary magnitude characteristics. In addition, Nyquist filters that involve time-domain constraints are designed. We present design examples of some of these filters.

Characterization of Linear-Phase Nonrecursive Filters

Nonrecursive linear-phase filters with N taps having an impulse response $h(n)$ ($n = 0$ to $N-1$) can be divided into four types as described below. Type 1 and Type 2 filters include those with symmetrical impulse responses ($h(n) = h(N-1-n)$). Type 1 filters have an odd number of taps while Type 2 filters have an even number of taps. On the other hand, Type 3 and Type 4 filters have an antisymmetrical impulse response ($h(n) = -h(N-1-n)$). Type 3 filters have an odd number of taps while Type 4 filters have an even number of taps. The frequency response of Type 1 and Type 2 filters can be expressed as $H(e^{j\omega}) = M(\omega)e^{-j(N-1)\omega/2}$ where

$$M(\omega) = \begin{cases} \sum_{n=0}^{(N-1)/2} b(n)\cos n\omega & \text{Type 1} \\ \sum_{n=1}^{N/2} b(n)\cos(n-1/2)\omega & \text{Type 2} \end{cases} \tag{1}$$

If N is odd, $b(0) = h[(N-1)/2]$ and $b(n) = 2h[(N-1)/2-n]$ for $1 \le n \le (N-1)/2$. If N is even, $b(n) = 2h(N/2-n)$ for $1 \le n \le N/2$.

The frequency response of Type 3 and Type 4 filters can be expressed as $H(e^{j\omega}) = jM(\omega)e^{-j(N-1)\omega/2}$ where

$$M(\omega) = \begin{cases} \sum_{n=1}^{(N-1)/2} b(n)\sin n\omega & \text{Type 3} \\ \sum_{n=1}^{N/2} b(n)\sin(n-1/2)\omega & \text{Type 4} \end{cases} \tag{2}$$

If N is odd, $b(n) = 2h[(N-1)/2-n]$ for $1 \le n \le (N-1)/2$. If N is even, $b(n) = 2h(N/2-n)$ for $1 \le n \le N/2$.

Objective Function

In this section, we introduce the objective function E_{mse} that reflects the weighted mean-square difference between the ideal amplitude response, $D(\omega)$, and the amplitude response of the filter, $M(\omega)$. The function is expressed as

$$E_{\text{mse}} = \frac{\alpha}{\pi}\int_P W_P(\omega)[D(\omega)-M(\omega)]^2\,d\omega + \frac{\beta}{\pi}\int_S W_S(\omega)M^2(\omega)\,d\omega \tag{3}$$

where P is the passband and S is the stopband. The quantities α and β reflect the relative emphasis given to the passband and stopband, respectively. On the other hand, $W_P(\omega)$ and $W_S(\omega)$ are nonnegative frequency domain weighting functions for the passband and stopband that can be used to emphasize certain frequencies over others.

For the case of Type 1 and Type 2 filters, we have $M(\omega) = \mathbf{b}^T\mathbf{c}(\omega)$, where

$$\mathbf{b} = \begin{cases} [b(0)\ \ b(1)\ \ \cdots\ \ b((N-1)/2)]^T & \text{Type 1} \\ [b(1)\ \ b(2)\ \ \cdots\ \ b(N/2)]^T & \text{Type 2} \end{cases} \tag{4}$$

and

$$\mathbf{c}(\omega) = \begin{cases} [1\ \ \cos\omega\ \ \cdots\ \ \cos(\frac{N-1}{2}\omega)]^T & \text{Type 1} \\ [\cos\frac{1}{2}\omega\ \ \cos\frac{3}{2}\omega\ \ \cdots\ \ \cos(\frac{N-1}{2}\omega)]^T & \text{Type 2} \end{cases} \tag{5}$$

Filter coefficients that are optimal in a weighted least-squares sense are obtained by minimizing E_{mse}. Consequently, we set $\frac{\partial E_{\text{mse}}}{\partial b(i)} = 0$ to obtain a system of linear equations $[\alpha\mathbf{Q} + \beta\mathbf{R}]\mathbf{b} = \alpha\mathbf{d}$, where

$$\mathbf{Q} = \int_P W_P(\omega)\mathbf{c}(\omega)\mathbf{c}^T(\omega)\,d\omega \tag{6}$$

$$\mathbf{R} = \int_S W_S(\omega)\mathbf{c}(\omega)\mathbf{c}^T(\omega)\,d\omega \tag{7}$$

and

$$\mathbf{d} = \int_P W_P(\omega)D(\omega)\mathbf{c}(\omega)\,d\omega \tag{8}$$

Similarly, for Type 3 and Type 4 filters, $M(\omega) = \mathbf{b}^T\mathbf{s}(\omega)$, where

$$\mathbf{b} = \begin{cases} [b(1)\ \ b(2)\ \ \cdots\ \ b((N-1)/2)]^T & \text{Type 3} \\ [b(1)\ \ b(2)\ \ \cdots\ \ b(N/2)]^T & \text{Type 4} \end{cases} \tag{9}$$

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and

$$\mathbf{s}(\omega) = \begin{cases} [\sin\omega \quad \sin 2\omega \quad \cdots \quad \sin(\frac{N-1}{2}\omega)]^T & \text{Type 3} \\ [\sin\frac{1}{2}\omega \quad \sin\frac{3}{2}\omega \quad \cdots \quad \sin(\frac{N-1}{2}\omega)]^T & \text{Type 4} \end{cases} \tag{10}$$

The system of equations resulting from the minimization of the objective function is $[\alpha\mathbf{Q} + \beta\mathbf{R}]\mathbf{b} = \alpha\mathbf{d}$, where

$$\mathbf{Q} = \int_P W_P(\omega)\mathbf{s}(\omega)\mathbf{s}^T(\omega)\, d\omega \tag{11}$$

$$\mathbf{R} = \int_S W_S(\omega)\mathbf{s}(\omega)\mathbf{s}^T(\omega)\, d\omega \tag{12}$$

and

$$\mathbf{d} = \int_P W_P(\omega)D(\omega)\mathbf{s}(\omega)\, d\omega \tag{13}$$

It can be noted that both $\mathbf{Q}$ and $\mathbf{R}$ are positive-definite real symmetric matrices. Thus, a unique solution is guaranteed for nonnegative values of α and β. In addition, the system of linear equations can be solved by a computationally efficient method, like the Cholesky decomposition, that avoids matrix inversion.

In comparing our approach with the eigenfilter method, we consider two aspects, namely, weighted mean-square error and computational complexity. Our approach formulates a better error measure than the eigenfilter method in that we explicitly minimize the weighted mean-square error between the ideal and the practical amplitude responses. In contrast, the eigenfilter method does not take the ideal response into account, but rather the frequency response of the practical filter at an arbitrary frequency. Consequently, the filter that is designed depends upon the the reference frequency chosen. By considering the ideal response and not requiring a reference frequency, our approach will result in a filter with a lower weighted mean-square error than its eigenfilter counterpart.

The difference in the computational complexity of the two methods primarily lies in the algorithms used to obtain the filter coefficients. Our approach merely involves the solution of a system of linear equations. Consequently, it is noniterative and does not require any tests for convergence. For the eigenfilter method, the filter coefficients are obtained by computing the eigenvector corresponding to the smallest eigenvalue of a positive-definite symmetric matrix $\mathbf{P}$. Such an eigenvector is computed by using the inverse power method where a system of linear equations is solved M times. The value of M is almost always greater than one and is influenced by the smallest two eigenvalues of $\mathbf{P}$. Thus, for finding the filter coefficients, our method is about M times faster than the eigenfilter method. Another aspect that influences the computational complexity is the evaluation of the entries of $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{d}$ for our method and $\mathbf{P}$ for the eigenfilter method. Although, for properly chosen weighting functions this does not take up the major computational burden, the number of multiplication, additions and trigonometric function evaluations is generally more for the eigenfilter method.

Common Frequency Selective Filters

In this section, we consider the design of some of the common frequency selective filters for which $D(\omega)$ is 1 in P and 0 in S. For these filters, the ideal frequency response must be necessarily real and, consequently, only Type 1 and Type 2 filters are considered. For a lowpass filter, $P = [0, \omega_p]$ and $S = [\omega_s, \pi]$. In the case of a bandpass filter, $P = [\omega_{p1}, \omega_{p2}]$ and $S = [0, \omega_{s1}] \cup [\omega_{s2}, \pi]$.

Example 1

Example 1 is a bandpass filter in which $N = 51, \omega_{s1} = 0.3\pi, \omega_{s2} = 0.8\pi$, $\omega_{p1} = 0.35\pi$, $\omega_{p2} = 0.7\pi$, $\alpha = 2/3$, $\beta = 1/3$ and $W_P(\omega) = W_S(\omega) \doteq 1$. Figure 1 shows the magnitude response of this filter.

We compare our design method with the eigenfilter approach and the MP algorithm based on three performance measures, namely, (1) the number of floating point operations (flops), (2) the weighted mean-square error E_{mse}, and (3) the peak error $E_{\text{peak}} = \max(A_1, A_2)$ where

$$A_1 = \max_P |W_P(\omega)[D(\omega) - M(\omega)]| \tag{14}$$

and

$$A_2 = \max_S |W_S(\omega)M(\omega)| \tag{15}$$

A comparison of the three methods with respect to the number of flops is shown in Table 1 and that with respect to E_{mse} and E_{peak} is shown in Table 2. It must be mentioned that the entries in Table 1 have been normalized relative to the number of flops in our method.

It can be seen from Table 1 that our method is computationally more efficient than both the eigenfilter method and the MP algorithm. As has been mentioned earlier, the eigenfilter method is iterative and entails more function evaluations as compared to our method. For the MP algorithm, the computational complexity stems from the fact that it uses the iterative Remez exchange algorithm.

Since our method explicitly minimizes the weighted mean-square error, it guarantees the lowest E_{mse} for a given set of filter specifications. This is exemplified by the results in Table 2. However, the differences in the value of E_{mse} between our method and the eigenfilter approach are small. Since the MP algorithm minimizes the peak error, E_{peak} is the lowest for this approach.

Differentiators

An ideal kth-order differentiator has a frequency response

$$H_I(e^{j\omega}) = \begin{cases} D(\omega)\ e^{jk\pi/2} & \omega \in \mathrm{P} \\ 0 & \omega \in \mathrm{S} \end{cases} \tag{16}$$

where $D(\omega) = (\omega/2\pi)^k$. For even k, only Type 1 and Type 2 differentiators can be designed while for odd k, one is restricted to the design of Type 3 and Type 4 filters.

There are two passband weights often considered, namely, $W_P(\omega) = 1$ (absolute error measure) and $W_P(\omega) = 1/D^2(\omega)$ (relative error measure). It has been shown in [7] that for $k > 1$, the relative error measure is not feasible due to a discontinuity in the integrand. A detailed comparison of the design of higher-order differentiators using our method, the eigenfilter approach [5] and the MP algorithm [8] in terms of the three performance measures mentioned earlier has been carried out in [7]. It is shown that minimizing E_{mse} using our approach leads to a lower mean-square error and is computationally more efficient than the eigenfilter and minimax methods.

Example 2

In this example, we use the objective function in Eq. (3) to design a first-order differentiator with 31 taps. The passband is given by $P = [0, 0.9\pi]$ and $W_P(\omega) = 1$. There is no stopband. Figure 2 shows the magnitude response of this filter while Tables 1 and 2 give the details of the comparison.

Design example 2 reflects a performance comparison using an absolute error measure. Now, consider the use of a relative error measure. In this case, the entries of $\mathbf{Q}$ and $\mathbf{d}$ have to be evaluated using numerical integration. Similarly, numerical integration has to be used to calculate the entries of the matrix involved in the eigenfilter design. The difference in the computational complexity is due to the difference in the methods of obtaining the filter coefficients. Also, our method yields a lower relative mean-square error. By specifying a nonconstant weighting function for the MP algorithm, no extra complexity is involved. Consequently, it is the most efficient method for designing differentiators with a relative error measure.

At this point, we have demonstrated that our technique leads to a lower weighted mean-square error and is computationally more efficient than the eigenfilter and minimax approaches. In the sequel, we show that our approach can accommodate a wide variety of filters just like the eigenfilter method. Therefore, the advantages we offer come with no compromise in diversity.

Nyquist Filters

In bandwidth efficient data transmission systems, lowpass Nyquist filters are used to avoid intersymbol interference. Cancellation of intersymbol interference implies that the output of a data receiver taken at symbol intervals depends only on its corresponding transmitted symbol. To accomplish this, time domain constraints in the form of zero crossings are imposed on the impulse response of the filter. Since these zero crossings are symmetric about a center coefficient, the filter has an odd number of taps. Hence, we consider Type 1 filters for the design. The time domain constraints are specified as $h(n) = 0$ for $n - (N-1)/2$ = a nonzero multiple of L where L is the zero crossing or symbol interval. The frequency domain specifications dictate a lowpass characteristic where the passband and stopband are given by $P = [0, \omega_p]$ and $S = [\omega_s, \pi]$, respectively, such that $\omega_p + \omega_s = 2\pi/L$. The ideal amplitude response is given by $D(\omega) = 1$ in P. Our method can easily accommodate the time domain constraints by taking only the terms in $M(\omega)$ that correspond to the nonzero impulse response coefficients to minimize E_{mse}. In fact, the dimension of the linear system of equations is now lower than what would be the case if the zero crossings were not imposed.

Example 3

For this example, we design a Nyquist filter with 39 taps and a symbol interval $L = 4$. The passband and stopband edges are given by $\omega_p = 0.2125\pi$ and $\omega_s = 0.2875\pi$. Also, $\alpha = \beta = 1$ and $W_P(\omega) = W_S(\omega) = 1$. The magnitude response of the filter is shown in Fig. 3.

Interpolated FIR Filters

The term interpolated FIR (IFIR) filter refers to the nonrecursive implementation of a linear-phase FIR filter with transfer function $H(z)$, as a cascade of two linear-phase filters with transfer functions $G(z^L)$ and $K(z)$. This requires about $1/L$ of the number of multiplications and additions of a conventional nonrecursive implementation of an equivalent FIR filter [9]. For the design of a lowpass $H(z)$, $D(\omega) = 1$ in $P = [0, \omega_p]$ and $S = [\omega_s, \pi]$ are specified. Consequently, $G(z)$ will have a passband $[0, L\omega_p]$ and a stopband $[L\omega_s, \pi]$. Although $G(z^L)$ has the proper band edges ω_p and ω_s, its additional passband images must be removed by a lowpass $K(z)$. It must be noted that $K(z)$ can have a wide transition band and hence, a low order for a specified stopband attenuation. For the formulation, $G(z)$ must be designed to satisfy the requirements for $H(z)$ assuming that $K(z)$ is known. Design of IFIR filters based on the MP algorithm [9] and the eigenfilter method [10] exist. Here, we demonstrate the design of a Type 1 IFIR filter $H(z)$ using our method. Let $H(z)$, $G(z)$ and $K(z)$ have N, I and J taps, respectively. The amplitude response of $G(z)$, having an impulse response $g(n)$, is expressed as $\mathbf{r}^T\mathbf{c}(\omega)$ where $\mathbf{r} = [r(0)\ \ r(1)\ \ \cdots\ \ r((I-1)/2)]^T$, $r(0) = g((I-1)/2)$ and $r(n) = 2g((I-1)/2-n)$ for $1 \leq n \leq (N-1)/2$.

In order to design $G(z)$, Eq. (3) has to be expressed in terms of $\mathbf{r}$ by transforming $\mathbf{b}$. Following the development in [10], we let $\mathbf{h}$ be the vector of impulse response coefficients of $H(z)$ so that $\mathbf{b} = \mathbf{Eh}$ where $\mathbf{E}$ is a $(N-1)/2+1$ by N matrix given by

$$\mathbf{E} = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 1 & 0 & 1 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & 0 \\ 1 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{17}$$

If $\mathbf{g_1}$ is the vector of impulse response coefficients of $G(z^L)$, we can write $\mathbf{h} = \mathbf{Kg_1}$ where $\mathbf{K}$ is a N by $(I-1)L+1$ Toeplitz matrix whose first row is $[k(0)\ 0\ 0\ \cdots\ 0]$ and first column is $[k(0)\ \ k(1)\ \ \cdots\ \ k(J-1)\ \ 0\ \ \cdots\ \ 0]$. If $\mathbf{g}$ is the vector of impulse response coefficients of $G(z)$, we can write $\mathbf{g_1} = \mathbf{Sg}$. The matrix S has a block structure

$$\mathbf{S} = \begin{bmatrix} \mathbf{S_1} \\ \mathbf{S_2} \\ \vdots \\ \mathbf{S_I} \end{bmatrix} \tag{18}$$

where the $\mathbf{S_k}$ ($k = 1$ to $I-1$) are L by I matrices whose entries are all zero except $S_k(1,k) = 1$. The last block, $\mathbf{S_I}$ is an I element row vector whose entries are all zero except the last element which equals 1. Finally, $\mathbf{g} = \mathbf{Dr}$ where D is an I by $(I-1)/2+1$ matrix given by

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & \cdots & 0.5 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0.5 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 0.5 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0.5 \end{bmatrix} \tag{19}$$

Thus, $\mathbf{b} = \mathbf{EKSDr} = \mathbf{Xr}$. By substituting $\mathbf{Xr}$ for $\mathbf{b}$, the objective function of Eq. (3) can be expressed as

$$\begin{aligned} E_{\text{mse}} &= \frac{\alpha}{\pi}\int_P W_P(\omega)[1 - \mathbf{r}^T\mathbf{X}^T\mathbf{c}(\omega)]^2\, d\omega \\ &+ \frac{\beta}{\pi}\int_S W_S(\omega)[\mathbf{r}^T\mathbf{X}^T\mathbf{c}(\omega)]^2\, d\omega \end{aligned} \tag{20}$$

By minimizing E_{mse} with respect to $\mathbf{r}$, the filter coefficients for $G(z)$ are obtained by solving a system of linear equations. Subsequently, the coefficients for $H(z)$ are obtained.

Example 4

A 49-tap IFIR lowpass filter with $\omega_p = 0.3\pi$, $\omega_s = 0.4\pi$, $\alpha = \beta = 0.5$ and $W_P(\omega) = W_S(\omega) = 1$ is designed. For $G(z)$ and $K(z)$, we choose $L = 2$, $I = 21$ and $J = 9$. The filter $K(z)$ is designed by our approach with a passband edge of 0.36π and a stopband edge of 0.55π. The weighting factors used are $\alpha = 1$, $\beta = 3$ and $W_P(\omega) = W_S(\omega) = 1$. The magnitude response of the IFIR filter is shown in Fig. 4.

Conclusions

In this paper, a least-squares approach to design a wide variety of linear-phase nonrecursive filters through a common framework is presented. Characteristics of this approach are that it (a) achieves a direct route that explicitly considers the ideal amplitude response in the design procedure, thereby resulting in filters that are optimal in the least-squares sense, (b) offers a closed-form solution for the filter coefficients, (c) is noniterative and computationally efficient in finding the filter coefficients and (d) is highly diverse in accommodating different filters including those with arbitrary magnitude responses and time domain constraints. The weighted mean-square error and the computational complexity achieved by our method is lower than that achieved by the eigenfilter and minimax methods.

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List of captions

1. Magnitude response of a bandpass filter with $\omega_{s1} = 0.3\pi$, $\omega_{s2} = 0.8\pi$, $\omega_{p1} = 0.35\pi$, $\omega_{p2} = 0.7\pi$, $\alpha = 2/3$, $\beta = 1/3$, $W_P(\omega) = W_S(\omega) = 1$ and $N = 51$.

2. Magnitude response of a first-order differentiator with passband given by $P = [0, 0.9\pi]$, $W_P(\omega) = 1$ and $N = 31$.

3. Magnitude response of a Nyquist filter with $\omega_p = 0.2125\pi$, $\omega_s = 0.2875\pi$, $\alpha = \beta = 1$, $W_P(\omega) = W_S(\omega) = 1$, $N = 39$ and $L = 4$.

4. Magnitude response of a 49-tap IFIR lowpass filter with $\omega_p = 0.3\pi$, $\omega_s = 0.4\pi$, $\alpha = \beta = 0.5$ and $W_P(\omega) = W_S(\omega) = 1$.

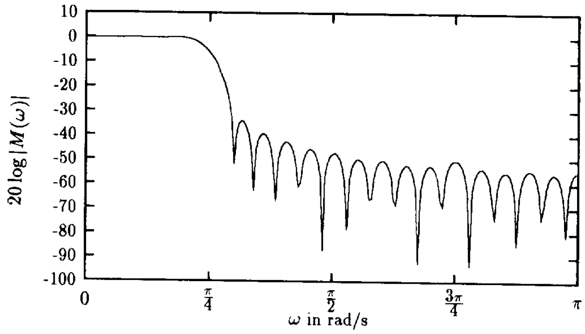

Figure 3:

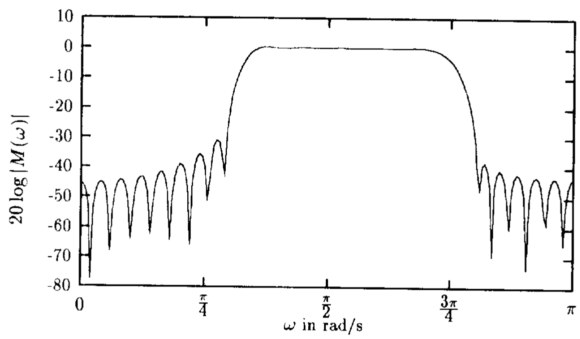

Figure 1:

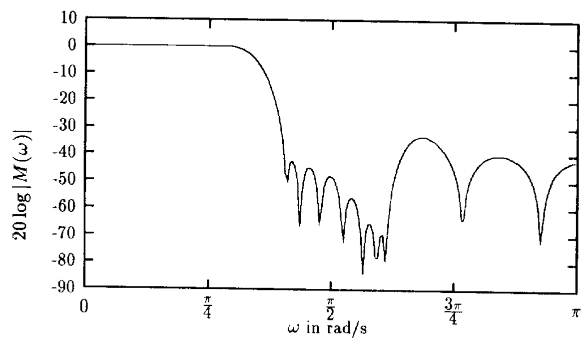

Figure 4:

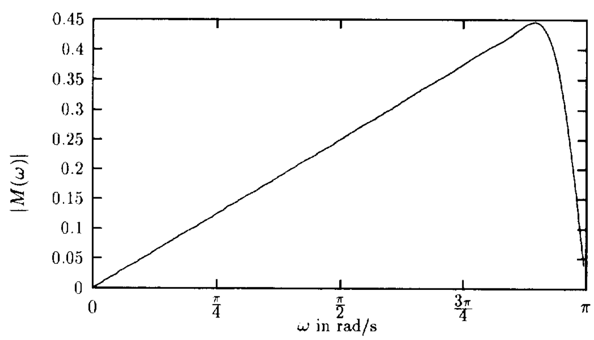

Figure 2:

Example	Floating point operations (flops) (Normalized relative to our method)		
	Our method	Eigenfilter method	MP method
1	1	4.80	5.94
2	1	3.44	12.82

Table 1: Comparison of the three methods with respect to the number of floating point operations.

Example	Our method		Eigenfilter method		MP method	
	E_{mse}	E_{peak}	E_{mse}	E_{peak}	E_{mse}	E_{peak}
1	3.840e-05	9.330e-02	3.946e-05	9.240e-02	1.982e-04	3.760e-02
2	2.729e-07	5.183e-03	2.739e-07	5.175e-03	5.426e-07	1.901e-03

Table 2: Comparison of the three methods with respect to E_{mse} and E_{peak}.