AN EFFICIENT LEAST-SQUARES APPROACH FOR THE DESIGN OF TWO-DIMENSIONAL LINEAR-PHASE NONRECURSIVE FILTERS

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ABSTRACT

A method is described which can be used to design two-dimensional nonrecursive linear-phase filters. The approach is based on formulating the absolute mean-square error between the amplitude responses of the practical and ideal digital filters as a quadratic function. The coefficients of the filters are obtained by solving a set of linear equations. This method leads to a lower mean-square error and is computationally more efficient than the eigenfilter method. The method is extended to the design of filters with time-domain constraints.

1. INTRODUCTION

The use of two-dimensional (2-D) nonrecursive filters is motivated by their inherent stability and their rendering of linear phase by the imposition of coefficient symmetry. This coefficient symmetry that achieves linear phase is important for image processing applications and for simplifying design and implementational complexity [1]. Additional coefficient symmetry can be imposed to further alleviate the design and computational effort [2]. Design approaches that are extensions of the approaches used for one-dimensional (1-D) filters include the use of windows and the frequency sampling technique [2]. The frequency transformation method starts with a 1-D linear phase filter designed by a 1-D technique and transforms it into a 2-D linear phase filter [2]. The transformation function is the Fourier transform of a 2-D zero-phase sequence. A well known example was introduced by McClellan [3]. Although designs based on the Chebyshev approximation problem exist [2], the methodology is not a simple extension of the Remez exchange algorithm. The eigenfilter approach proposed in [4] for the design of 1-D filters has recently been extended to the design of 2-D filters in [5]. In this method, an error function based on the difference between a desired response and the amplitude response of the practical filter is formulated. The desired response is equal to a scaled version of the ideal response where the scaling factor depends on the amplitude response of the designed filter at an arbitrary 2-D frequency. This is done to set up the error function in a quadratic form in order that the filter coefficients are found by computing the eigenvector corresponding to the smallest eigenvalue of a real, symmetric and positive-definite matrix.

In this paper, the least-squares approach for the design of 1-D nonrecursive linear-phase filters described in [6]-[8] is extended to the design of 2-D nonrecursive linear-phase filters. The procedure involves formulating the error between the practical and ideal responses as a quadratic function. The explicit inclusion of the ideal amplitude response in the error function leads to a more meaningful formulation than the eigenfilter method and does not necessitate the use of a reference frequency. The coefficients of the filter are obtained by solving a system of linear equations. By way of some design examples, our method is compared with the eigenfilter approach in terms of several performance measures and it is shown that our method is superior to the eigenfilter approach. Furthermore, we have extended our method to the design of 2-D minimum-energy and 2-D Nyquist filters.

2. QUADRANTALLY-SYMMETRIC FILTERS

A 2-D nonrecursive filter with N_1 by N_2 taps can be represented by the transfer function

$$H(z_1, z_2) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} h(n_1, n_2)\, z_1^{-n_1}\, z_2^{-n_2} \quad (1)$$

where $h(n_1, n_2)$ is the impulse response of the filter. By incorporating the linear-phase symmetry constraints, the frequency response of a 2-D filter is given by

$$H\left(e^{j\omega_1}, e^{j\omega_2}\right) = M(\omega_1, \omega_2) e^{-j\left(\frac{N_1-1}{2}\right)\omega_1} e^{-j\left(\frac{N_2-1}{2}\right)\omega_2} \quad (2)$$

where $M(\omega_1, \omega_2)$ is the amplitude response. For the case when the filter has quadrantal symmetry, the following condition holds

$$h(n_1, n_2) = h(N_1 - 1 - n_1, n_2) = h(n_1, N_2 - 1 - n_2) \quad (3)$$

for $0 \le n_1 \le N_1 - 1$ and $0 \le n_2 \le N_2 - 1$. When N_1 and N_2 are both odd, the amplitude response is given by

$$M(\omega_1, \omega_2) = \sum_{n_1=0}^{\frac{N_1-1}{2}} \sum_{n_2=0}^{\frac{N_2-1}{2}} a(n_1, n_2)\, \cos(n_1\omega_1)\cos(n_2\omega_2)$$

The coefficients $a(n_1, n_2)$ are related to the filter coefficients $h(n_1, n_2)$. On the other hand, when N_1 and N_2 are both even,

$$M(\omega_1, \omega_2) = \sum_{n_1=0}^{\frac{N_1}{2}} \sum_{n_2=0}^{\frac{N_2}{2}} b(n_1, n_2)\, \cos(n_1'\omega_1)\cos(n_2'\omega_2)$$

where $n_1' = n_1 - 0.5$ and $n_2' = n_2 - 0.5$. Again, the coefficients $b(n_1, n_2)$ are related to $h(n_1, n_2)$. Other cases of symmetry for N_1 odd and N_2 even and N_1 even and N_2 odd can be obtained in a similar manner.

The ideal linear-phase frequency response can be written as

$$H_I(e^{j\omega_1}, e^{j\omega_2}) = D(\omega_1, \omega_2) e^{-jQ_1\omega_1} e^{-jQ_2\omega_2} \quad (4)$$

By comparing (2) and (4), we note that a 2-D nonrecursive filter can be designed whose amplitude response approximates any arbitrary desired characteristic $D(\omega_1, \omega_2)$.

3. ERROR FUNCTION MINIMIZATION

The mean-square error between $D(\omega_1,\omega_2)$ and $M(\omega_1,\omega_2)$ can be expressed as

$$E_{mse} = \alpha \int\int_P [D(\omega_1,\omega_2) - M(\omega_1,\omega_2)]^2 \ d\omega_1 \ d\omega_2$$
$$+\beta \int\int_S M^2(\omega_1,\omega_2) \ d\omega_1 \ d\omega_2 \tag{5}$$

where P is the passband and S is the stopband in the (ω_1,ω_2) plane. As can be noted from Eq. (5), $D(\omega_1,\omega_2)$ is zero in the stopband. The quantities α and β reflect the relative emphasis given to the passband and stopband, respectively. By minimizing the error function with respect to the filter coefficients, the required filter can be designed.

For the case when N_1 and N_2 are odd, let the amplitude response be given by $M(\omega_1,\omega_2) = \mathbf{a}^T \mathbf{c}(\omega_1,\omega_2)$ where

$$\mathbf{a} = \begin{bmatrix} a(0,0) \\ a(0,1) \\ \vdots \\ a\left(0,\frac{N_2-1}{2}\right) \\ a(1,0) \\ a(1,1) \\ \vdots \\ a\left(1,\frac{N_2-1}{2}\right) \\ \vdots \\ a\left(\frac{N_1-1}{2},0\right) \\ a\left(\frac{N_1-1}{2},1\right) \\ \vdots \\ a\left(\frac{N_1-1}{2},\frac{N_2-1}{2}\right) \end{bmatrix}$$

and

$$\mathbf{c}(\omega_1,\omega_2) = \begin{bmatrix} 1 \\ \cos(\omega_2) \\ \vdots \\ \cos\left[\left(\frac{N_2-1}{2}\right)\omega_2\right] \\ \cos(\omega_1) \\ \cos(\omega_1)\cos(\omega_2) \\ \vdots \\ \cos(\omega_1)\cos\left[\left(\frac{N_2-1}{2}\right)\omega_2\right] \\ \vdots \\ \cos\left[\left(\frac{N_1-1}{2}\right)\omega_1\right] \\ \cos\left[\left(\frac{N_1-1}{2}\right)\omega_1\right]\cos(\omega_2) \\ \vdots \\ \cos\left[\left(\frac{N_1-1}{2}\right)\omega_1\right]\cos\left[\left(\frac{N_2-1}{2}\right)\omega_2\right] \end{bmatrix}$$

The mean-square error given by (5) can be rewritten as

$$E_{mse} = \alpha\int\int_P \left[D^2(\omega_1,\omega_2) - 2D(\omega_1,\omega_2)\mathbf{a}^T\mathbf{c}(\omega_1,\omega_2)\right.$$
$$\left. + \mathbf{a}^T\mathbf{c}(\omega_1,\omega_2)\mathbf{c}^T(\omega_1,\omega_2)\mathbf{a}\right] \ d\omega_1 \ d\omega_2$$
$$+\beta\int\int_S \mathbf{a}^T\mathbf{c}(\omega_1,\omega_2)\mathbf{c}^T(\omega_1,\omega_2)\mathbf{a} \ d\omega_1 \ d\omega_2$$

In minimizing E_{mse}, we set $\frac{\partial E_{mse}}{\partial \mathbf{a}} = 0$ to obtain a system of linear equations $(\alpha\mathbf{Q} + \beta\mathbf{R})\mathbf{a} = \alpha\mathbf{d}$ where

$$\mathbf{Q} = \int\int_P \mathbf{c}(\omega_1,\omega_2)\mathbf{c}^T(\omega_1,\omega_2) \ d\omega_1 \ d\omega_2 \tag{6}$$

$$\mathbf{R} = \int\int_S \mathbf{c}(\omega_1,\omega_2)\mathbf{c}^T(\omega_1,\omega_2) \ d\omega_1 \ d\omega_2 \tag{7}$$

$$\mathbf{d} = \int\int_P D(\omega_1,\omega_2)\mathbf{c}(\omega_1,\omega_2) \ d\omega_1 \ d\omega_2 \tag{8}$$

A similar development exists when N_1 and N_2 are both even such that a system of linear equations results. It can be noted from the above equations that $\mathbf{Q}$ and $\mathbf{R}$ are positive-definite, real and symmetric matrices. Consequently, the system of linear equations can be solved by a computationally efficient method, like the Cholesky decomposition, that avoids matrix inversion [9].

4. DESIGN EXAMPLES

In this section, we provide three design examples in which the entries of $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{d}$ are obtained either in closed form or by numerical integration. In the first two examples, these entries are obtained in closed form while in the last example, the entries are obtained by numerical integration. For the sake of comparison, the examples chosen here are the same as those presented in [5]. For all the designs, $\alpha = \beta = 1$ and $N = N_1 = N_2$.

Example 1 (*Rectangular lowpass filter*)
The desired amplitude response for this filter is given by

$$D(\omega_1,\omega_2) = \begin{cases} 1 & P: & 0 \le \omega_1 \le \omega_{p1} \\ & & 0 \le \omega_2 \le \omega_{p2} \\ 0 & S: & \omega_{a1} \le \omega_1 \le \pi \\ & & \omega_{a2} \le \omega_2 \le \pi \end{cases}$$

For this design $\omega_{p1} = \omega_{p2} = 0.4\pi$, $\omega_{a1} = \omega_{a2} = 0.6\pi$ and $N = 27$. Figure 1 shows the magnitude response of the designed filter.

Example 2 (*Fan filter*)
For this filter the desired amplitude response is given by

$$D(\omega_1,\omega_2) = \begin{cases} 1 & P: & 0 \le \omega_1 \le \pi \\ & & \omega_1 \le \omega_2 \le \pi \\ 0 & S: & \omega_a \le \omega_1 \le \pi \\ & & 0 \le \omega_2 \le \pi - \omega_a \end{cases}$$

Here, N has been chosen to be equal to 23 and $\omega_a = 0.16\pi$. The magnitude response of the designed filter is shown in Fig. 2.

Example 3 (*Minimum energy filter*)
A minimum energy filter is one which yields the smallest stopband energy subject to a constraint that is imposed to avoid a zero solution vector. Since our error function does not accommodate constraints, we transform the problem to that of an unconstrained minimization. The idea is to make the amplitude response at zero frequency close to unity and minimize the stopband energy. The modified error function is given by

$$E_{mse} = \alpha\left[1 - M(0,0)\right]^2 + \beta\int\int_S M^2(\omega_1,\omega_2) \ d\omega_1 \ d\omega_2 \tag{9}$$

The filter coefficients are again obtained by solving a system of equations. In this example, a minimum-energy filter is designed for which the stopband extends outward of a circular region of radius equal to 0.3π. Figure 3 shows the magnitude response of the designed filter where $N = 15$.

5. PERFORMANCE RESULTS

In this section, we compare our design method with the eigenfilter approach from two points of view, namely, the number of floating point operations (flops), and the mean-square error E_{mse}. The reference frequencies for all the designs using the eigenfilter method have been chosen as in [5].

5.1. Computational Complexity

For our method, the filter parameters are obtained by a system of linear equations involving a positive-definite matrix $F = \alpha Q + \beta R$ and a right hand side vector $g = \alpha d$. It can be seen that F is a real, symmetric, and positive-definite matrix. Therefore, F can be decomposed as $F = LL^T$ where L is a real lower triangular matrix [9]. Consequently, the system of linear equations can be written as $LL^T a = g$. By letting $v = L^T a$, we get $g = Lv$. Given L and g, we can obtain v by recursively solving a set of linear equations. Let l_{ij} be the element in the ith row and jth column of L. It can be shown that

$$v(n) = \frac{1}{l_{nn}} \left\{ g(n) - \sum_{j=0}^{n-1} l_{nj} v(j) \right\} \qquad (10)$$

for $n = 0, 1, \cdots, N_t - 1$ where N_t is the dimension of the system. Since F is positive-definite, the l_{nn} in the above equation are nonzero. We first solve for $v(0)$ and then recursively obtain $v(n)$. A total of $N_t(N_t - 1)/2$ multiplications, N_t divisions, and $N_t(N_t - 1)/2$ additions are required to compute v. Similarly, we can recursively obtain the vector a for a given v and L. The total time required to obtain the solution is

$$T_o = T_{chol} + T_{eqt} \qquad (11)$$

where T_{chol} is the time required to decompose F and

$$T_{eqt} = (T_a + T_m)N_t(N_t - 1) + 2N_t T_d \qquad (12)$$

Here T_a, T_m, T_d, are, respectively, the time required for one real addition, multiplication, and division.

In the eigenfilter approach, the total mean-square error is formulated as $E_{mse} = a^T P a$. It can be noted that P is a real, symmetric, and positive-definite matrix. The coefficients of the filters are obtained as the eigenvector corresponding to the smallest eigenvalue of P. In order to compute the smallest eigenvalue and its corresponding eigenvector, generally an iterative inverse power method is used [4]. At the $(k + 1)$th iteration, a vector x_{k+1} is computed from the previous iterate x_k as

$$y_{k+1} = P^{-1} x_k \qquad (13)$$

$$x_{k+1} = y_{k+1} / \| y_{k+1} \| \qquad (14)$$

where $\| y_{k+1} \|$ denotes the L_2 norm of y_{k+1}. If $\| x_{k+1} - x_k \| < \epsilon$ (typically ϵ is about 10^{-6}), then x_{k+1} is a good approximation of the eigenvector corresponding to the smallest eigenvalue. We can rewrite (13) as $x_k = P y_{k+1}$. Using the technique described above for solving a system of linear equations, we can obtain y_{k+1} and subsequently x_{k+1}.

It can now be seen that the eigenfilter method requires solving a system of linear equations several times before obtaining the eigenvector corresponding to the smallest eigenvalue. On the contrary, our approach requires solving a system of linear equations only once. If M is the number of iterations required in the eigenfilter method, then the total time taken for obtaining the filter coefficients using the eigenfilter method is

$$T_e = T_{chol} + M(T_{eqt} + T_{nd}) \qquad (15)$$

where T_{nd} is the time taken to obtain x_{k+1} from y_{k+1} using Eq. (14). The value of M increases as the ratio λ_2/λ_1, where λ_1 is the smallest eigenvalue and λ_2 is the next smallest eigenvalue, decreases. If the ratio is too small, it may not even be possible to evaluate the smallest eigenvalue and its corresponding eigenvector using the inverse power method (the inverse power method may not converge).

The other aspect that influences the computational complexity is in finding the entries of Q, R and d for our approach and those of P for the eigenfilter approach. The expressions for Q and R are independent of $D(\omega_1, \omega_2)$. However, the use of either a closed-from expression or numerical integration in finding the entries of Q and R is purely dictated by the passband and stopband regions. On the other hand, the expression for d is influenced by both $D(\omega_1, \omega_2)$ and the passband region. Even for a simple passband region (like a rectangular shape), the nature of $D(\omega_1, \omega_2)$ may mandate numerical integration for determination of the entries. For the eigenfilter approach, the matrix P is influenced by $D(\omega_1, \omega_2)$, the passband region and the stopband region. For the designs in which the entries of Q, R and d for our method and P for the eigenfilter method are evaluated in closed form, the saving in flops is significantly high. In the case of design example 1, the eigenfilter method requires 4.38 times the number of flops taken by our method. In the case of design example 2, the eigenfilter method requires 2 times the number of flops taken by our method. This is due to the fact that more functions have to be evaluated in the eigenfilter method as compared to our method. Moreover, the eigenfilter method is iterative while our method is not. When the entries of Q, R and d for our method and P for the eigenfilter method are evaluated using numerical integration, the saving in flops is not so significant. The computational complexity for the design of minimum-energy filters is almost the same for both our approach and the eigenfilter method. The small saving in flops is due to the eigenfilter method being iterative.

5.2. Error Measure

Our method formulates a better error measure than the eigenfilter method in that we explicitly minimize the mean-square error between the ideal and the practical amplitude responses. In contrast, the eigenfilter method does not take the ideal response into account but rather the frequency response of the practical filter at an arbitrary reference frequency. In fact, the filter that is designed depends upon the reference frequency. The performance comparisons confirm that our method leads to a lower mean-square error than the eigenfilter method although the differences are generally small. The most glaring difference is for the rectangular lowpass filter in that for our method, $E_{mse} = 1.6e - 05$ and for the eigenfilter approach, $E_{mse} = 2.1e - 05$.

6. TWO-DIMENSIONAL NYQUIST FILTERS

A 1-D zero-phase Nyquist filter with an impulse response $h(n)$ has an odd number of taps, is lowpass, and has time-domain constraints in the form of equally spaced zero crossings about $n = 0$ [4][8]. For the case of an N_1 by N_2 2-D zero-phase Nyquist filter with an impulse response $h(n_1, n_2)$, N_1 and N_2 are both odd and the time-domain constraints are of the form $h(t_{11}n_1 + t_{12}n_2, t_{21}n_1 + t_{22}n_2) = 0$ under the conditions that $(n_1, n_2) \neq (0, 0)$ and the matrix

$$T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \qquad (16)$$

is nonsingular and has only integer entries [10]. The frequency domain requirement is that the Nyquist filter be lowpass with $D(\omega_1, \omega_2) = 1$ in a passband region P and have a stopband region S. The time-domain constraints defined above are consistent with a zero-phase characteristic.

However, to achieve quadrantal symmetry, additional conditions on **T** must be specified. One sufficient condition for a Nyquist filter to have quadrantal symmetry is that **T** be diagonal. For the design, as in the 1-D case [8], the imposed zero-valued impulse response coefficients do not appear in the minimization of E_{mse}. As a consequence, the dimension of the system of equations to be solved is reduced.

Example 4 (*Two-dimensional Nyquist filter*)

In this example. we design a Nyquist filter with parameters $N_1 = N_2 = 23$, $t_{11} = 3$, $t_{22} = 4$ and $t_{12} = t_{21} = 0$. The passband region is given by $0 \leq \omega_1 \leq \pi/6$ and $0 \leq \omega_2 \leq \pi/2$. The stopband region is given by $\pi/8 \leq \omega_1 \leq \pi$ and $3\pi/8 \leq \omega_2 \leq \pi$. Figure 4 shows the magnitude response of the filter.

7. CONCLUSIONS

In this paper, a method to design 2-D nonrecursive linear-phase filters has been presented. In this method, we explicitly minimize the absolute mean-square error between the ideal and actual frequency responses. This leads to a closed form solution for the filter coefficients in terms of a system of linear equations. The filter coefficients are found in a noniterative and computationally simple manner. It has been shown that the filters designed using our method has lower mean-square error as compared to that designed using the eigenfilter method. Moreover, the computational complexity in our method is significantly lower than in the eigenfilter method. Finally, our method has been extended to the design of filters with time-domain constraints.

8. REFERENCES

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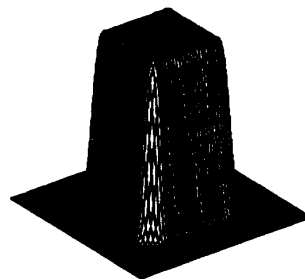
Figure 1: Magnitude response of a 27×27 rectangular lowpass filter with $\omega_{p1} = \omega_{p2} = 0.4\pi$ and $\omega_{a1} = \omega_{a2} = 0.6\pi$.
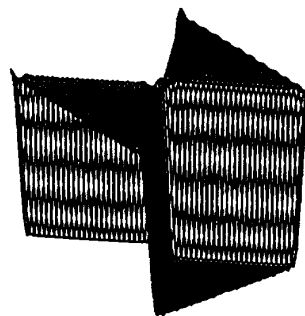

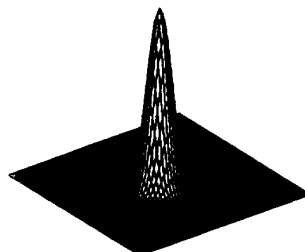Figure 2: Magnitude response of a 23×23 fan filter with $\omega_a = 0.16\pi$.

Figure 3: Magnitude response of a 15×15 minimum energy filter for which the stopband extends outward of a circular region of radius equal to 0.3π.
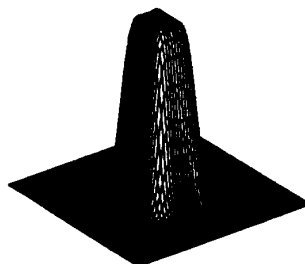

Figure 4: Magnitude response of a 23×23 Nyquist filter.