# GENERATION OF ANALOG AND DIGITAL TRANSFER FUNCTIONS HAVING A MONOTONIC MAGNITUDE RESPONSE

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**Abstract**

*A new method for the generation of transfer functions of filters resulting in monotonic magnitude responses is proposed. Not only filters having the existing monotonic responses can be obtained using this approach, but additional filters having such responses can be generated, thereby originating a large family of such filters both in the analog and the digital domains.*

## I. Introduction

Recently, low-pass filters having a monotonic magnitude response in the pass-band have been proposed [1]. The characteristics of these filters are intermediate between Butterworth and Papoulis filters [2.3,4]. However, they generate only certain distinct types of monotonic response. It is also shown [5] by proper perturbation of the pole locations of a Butterworth filter, monotonic magnitude response can still be maintained.

In this paper, it is shown how analog and digital filters having several monotonic magnitude responses can be generated in a general fashion.

## 2. Generation of Transfer functions

Our approach starts by first obtaining an analog low-pass filter having a monotonic magnitude response as follows: One can start with

$$f(\omega^2) = \omega^{2n} f_1(\omega^2) \qquad ..(1)$$

where n is a positive integer or zero and $f_1(\omega^2)$ is any polynomial that is always positive in ω. It is always possible to find this type of polynomial, because its roots can be either complex conjugate or lie on the negative real axis and they can be multiple. The value of n determines the extent of the maximally flat behavior of the response at ω = 0. We present here two methods of generating the required magnitude response which is monotonic.

**Method I** :

We formulate

$$g_1(x) = \int f_1(x)\,dx \qquad ..(2)$$

where $x = \omega^2$.and the constant of integration is zero. It is readily seen that $g_1(x)$ is always positive when $\omega > 0$ and could be zero when $\omega = 0$. The squared magnitude response of the resulting low-pass filter is

$$|H_1(\omega)|^2 = \frac{1}{1+g_1(\omega^2)} \qquad ..(3a)$$

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As an alternative, one can put n = 0 in (1) and then modify (3a) as

$$|H_2(\omega)|^2 = \frac{1}{1+\omega^{2n}\, g_1(\omega^2)} \qquad ..(3b)$$

which also gives monotonic magnitude responses having zero slopes at the origin, having an order equal to n.

**Method II :**

In this method, we formulate

$$g_2(y) = \int\int f(y)\,dy \qquad ..(4)$$

where y = ω and the constant of integration is zero. The squared magnitude response of the resulting low-pass filter is

$$|H_3(\omega)|^2 = \frac{1}{1+g_2(\omega^2)} \qquad ..(5a)$$

which, in turn, is also monotonic. Just as in the previous method, we can initially put n = 0 in (1), carry out the required integrations and then modify (5a) as

$$|H_4(\omega)|^2 = \frac{1}{1+\omega^{2n}\, g_2(\omega^2)} \qquad ..(5b)$$

which also exhibits the property of zero slope at the origin.

The basic difference between the two methods is that, in the first case, integration is carried out with respect to $\omega^2$ once and in the second case, the integration is carried out twice with respect to ω. The two methods give rise to different filters. This could be considered as the starting point for the next stage of integration and the process can be repeated. Also, at any stage, one can choose either Method I or Method II. Thus, one can immediately see the large number of possibilities.

The next step is to obtain the corresponding low-pass analog transfer function H(s). This is accomplished by using the relationship that the squared magnitude response is equal to H(s).H(-s) evaluated at s = jω. Well known procedures permit us to realize singly-terminated or doubly-terminated network realization of the low-pass filter H(s). Appropriate transformations of the s-variable [2] result in high-pass, band-pass and band-elimination analog filters.

A family of digital low-pass filters H(z) having monotonic responses is obtained by the application of the general bilinear transformation [6]

$$s = \frac{z-a}{z+1} \qquad ..(6)$$

with $0 < a \le 1$ for assuring stability. Appropriate transformations of the z-variable will result in high-pass, band-pass and band-elimination digital filters.

### Numerical Example

The above techniques are illustrated by a numerical example. Consider the second-order Bessel filter given by

$$f_1(\omega^2) = \frac{1}{\frac{\omega^4}{9}+\frac{\omega^2}{3}+1} \qquad ..(7)$$

Let n = 0 so that f($\omega^2$) = . $\frac{\omega^4}{9}+\frac{\omega^2}{3}+1$.

**Method I** : The magnitude response obtained by the first method is given by

$$|H_1(\omega)|^2 = \frac{1}{\frac{\omega^6}{27}+\frac{\omega^4}{6}+\omega^2+1} \qquad ..(8)$$

for which the -3-dB point is obtained at ω = 0.9247. The corresponding transfer function is given by

$$H_1(s) = \frac{5.19961}{s^3+4.681s^2+8.7016s+5.19961} \qquad ..(9)$$

At this stage, two other modifications are possible and they are:

(a) The value of n could be different from zero. For purposes of illustration, we let n = 2. The corresponding magnitude function is given by

$$|H_{a1}(\omega)|^2 = \frac{1}{\frac{\omega^{10}}{45}+\frac{\omega^8}{12}+\frac{\omega^6}{3}+1} \quad ..(10)$$

for which the -3-dB response is obtained at $\omega = 1.1315$.

(b) Instead, the factor $\omega^4$ could be appropriately added after the integration is carried out. In such a case, the corresponding magnitude function as

$$|H_{b1}(\omega)|^2 = \frac{1}{\frac{\omega^{10}}{27}+\frac{\omega^8}{6}+\omega^6+1} \quad ..(11)$$

for which the -3-dB response is obtained at $\omega = 0.9714$. Fig.1 gives the magnitude responses of (8), (10) and (11).

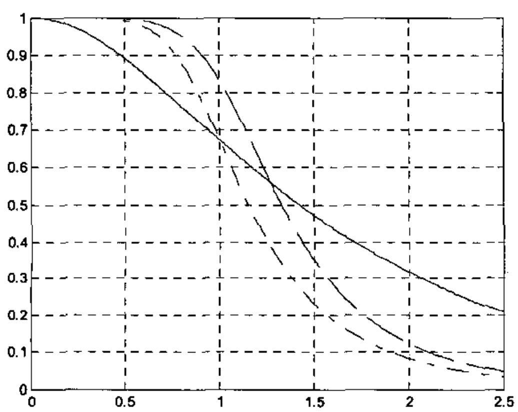


Fig.1: Magnitude responses of (8) {shown as _____}, of (10) {shown as ------) and that of (11) {shown as --.--.}.

**Method II** : The magnitude response obtained by this method is given by

$$|H_2(\omega)|^2 = \frac{1}{\frac{\omega^6}{270}+\frac{\omega^4}{36}+\frac{\omega^2}{2}+1} \quad ..(12)$$

Its -3-dB response is obtained at $\omega = 1.3348$. The corresponding transfer function is given by

$$H_2(s) = \frac{11.6186}{s^3+6.4199s^2+18.8577s+11.6186} \quad ..(13)$$

Just as in the case of Method I, one can incorporate the two modifications. They are:

(a) The value of n could be other than zero. For purposes of illustration, we let n = 2 and the corresponding magnitude function is given by

$$|H_{a2}(\omega)|^2 = \frac{1}{\frac{\omega^{10}}{810}+\frac{\omega^8}{168}+\frac{\omega^6}{30}+1} \quad ..(14)$$

for which the -3-dB response is obtained at $\omega = 1.6114$

(b) Instead, the factor $\omega^4$ could be appropriately added after the integration is carried out. In such a case, we get the corresponding magnitude function as

$$|H_{b2}(\omega)|^2 = \frac{1}{\frac{\omega^{10}}{270}+\frac{\omega^8}{36}+\frac{\omega^6}{2}+1} \quad ..(15)$$

for which the -3-dB response is obtained at $\omega = 1.1083$. Fig.2 gives the magnitude responses of (12), (14) and (15).

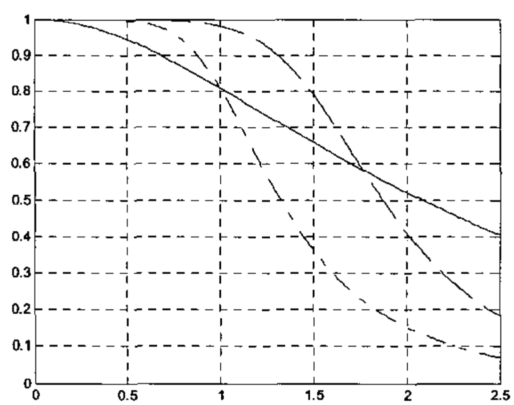


Fig.2: Magnitude responses of (12) {shown as _____}, of (14) {shown as ------) and that of (15) {shown as --.--.}.

## 3. Conclusions

A new method of generating transfer functions which result in strictly monotonic responses has been proposed. It is required that one has to start from a polynomial which is strictly positive for all finite frequencies, the order being immaterial. Two methods of integration are proposed, one with respect to $\omega^2$ and another with respect to $\omega$. The constant of integration is always adjusted to be unity at $\omega = 0$. Also, the slope at the origin can always be made zero and the degree of maximal flatness at the origin can be controlled independently of the order of the starting polynomial. A very large number of such responses (almost infinite) can be generated starting from a low order like two. The only requirement is that when once a particular response is generated, the bandwidth (or -3-dB point) has to be computed. It is also shown that both analog and digital filters are possible. A numerical example is given to illustrate the method.

## References


[1] I.M.Filanovsky, A generalization of filters with monotonic magnitude-frequency response, *IEEE Transactions on Circuits and Systems I : Fundamental Theory and Applications*, Vol.46, No.11, pp.1382-1385, November 1999.

[2] M.E.Van Valkenburg, ANALOG FILTER DESIGN, *Holt, Rinehart and Winston*, 1982, Ch.6.

[3] A.Papoulis, Optimum filters with monotonic response, *Proc.IRE*, pp.606-609, 1958.

[4] A.Papoulis, On monotonic response filters, *Proc.IRE*, pp.332-333,1958.

[5] K.R.Pai, K.V.V.Murthy and V.Ramachandran, Complementary pole-pair filters and pole-parameter transformations, *Journal of Circuits, Systems and Computers*, Vol.6, No.4, pp.319-350, 1996.

[6] C.S.Gargour, V.Ramachandran, Ravi P.Ramachandran and F.S.Awad, Variable magnitude characteristics of 1-D IIR discrete filters by a generalized bilinear transformation, *43rd Midwest Symposium on Circuits and Systems*, Michigan State University, Michigaan, U.S.A., August 8-11, 2100, (Four pages).