DESIGN OF MECHANICAL SYSTEMS HAVING MAXIMALLY FLAT RESPONSE AT LOW FREQUENCIES

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Abstract

This paper discusses a method of designing translational and rotational mechanical systems having maximally flat response at the lower end of the frequency spectrum.

Introduction

One of the methods of analyzing a system is to obtain an analogous electrical network and then use known analysis techniques to obtain the required information [1]. The system may be mechanical (translational and rotational), fluid, thermal or of any other type. The analysis can be carried out either in the time-domain or in the frequency-domain. Also, Laplace-transform techniques can be conveniently used in all cases, when the system considered is linear [2].

It is the purpose of this paper to show that the same concept can be applied in the design of such systems. Specifically, since a considerable amount of literature exists in the design of electrical low-pass filters, it can be made use of in the design of similar mechanical systems also.

Magnitude of the Transfer function

One starts with the magnitude of the transfer function

$$\left|H(j\Omega_0)\right| = \left|\frac{V_{out}}{V_{in}}\right| = \frac{1}{\sqrt{1+\Omega_0^{2n}}} \qquad (1)$$

where V_{in} is the input velocity,
V_{out} is the output velocity,
Ω_0 is the normalized frequency = $\frac{w_0}{w_c}$,
ω_0 is the operating frequency,
ω_c is the cut-off frequency,
and n is the order of the system.

Since the velocity is an across-variable, the analogous variable in the electrical system is the voltage and therefore Eq.(1) can be considered to be a voltage transfer function. This permits one to use the theory of electrical filters to study the properties of Eq.(1) and its design.

Some important properties of the magnitude transfer function considered.

Property 1 : At zero frequency, that is, at $\Omega_0 = 0$, the response is always unity.

Property 2 : At unity normalized frequency, that is, at $\Omega_0 = 1$, (which is the cut-off frequency), the response is

always $\frac{1}{\sqrt{2}}$, irrespective of the order of the system.

Property 3 : It is readily verified that the first (2n-1) derivatives at $\Omega_0 = 0$ will always be zero. This gives rise to the maximally flat response in the neighbourhood of $\Omega_0 = 0$.

These three properties do not give any information regarding the selection of order n.

Property 4 : The magnitude response is monotonically decreasing.

Property 5 : It can be shown that

$$\left.\frac{d\left|H(j\Omega_0)\right|}{d\Omega_0}\right|_{\Omega_0=1} = -0.3535n \qquad (2)$$

This gives the slope of the magnitude response at the cut-off frequency. This is a possibility for the specification of the system. Note that n has to be an integer. If n comes out to be a fraction, it is converted into the next higher integer.

Alternatively, the order n could be determined by specifying the magnitude response at a frequency $\Omega_0 > 1$. In this case also, if n comes out to be a fraction, it is converted into the next higher integer. Table I gives the magnitude of the transfer functions for different orders for $\Omega_0 = 1.5$ and $\Omega_0 = 2.0$.

Table I

Magnitudes of the transfer function for different orders at $\Omega_0 = 1.5$ and $\Omega_0 = 2.0$ radians.

Order	$\Omega_0 = 1.5$		$\Omega_0 = 2.0$	
	Magni-tude	in dB	Magni-tude	in dB
1	0.5547	-5.12	0.4472	-7.00
2	0.4061	-7.83	0.2425	-12.31
3	0.2840	-10.93	0.1240	-18.12
4	0.1938	-14.25	0.0629	-24.03
5	0.1306	-17.68	0.0315	-30.04
6	0.0875	-21.16	0.0223	-33.05

Determination of the transfer function

After the determination of n, the order of the system, the next step is to generate the appropriate transfer function. This is carried out with the help of the complex frequency given by

$$S_0 = \sigma_0 + j\Omega_0 \qquad (3)$$

and S_0 can also be considered as a Laplace-transform variable. This permits one to write

$$H(S_0) \,.\, H(-S_0) = \left|H(j\Omega_0)\right|^2_{\Omega_0^2 = -S_0^2} = \frac{1}{1+\left(-S_0^2\right)^n} \qquad (4)$$

It is evident that when the denominator is equated to zero, it contains 2n roots $S_1, S_2, \ldots, S_{2n-1}, S_{2n}$, each one having magnitude of unity. In other words, all of them lie on the unit circle in the complex-plane S_0. Since the function has to be stable, $H(S_0)$ shall contain all its poles in the left-half of the S_0-plane, or

$$H(S_0) = \frac{1}{D(S_0)} \qquad (5)$$

where $D(S_0)$ is a strictly Hurwitz polynomial. It is readily shown that the roots of the equation

$$1 + (-1)^n S_0^{2n} = 0 \qquad (6a)$$

are given by

$$S_{0k} = e^{j\frac{p(2k-1)}{2n}}, \; k = 1,2,\ldots,2n \qquad (6b)$$

From these roots,

$$D(S_0) = a_0 + a_1S_0 + a_2S_0^2 + \ldots + a_kS_0^k + .. \qquad (6c)$$

can be readily obtained and these are given upto n = 6 in Table 2.

For details, please see [3]

Table 2
Coefficients of the denominator polynomial of the transfer function upto order 6.

Order	
1	$a_1 = a_0 = 1$
2	$a_2 = a_0 = 1$, $a_1 = 1.414214$
3	$a_3 = a_0 = 1$ $a_2 = a_1 = 2$
4	$a_4 = a_0 = 1$ $a_3 = a_1 = 2.613216$ $a_2 = 3.414214$
5	$a_5 = a_0 = 1$ $a_4 = a_1 = 3.236068$ $a_3 = a_2 = 5.236068$
6	$a_6 = a_0 = 1$ $a_5 = a_1 = 3.863703$ $a_4 = a_2 = 10.097835$ $a_3 = 9.14162$

Implementation of the generated Transfer Function

We are mainly interested in a mechanical system, which results in the generated transfer function. One possibility is to design a passive electrical network and then obtain the corresponding mechanical system by appropriate modeling. One such network is shown in Fig.1.

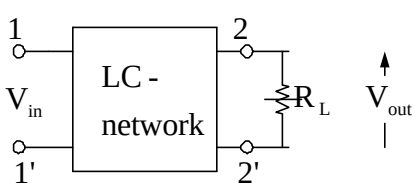


Fig.1: The passive electrical network considered.

The LC-network consists of only inductors and capacitors and the mutual inductances are absent. (There are other possibilities and they are not considered here). It can be readily shown that

$$H(S_0) = \frac{V_{out}}{V_{in}} = \frac{-y_{12}}{y_{22} + R_L} \qquad (7)$$

where

y_{22} = input admittance at 2-2' with 1-1' short-circuited and R_L removed,

and

$-y_{12}$ = feedback transfer admittance between 2-2' and 1-1' with R_L removed and 1-1' short-circuited.

It can be shown that y_{22} and $-y_{12}$ are the ratio of even to odd polynomials [3], because they are obtained for LC-networks only.

Writing

$$D(S_0) = M(S_0) + N(S_0) \qquad (8)$$

where $M(S_0)$ = Even part of $D(S_0)$,

and $N(S_0)$ = Odd part of $D(S_0)$.

It can be shown that

$$-y_{12} = \frac{1}{N(S_0)} \qquad (9)$$

$$-y_{22} = \frac{M(S_0)}{N(S_0)} \qquad (10)$$

with $R_L = 1\Omega$ (taken without any loss of generality).

Expanding $\frac{M(S_0)}{N(S_0)}$ around $S_0 = \infty$ into a continued fraction expansion, the required LC-network is obtained. Once the electrical network is realized, the corresponding mechanical system is obtained by replacing the resistor by a damper, the inductor by a spring and the capacitor by a mass (for translational systems) and a flywheel (for rotational systems).

This is illustrated by the following two examples where the orders considered are n = 3 and n = 4 respectively.

Example 1

Let the order n be 3. The transfer function will be

$$H_3(S_0) = \frac{1}{S_0^3 + 2S_0^2 + 2S_0 + 1}$$

This yields

$$-y_{12} = \frac{1}{S_0^3 + 2S_0}$$

and

$$y_{22} = \frac{2S_0^2 + 1}{S_0^3 + 2S_0}$$

y_{22} has to be expanded into continued fractions and this yields

$$y_{22} = (0)S_0 + \cfrac{1}{\cfrac{S_0}{2} + \cfrac{1}{\cfrac{4}{3}S_0 + \cfrac{1}{\cfrac{3}{2}S_0}}}$$

This results in the electrical network shown in Fig.2(a).

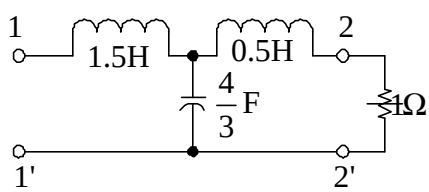
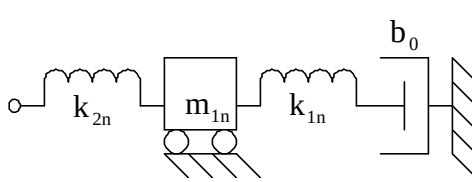


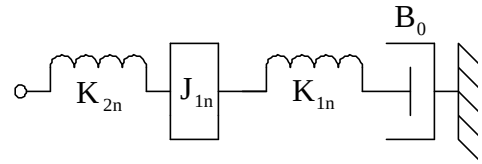
Fig.2(a): A third-order low-pass filter having maximally flat response around $\omega_0 = 0$.

The corresponding normalized mechanical translation system is shown in Fig.2(b) and the normalized mechanical rotational system is shown in Fig.2(c).



where $k_{1n} = 2$, $k_{2n} = \frac{2}{3}$, $m_{1n} = \frac{4}{3}$ and $b_0 = 1$.

Fig.2(b): A third-order normalized mechanical translational system corresponding to the electrical network of Fig.2(a).



where $K_{1n} = 2$, $K_{2n} = \frac{2}{3}$, $J_{1n} = \frac{4}{3}$ and $B_0 = 1$.

Fig.2(c): A third-order normalized mechanical rotational system corresponding to the electrical network of Fig.2(a).

Example 2

Let the order n be 4. The transfer function will be

$$H_4(S_0) = \frac{1}{S + a_3S + a_2S + a_1S + 1}$$

where $a_3 = a_1 = 2.613126$,
and $a_2 = 3.414214$.

This yields

$$-y_{12} = \frac{1}{2.613123S_0^3 + 2.613126S_0}$$

and

$$y_{22} = \frac{S_0^4 + 3.414214S_0^2 + 1}{2.613123S_0^3 + 2.613126S_0}$$

The continued fraction of y_{22} around $S_0 = \infty$ yields

$$y_{22} = C_4S_0 + \cfrac{1}{L_3S_0 + \cfrac{1}{C_2S_0 + \cfrac{1}{L_1S_0}}}$$

where $L_1 = 1.530734$,
$C_2 = 1.577161$,
$L_3 = 1.082392$,
$C_4 = 0.382638$.

This results in the electrical network shown in Fig.3(a).

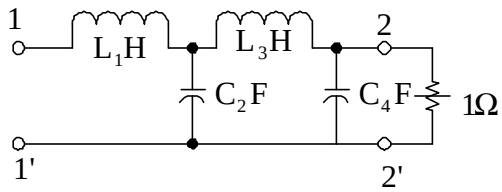
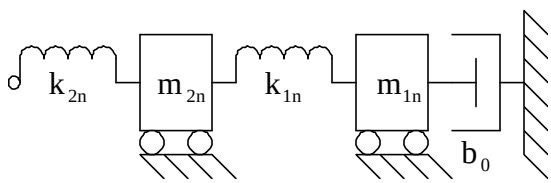


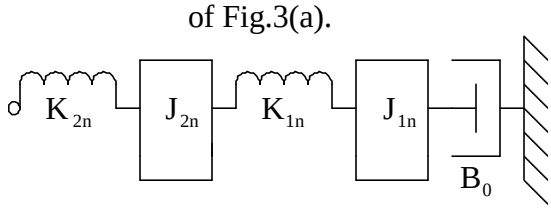
Fig.3(a): A fourth-order low-pass filter having maximally flat response around $\omega_0 = 0$.

The corresponding normalized mechanical translational system is shown in Fig.3(b) and the normalized mechanical rotational system is shown in Fig.3(c).



where $k_{1n} = 0.9238797$, $k_{2n} = 0.6532813$, $m_{1n} = 0.38263$, $m_{2n} = 1.577161$, $b_0 = 1$.

Fig.3(b): A fourth-order normalized mechanical translational system corresponding to the electrical network of Fig.3(a).



where $K_{1n} = 0.92388$, $K_{2n} = 0.6532813$, $J_{1n} = 0.38263$, $J_{2n} = 1.577161$, $B_0 = 1$.

Fig.3(c): A fourth-order normalized mechanical rotational system corresponding to the electrical network of Fig.3(a).

Scaling

Having obtained the normalized values in the design, one has to denormalize them in order to obtain the actual values. This process is known as **scaling**. There are two types of scaling, namely (a) **Impedance scaling** and (b) **Frequency scaling**. Both these can be carried out independently or simultaneously, depending on the requirements. They are discussed below:

(a) Impedance scaling **:** This is necessitated by the fact that b_0 or B_0 is taken to be unity initially. By impedance scaling, the impedance of every component has to be scaled by the same quantity. However, the transfer function remains unaltered. Let α be the impedance scaling factor. It is readily seen that b_0 and B_0 become $\frac{b_0}{a}$ and $\frac{B_0}{a}$ respectively. The impedances $\frac{S_0}{k}$ and $\frac{S_0}{K}$ becomes $\frac{a\,S_0}{k}$ and $\frac{a\,S_0}{K}$ respectively. These yield the results that the spring constants becomes $\frac{k}{a}$ and $\frac{K}{a}$ respectively. Considering the masses, the impedances $\frac{1}{S_0 m}$ and $\frac{1}{S_0 J}$ become $\frac{a}{S_0 m}$ and $\frac{a}{S_0 J}$ respectively, giving us the new values as $\frac{m}{a}$ and $\frac{J}{a}$ after impedance scaling. These new values may be further subjected to frequency scaling discussed below.

(b) Frequency scaling : This is required because the operating frequency has been normalized to unity. We can readily put $S_0 = \frac{s_0}{b}$, where β is the frequency scaling factor. The quantities $\frac{b_0}{a}$ and $\frac{B_0}{a}$ remain unchanged. By a similar treatment as given in the case of

Impedance scaling, it can be shown that the quantities k, K, m and J become $k\beta$, $K\beta$, $\frac{m}{b}$ and $\frac{J}{b}$ respectively.

(c) Combined Impedance and Frequency scaling :

Both the scaling operations can be combined and the final results are given in Table III. The quantities in column 3 are given after they are subjected to frequency scaling only. The quantities in column 2 can be obtained from those in column 4 are obtained by putting $\alpha = 1$ and those in column 3 are obtained by putting $\beta = 1$ in those given in column 4.

Table 3

Effect of scaling on the component values

Quantity	Imped-ance scaling	Frequ-ency scaling	Comb-ined scal-ings
b_0	$\frac{b_0}{a}$	b_0	$\frac{b_0}{a}$
B_0	$\frac{B_0}{a}$	B_0	$\frac{B_0}{a}$
k	$\frac{k}{a}$	$k\beta$	$\frac{kb}{a}$
K	$\frac{K}{a}$	$K\beta$	$\frac{Kb}{a}$
m	$\frac{m}{a}$	$\frac{m}{b}$	$\frac{m}{ab}$
J	$\frac{J}{a}$	$\frac{J}{b}$	$\frac{J}{ab}$

By using these scalings, we shall now determine the actual values to be used for the two examples considered above.

Example 3

In this example, we shall consider the scaling of a third order translational system considered in Fig.2(a) with $b_0 = \frac{100}{3}$ N.s/m and a cut-off frequency of 5 radians/second. This gives $\alpha = \frac{3}{100}$ and $\beta = 5$. The various denormalized component values come out to be: $k_{1n} \rightarrow \frac{1000}{3}$ N/m, $k_{2n} \rightarrow \frac{1000}{9}$ N/m and $m_{1n} \rightarrow \frac{80}{9}$ kg. If the fourth order system considered in Fig.3(a) is considered with the same scaling factors, the various denormalized will be $k_{1n} \rightarrow$ 153.98 N/m, $k_{2n} \rightarrow$ 108.88 N/m, $m_{1n} \rightarrow$ 2.5509 kg and $m_{2n} \rightarrow$ 10.5144 kg.

Example 4

In this example, we shall consider the denormalization of the rotational system considered in Fig.2(c). Let B_0 = 20 N.m.s and the cut-off frequency be 5 radians/second. The various component values come out to be $K_{1n} \rightarrow$ 200 N.m/radians, $K_{2n} \rightarrow \frac{200}{3}$ N.m/radians, and $J_{1n} \rightarrow$ 16 kg.m^2. When the fourth order rotational system given in Fig.3(c) is considered, the various denormalized component values come out to be $K_{1n} \rightarrow$ 30.796 N.m/radians, $K_{2n} \rightarrow$ 21.776 N.m/radians, $J_{1n} \rightarrow$ 1.13052 kg.m^2 and $J_{2n} \rightarrow$ 7.308644 kg.m^2.

The responses for the orders 3 and 4 are shown in Fig.4.

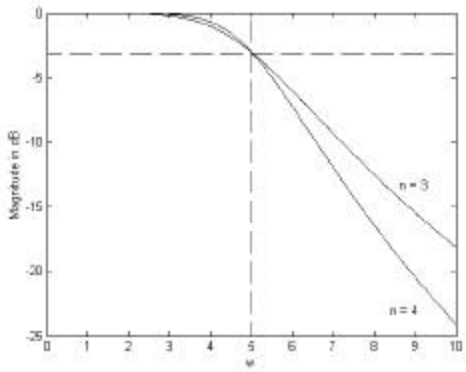


Fig.4: Magnitude responses for n = 3 and n = 4.

Conclusions

In this paper, it is shown that the design method used in electrical wave filters can be effectively used in the design of mechanical low-pass systems. Specifically, maximally-flat response near the zero frequency has been considered and the corresponding mechanical system has been designed. Though lower order systems are considered, the same concept can be used to design higher order systems as well. It is envisaged that the same concept can be used when other types of responses are considered.

References

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