SOME PROPERTIES OF THE Z-DOMAIN CONTINUED FRACTION EXPANSIONS OF 1-D DISCRETE REACTANCE FUNCTIONS

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ABSTRACT

The denominator polynomial of a given causal stable z-domain transfer function is modified so that the magnitude of the frequency response remains the same. This simple modification permits an infinite number of decompositions of the modified denominator into a mirror-image polynomial (MIP) and an anti-mirror-image polynomial (AMIP). Two types of Discrete Reactance Functions (DRF) are constructed. From these DRFs, continued fraction expansions (CFE) are considered and some properties are obtained. These properties indicate whether the original denominator polynomial has all its roots within the unit circle (is minimum phase) or not.

1. INTRODUCTION

The stability of a one-dimensional (1-D) discrete transfer function is determined in several ways. A popular approach is the Jury criterion [1] Another approach is to decompose the nth order denominator polynomial $D_n(z)$ as a sum of a mirror-image polynomial (MIP) and an anti-mirror-image polynomial (AMIP) and based on their properties, the stability is determined. One of the decompositions is due to Schussler [2]. This is used in the implementation of a stability test [3]. Another decomposition is due to Davis [4] which can also be used to determine the stability of the system [5]. In this paper, it is shown that several other new possibilities of decomposing $z^{q}D_{n}(z)$ into a sum of a MIP and an AMIP exist and some of the properties of such polynomials are discussed especially in terms of stability. For the stability checks, two new types of Discrete Reactance Functions (DRF) are constructed and new continued fraction expansions (CFE) are considered. It is noted that for a given numerator polynomial in the transfer function, the magnitude response is unaltered, when the denominator is either $D_n(z)$ or $z^q D_n(z)$.

2. MIRROR-IMAGE AND ANTI-MIRROR-IMAGE POLYNOMIALS

Let

$$D_n(z) = \sum_{i=0}^n d(i) z^i \tag{1}$$

be a minimum phase polynomial in that all its roots are within the unit circle in the z-plane. Then, Jury's necessary conditions [1],

namely,

$$D_n(1) > 0$$
 (2)
 $(-1)^n D_n(-1) > 0$ (3)

are satisifed. One can formulate a stable allpass transfer function given by

1

$$H_{n+q}(z) = \frac{z^n D_n(z^{-1})}{z^q D_n(z)}$$
(4)

where $q \ge 0$ and n + q is the order of $H_{n+q}(z)$. Based on $D_n(z)$, we can now define a mirror-image-polynomial (MIP) $M_{n+q}(z)$ and an anti-mirror-image-polynomial (AMIP) $A_{n+q}(z)$ as follows:

$$M_{n+q}(z) = \frac{1}{2} [z^q D_n(z) + z^n D_n(z^{-1})]$$
(5)

$$\mathbf{A}_{n+q}(z) = \frac{1}{2} [z^q D_n(z) - z^n D_n(z^{-1})]$$
(6)

In general, a polynomial of order n, namely, $P_n(z)$ is an MIP if $P_n(z) = z^n P_n(z^{-1})$. Similarly, $P_n(z)$ is an AMIP if $P_n(z) = -z^n P_n(z^{-1})$. Obviously, $M_{n+q}(z)$ is an (n+q)th order polynomial obtained as half the sum of the denominator and the numerator polynomials of $H_{n+q}(z)$. Similarly, $A_{n+q}(z)$ is an (n+q)th order polynomial obtained as half the difference of the denominator and the numerator polynomials of $H_{n+q}(z)$. It is clear that, depending on the value of q chosen, an infinite number of pairs of MIPs and AMIPs is obtained. Reconstruction of $D_n(z)$ is possible as

$$D_n(z) = z^{-q} (M_{n+q}(z) + A_{n+q}(z))$$
(7)

We will now consider some properties of these polynomials. The proofs of the various results are not given due to space considerations.

3. PROPERTIES AND IDENTIFICATIONS OF MIP AND AMIP

First, we have introduced a novel general MIP and AMIP identification given in Eq. (5) and Eq. (6). The general factorization properties of $M_{n+q}(z)$ and $A_{n+q}(z)$ are given as:

Case 1: Suppose |q - n| is odd. Since (z + 1) is only a factor of $M_{n+q}(z)$ and (z-1) is only a factor of $A_{n+q}(z)$, one can write

$$M_{n+q}(z) = (z+1)M_{n+q-1}^{(-1)}(z)$$
(8)

$$A_{n+q}(z) = (z-1)M_{n+q-1}^{(1)}(z)$$
(9)

where $M_{n+q-1}^{(-1)}(z)$ and $M_{n+q-1}^{(1)}(z)$ are different MIP's of order n + q - 1. The superscript (-1) indicates that the MIP was derived from an MIP obtained from $D_n(z)$ using Eq. (5) with the root at z = -1 removed or deconvolved. Similarly, the superscript 1 indicates that the MIP was derived from an AMIP obtained from $D_n(z)$ using Eq. (6) with the root at z = 1 deconvolved.

Case 2: Suppose |q - n| is even. Since (z + 1) and (z - 1) are only factors of $A_{n+q}(z)$, one can write

$$A_{n+q}(z) = (z-1)M_{n+q-1}^{(1)}(z)$$
(10)

$$= (z+1)A_{n+a-1}^{(-1)}(z)$$
(11)

where $M_{n+q-1}^{(1)}(z)$ is an MIP of order n + q - 1 and $A_{n+q-1}^{(-1)}(z)$ is an AMIP of order n + q - 1. Roots at $z = \pm 1$ are approxiately deconvolved.

The new identification and general factorization properties we give above have special cases. First, consider q = 0. For n odd, the factorization given in Case 1 above holds in that

$$M_n(z) = \frac{1}{2} [D_n(z) + z^n D_n(z^{-1})]$$
(12)

$$= (z+1)M_{n-1}^{(-1)}(z)$$
(13)

and

$$A_n(z) = \frac{1}{2} [D_n(z) - z^n D_n(z^{-1})]$$
(14)

$$= (z-1)M_{n-1}^{(1)}(z)$$
(15)

Similarly, for n even, the factorization given in Case 2 above holds in that

$$A_n(z) = \frac{1}{2} [D_n(z) - z^n D_n(z^{-1})]$$
(16)

$$= (z-1)M_{n-1}^{(1)}(z)$$
(17)

$$= (z+1)A_{n-1}^{(-1)}(z)$$
 (18)

The special case q = 0 is Schussler's identification [2]. Reconstruction is possible using $D_n(z) = M_n(z) + A_n(z)$. For $D_n(z)$ to be minimum phase, it is necessary and sufficient that (1) $|d_n| > |d_0|$ and (2) the roots of the MIP $M_n(z)$ and the AMIP $A_n(z)$ are on the unit circle, simple and interlace. This has been further implemented as a part of a stability test [3] and a subsequent formulation for the design of optimum least-squares infinite impulse response filters in which stability is guaranteed [6].

Another special case arises when q = 1. For *n* even and *n* odd, the factorizations given in Cases 1 and 2 respectively, are applicable. For even *n*,

$$M_{n+1}(z) = \frac{1}{2} [z D_n(z) + z^n D_n(z^{-1})]$$
(19)

$$= (z+1)M_n^{(-1)}(z)$$
 (20)

and

$$A_{n+1}(z) = \frac{1}{2} [z D_n(z) - z^n D_n(z^{-1})]$$
(21)

$$= (z-1)M_n^{(1)}(z)$$
 (22)

For n odd,

$$A_{n+1}(z) = \frac{1}{2} [z D_n(z) - z^n D_n(z^{-1})]$$
(23)

$$= (z-1)M_n^{(1)}(z)$$
 (24)

$$= (z+1)A_n^{(-1)}(z)$$
 (25)

Reconstruction is possible using $D_n(z) = z^{-1}(M_{n+1}(z)+A_{n+1}(z))$. Again, for a minimum phase $D_n(z)$, it is necessary and sufficient that the roots of the MIP $M_{n+1}(z)$ and the AMIP $A_{n+1}(z)$ are on the unit circle, simple and interlace. This special case is the line spectral frequency (LSF) formulation commonly used in speech processing. The LSFs were first introduced by Itakura [7] as the angular frequencies of the unit circle roots of the MIP $M_{n+1}(z)$ and the AMIP $A_{n+1}(z)$ neglecting the roots at $z = \pm 1$. The polynomial $D_n(z)$ is derived by linear predictive analysis [8]. The LSFs are commonly used in speech coding due to their intimate relationship with the speech spectral envelope thereby making them conducive for transmission at low bit rates [9][10][11].

The case q = 1 has also resulted in Davis' identification of $D_n(z)$ as the sum of an MIP and an AMIP of different orders derived from the Schussler and LSF formulations [4]. This has also been implemented as a part of a stability test [5]. For even *n*, Davis' identification is

$$D_n(z) = M_n^{(-1)}(z) + A_{n-1}^{(-1)}(z)$$
(26)

where $M_n^{(-1)}(z)$ is defined in Eq. (20) and $A_{n-1}^{(-1)}(z)$ is defined in Eq. (18). For odd n, Davis' identification is

$$D_n(z) = M_{n-1}^{(-1)}(z) + A_n^{(-1)}(z)$$
(27)

where $M_{n-1}^{(-1)}(z)$ is defined in Eq. (13) and $A_n^{(-1)}(z)$ is defined in Eq. (25).

4. STABILITY CONSIDERATIONS BASED ON PROPERTIES OF MIP AND AMIP

Here, we introduce new results based on $M_{n+q}(z)$ and $A_{n+q}(z)$ defined in Eq. (5) and Eq. (6).

Theorem 1: The magnitude of the allpass function $H_{n+q}(z)$ (see Eq. (4)) on various circles in the z-plane of different radii r is given as

$$|H_{n+q}(r\epsilon^{j\omega})| \text{ is } \begin{cases} >1 & \text{for } r < 1\\ =1 & \text{for } r = 1\\ <1 & \text{for } r > 1 \end{cases}$$
(28)

This result leads to Theorem 2.

Theorem 2: Let

$$R_{n+q}(z) = \frac{M_{n+q}(z)}{A_{n+q}(z)}$$
(29)

Then,

$$\operatorname{Re}(R_{n+q}(re^{j\omega})) \text{ is } \begin{cases} < 0 & \text{for } r < 1 \\ = 0 & \text{for } r = 1 \\ > 0 & \text{for } r > 1 \end{cases}$$
(30)

where Re() denotes the real part.

Theorem 2 shows that $R_{n+q}(z)$ is a Positive Exterior Function (PEF). Therefore, $R_{n+q}(z)$ contains all its poles and zeros on the unit circle, which are simple and interlace [12]. The PEF $R_{n+q}(z)$ and its reciprocal $1/R_{n+q}(z)$ are Discrete Reactance Functions (DRFs). This permits us to define two types of DRFs, namely:

1. Type A: The degrees of the numerator and the denominator polynomials are the same. The DRFs are $M_{n+q}(z)/A_{n+q}(z)$ and its reciprocal for various values of $q \ge 0$.

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Figure 1: Roots of $D_3(z)$ (denoted by a +), $M_5(z)$ (denoted by a \bigcirc) and $A_5(z)$ (denoted by an X).

2. Type B: The degree difference between the numerator and the denominator polynomials is one. For any $q \ge 1$, the various DRFs are $M_{n+q}(z)/A_{n+q-1}(z), M_{n+q}(z)/A_{n+q+1}(z), M_{n+q+1}(z), M_{n+q+1}(z)/A_{n+q}(z)$ and their reciprocals.

Each type of DRF can be expanded into continued fractions (CFEs).

Theorem 3: Type A DRF permits the following forms of CFEs: CFE1:

$$a_{1}\frac{z-1}{z+1} + \frac{1}{a_{2}\frac{z-1}{z+1} + \frac{1}{a_{3}\frac{z-1}{z+1} + \frac{1}{\cdots a_{n+q}\frac{z-1}{z+1}}}$$
(31)

CFE2:

$$\frac{z+1}{z-1} + \frac{1}{b_2 \frac{z+1}{z-1} + \frac{1}{b_3 \frac{z+1}{z-1} + \frac{1}{\cdots b_n + a} \frac{z+1}{z+1}}}$$

CFE3: When n + q is even, we get

 b_1

$$c_{1}\frac{z+1}{z-1} + c_{2}\frac{z-1}{z+1} + \frac{1}{c_{3}\frac{z+1}{z-1} + c_{4}\frac{z-1}{z+1} + \frac{1}{\cdots c_{n+q-1}\frac{z+1}{z-1} + c_{n+q}\frac{z-1}{z+1}}}$$
(33)

CFE3: When n + q is odd, we get

$$d_0 \frac{z+1}{z-1} + \frac{1}{d_1 \frac{z+1}{z-1} + d_2 \frac{z-1}{z+1} + \frac{1}{\cdots d_{n+q-2} \frac{z+1}{z+1} + d_{n+q-1} \frac{z-1}{z+1}}}$$
(34)

The CFEs CF1 and CF2 have been indicated in [13]. Stability of $1/D_n(z)$ is ensured when each of the set of coefficients a_i, b_i , and either of c_i or d_i is positive. Each individual set of coefficients provide the necessary and sufficient conditions for stability. Hence, any one CFE can be used. Note also that some of the coefficients may not exist. For example, if z + 1 is not a factor of the denominator polynomial of the DRF, a_1 and c_2 will not exist. In this situation, the CFE can still be considered or the CFE of the reciprocal can be considered. Similarly, if z - 1 is not a factor of the denominator polynomial of the DRF, b_1, c_1 and d_0 will not exist. Again,

DRF	CFE1	CFE2
$M_{3}(z)/A_{3}(z)$	a_1 does not exist	$b_1 = 1$
	$a_2 = 1/9$	$b_2 = 7/8$
	$a_3 = 81/56$	$b_{3} = 8$
	$a_4 = 56/63$	
$M_4(z)/A_4(z)$	$a_1 = 1/10$	$b_1 = 1/2$
	$a_2 = 50/73$	$b_2 = 4/11$
	$a_3 = 1387/840$	$b_3 = 121/96$
	$a_4 = 336/511$	$b_4 = 96/11$
$M_5(z)/A_5(z)$	a_1 does not exist	$b_1 = 1/3$
	$a_2 = 1/11$	$b_2 = 63/64$
	$a_3 = 121/256$	$b_3 = 128/93$
	$a_4 = 2048/1793$	$b_4 = 961/576$
	$a_5 = 3214849/1951488$	$b_5 = 576/31$
	$a_6 = 243936/138061$	

Table 1: Coefficients of CFE1 and CFE2 for three different Type A DRFs corresponding to q = 0, 1 and 2.

the CFE can still be considered or the CFE of the reciprocal can be considered. Whether the first coefficient exists or not, the number of coefficients in the CFE will still be equal to the order of the MIP or AMIP.

Theorem 4: Type B DRF permits the following form of CFE: CFE4:

$$e_1(z-1) + \frac{1}{e_2(1-z^{-1}) + \frac{1}{e_3(z-1) + \frac{1}{e_3(z-1) + \frac{1}{e_2(1-z^{-1}) + \cdots}}}$$
(35)

The number of coefficients in the CFE is equal to the order of the MIP or AMIP, whichever is higher. If there is a common factor between the MIP and AMIP, the number of coefficients of the CFE reduces by one. The set of coefficients ϵ_i being positive provides only a necessary condition for stability of $1/D_n(z)$.

Numerical Example: Consider the minimum phase polynomial $D_3(z) = 3z^3 + 2z^2 + z + 1$. For q = 0, the MIP and AMIP are given by

$$M_{3}(z) = 4z^{3} + 3z^{2} + 3z + 4$$
(36)

$$2A_3(z) = 2z^3 + z^2 - z - 2$$
(37)

For q = 1, the MIP and AMIP are given by

2

$$2M_4(z) = 3z^4 + 3z^3 + 2z^2 + 3z + 3$$
(38)

$$2A_4(z) = 3z^4 + z^3 - z - 3 \tag{39}$$

For q = 2, the MIP and AMIP are given by

$$2M_{5}(z) = 3z^{5} + 2z^{4} + 2z^{3} + 2z^{2} + 2z + 3$$
(40)

$$2A_5(z) = 3z^5 + 2z^4 - 2z - 3 \tag{41}$$

Figure 1 shows the roots of $D_3(z)$, $M_5(z)$ and $A_5(z)$. Note how the roots of $A_5(z)$ and $M_5(z)$ interlace on the unit circle.

The coefficients of the CFEs for the DRFs of Type A are given in Tables 1 and 2. It is readily observed that all the coefficients of the CFEs are positive, thereby showing that the roots of $D_2(z)$ are contained within the unit circle. Note also that since z + 1 is not a factor of $A_2(z)$ and $A_5(z)$, a_1 does not exist in either case. However, the number of CFE coefficients is still equal to the order of the MIP or AMIP.

(32)

DRF	CFE3
$M_3(z)/A_3(z)$	$d_0 = 1, d_1 = 7/8, d_2 = 1/8$
$M_4(z)/A_4(z)$	$c_1 = 1/2, c_2 = 1/11,$
	$c_3 = 143/108, c_4 = 121/108$
$M_5(z)/A_5(z)$	$d_0 = 1/3, d_1 = 63/64, d_2 = 3/32,$
,	$d_3 = 128/81, d_4 = 64/81$

Table 2: Coefficients of CFE3 for three different Type A DRFs corresponding to q = 0, 1 and 2.

The coefficients of the CFE for the DRF of Type B are given in Table 3. It is readily observed that all the coefficients of the CFE are positive which is a necessary condition for the roots of $D_3(z)$ to be within the unit circle. Note also that $A_4(z)/M_3(z)$ and $M_5(z)/A_4(z)$ each have a common factor of z + 1 thereby reducing the number of CFE coefficients by one.

Theorem 5: The MIPs and AMIPs defined in Eq. (5) and Eq. (6) have the following properties:

- 1. $M_{n+q+1}(z) = (z+1)M_{n+q}(z) + (z-1)A_{n+q}(z)$
- 2. $A_{n+q+1}(z) = (z-1)M_{n+q}(z) + (z+1)A_{n+q}(z)$
- 3. $M_{n+q+1}(z) + A_{n+q}(z) = z^{q-1}(z+1)D_n(z)$

This allows us to easily calculate the MIPs and AMIPs of higher order from the lower order MIPs and AMIPs.

5. SUMMARY AND CONCLUSIONS

It is shown in this paper that a polynomial of the type $z^q D_n(z)$ (containing all its zeros within the unit circle) can be decomposed into a sum of an MIP and an AMIP. Such a polynomial is considered, because the magnitude response of any transfer function containing such a denominator polynomial remains invariant to the value of q. For each value of q, there exists an MIP and AMIP pair whose roots are on the unit circle, are simple and interlace. The value q = 0 corresponds to Schussler's identification. The value q = 1corresponds the LSF formulation and Davis's identification. These are particular cases of the new general MIP and AMIP decomposition presented here. From the various MIPs and AMIPs obtained, a number of Discrete Reactance Functions (DRFs) broadly classified as two types (Type A and Type B) can be constructed. From the DRFs, new Continued Fraction Expansions (CFE) are possible. In the Type A DRF, if the coefficients of the CFE are positive, the polynomial contains all its roots within the unit circle. This is not always the case for the Type B DRF. The coefficients of the CFE being positive constitutes only a necessary condition for minimum phase. Additional conditions have to be satisfied in order to ensure that the roots of the polynomial are contained within the unit circle [5][14].

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DRF	CFE
$A_4(z)/M_3(z)$	$\epsilon_1 = 3/4, \epsilon_2 = 16/7, \epsilon_3 = 1/4$
$M_4(z)/A_3(z)$	$\epsilon_1 = 2/3, \epsilon_2 = 4/9, \epsilon_3 = 81/14,$
	$\epsilon_4 = 49/963$
$\overline{M_5(z)}/A_4(z)$	$\epsilon_1 = 1, \epsilon_2 = 3/4, \epsilon_3 = 16/7,$
	$e_4 = 1/4$
$A_5(z)/M_4(z)$	$e_1 = 1, e_2 = 3/2, e_3 = 4/9,$
	$\epsilon_4 = 81/14, \epsilon_5 = 49/963$

Table 3: Coefficients of the CFE for the Type B DRF

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