On the Properties And Design of Stable IIR Transfer Functions Generated Using Fibonnaci Numbers

C. S. Gargour
Electrical Engineering Department
Ecole de Technologie Supérieure,
(ETS) Univ. of Quebec
Montreal QC, Canada, H2I 2C8,
(gargour@ele.etsmtl.ca)

V. Ramachandran
Electrical Engineering Department
Concordia University
Montreal, QC, Canada, H3G M8
(kamala@ece.concordia.ca

Ravi P. Ramachandran
Electrical Engineering Department
Rowan University, Glassboro, New
Jersey, 08028, U.S.A.
(ravi@rowan.edu)

***Abstract*—This paper considers z-domain transfer functions whose denominator polynomial possesses the property that the coefficient of z^i is greater than the coefficient of z^{i-1}. Such transfer functions can be shown to be always stable and their denominator polynomials can be formed as a finite length time reversed Fibonacci sequence of numbers. Appropriate numerator polynomials can be configured to design lowpass, highpass, bandpass, band-elimination and allpass IIR filters. It is observed that the phase response closely approximates a linear behavior. This study is on the design of IIR filters using Fibonacci numbers. The advantages are that frequency selective filters with an approximately linear phase characteristic can be obtained with neither a stability test nor an analog prototype,**

***Keywords*—fibonacci, filters, iir, stable, transfert function**

I. INTRODUCTION

The different types of discrete time filters and methods of generating stable transfer functions have been widely discussed in [1]. In this paper, we consider IIR stable transfer functions whose denominator polynomial is generated using a finite length sequence of Fibonacci numbers in reverse order. In Section 2, this method is shown to guarantee a stable transfer function. By appropriately configuring the numerator polynomial, various frequency selective filters are designed.

The advantages of this approach are that (1) it does not require an analog prototype, (2) the phase response is approximately linear, (3) stability is guaranteed and (4) a sophisticated optimization algorithm is not required, The main drawback is that the order of the filter cannot be directly calcualated from a prescribed passband ripple and stopband attenuation.

It is noted that Fibonacci based impulse response filters have been reported in [2]. Other methods of IIR filter design include least-squares design for optimizing magnitude and phase with a pole radius constraint [3], a linear programming approach [4], with a prescribed stability margin [5] and multiple criterion optimization [6].

II. STABILITY ANALYSIS AND DESIGN EXAMPLES

Suppose the transfer function has a denominator polynomial D(z) with increasing coefficients. The proof of Theorem 1 demonstrates that D(z) is minimum phase..

Theorem 1 : The polynomial

$$D(z) = a_N z^N + a_{N-1} z^{N-1} + a_{N-2} z^{N-2} + \ldots + a_{N-k+1} z^{N-k+1} + a_{N-k} z^{N-k} + \ldots + a_2 z^2 + a_1 z + a_0 \quad (1)$$

possesses all its roots located within the unit circle, when the coefficients satisfy the condition

$$a_N > a_{N-1} > \ldots > a_{N-k+1} > a_{N-k} > \ldots > a_0 > 0 \quad (2a)$$

$$\text{or } a_{N-k+1} > a_{N-k}, \; k = 1, 2, \ldots, N-1, N \quad (2b)$$

Proof : The proof is based on Jury's stability conditions [7].

(a) It is required that $D(1) > 0$ (3)

This is readily verified.

(b) It is required that $(-1)^N D(-1) > 0$ (4)

We have to consider two different cases:

(i) <u>N even</u> : This means that D(z) contains an odd number of terms and D(-1) can be regrouped as follows:

$(a_N - a_{N-1}) + (a_{N-2} - a_{N-3}) + \ldots + (a_2 - a_1) + a_0$

which is always positive.

(ii) <u>N odd</u> : This means that D(z) contains even number of terms and D(-1) can be regrouped as follows:

$(-a_N + a_{N-1}) + (-a_{N-2} + a_{N-3}) + \ldots + (-a_1 + a_0)$

Obviously, this quantity is negative and (4) is readily satisfied in this case.

(c) Now, Jury's table is constructed as follows:

$$\begin{array}{cccccccc} a_N & a_{N-1} & a_{N-2} \ldots & a_{N-k} & a_{N-k-1} \ldots & a_2 & a_1 & a_0 \\ a_0 & a_1 & a_2 \ldots & a_k & a_{k+1} \ldots & a_{N-2} & a_{N-1} & a_N \\ \hline b_{N-1} & b_{N-2} & b_{N-3} \ldots & b_{N-k} & b_{N-k\_1} \ldots & b_2 & b_1 & b_0 \end{array}$$

Where $b_{N-1} = a_N^2 - a_0^2$,

$b_{N-2} = a_N a_{N-1} - a_1 a_0$,

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$b_{N-k} = a_N a_{k+1} - a_{N-k-1} a_0$,

$b_{N-k-1} = a_N a_k - a_{N-k} a_0,$

..

$b_1 = a_N a_2 - a_{N-2} a_0,$

$b_0 = a_N a_1 - a_{N-1} a_0.$

Consider the quantity

$b_{N-k} - b_{N-k-1} = a_N(a_{K+1} - a_k) + a_0(a_{N-k} - a_{N-k-1})$

which is always positive. Therefore, one can conclude

$b_{N-1} > b_{N-2} > b_{N-3} > . . > b_{N-k} > b_{N-k-1} > . .> b_2 > b_1 > b_0 > 0$

Continuing this process of constructing Jury's table, it is seen that, at every step, the inequalities of the type in (2a) are always satisfied. Therefore, D(z) contains all its roots within the unit circle.

It is known that if D(z) has its roots within the unit circle, D(-z) also has its roots within the unit circle.

Table 1: Polynomial coefficients and their roots

Order	Coefficients values (high order to low order)	Roots
2	2 1 1	-0.2500 ± j0.6614
3	3 2 1 1	-0.7639 -0.0586 ± j0.6495
4	5 3 2 1 1	-0.7188 -0.5337 ± j0.4583 0.2337 ± j0.5912
5	8 5 3 2 1 1	-0.7188 -0.2908 ± j0.5967 0.3377 ± j0.5298
6	13 8 5 3 2 1 1	-0.6054 ± j0.3255 0.4053 ± j017524 -0.1076 ± j0.6371
7	21 13 8 5 3 2 1 1	-0.6898 -0.4451 ± j0.4916 0.4512 ± j0.4287 0.0293 ± j0.6359
8	34 21 13 8 5 3 2 1 1	0.4838 ± j0.3895 -0.6271 ± j0.2478 -0.2985 ± j0.5763 0.1329 ± j0.6168
9	55 34 21 13 8 5 3 2 1 1	-0.6737 -0.5193 ± j0.4062 -0.1731 ± 0.6164 0.5078 ± j0.3562 0.2124 ± j0.5904
10	89 55 34 21 13 8 5 3 2 1 1	-0.6343 ± j0.1988 - 0.4069± j0.5052 -0.0682 ± j0.6309 0.5259 ± j0.3278 0.2745 ± j05.616

One can use the result of Theorem 1 to construct IIR stable transfer functions. Table 1 gives such functions when Fibonacci numbers are used as the polynomial coefficients of D(z). The table also contains the poles of such transfer functions. Orders up to 10 are considered.

Figure 1(a) shows the magnitude response and Figure 1(b) gives the phase response of a typical lowpass IIR filter of order 8, whose transfer function is given by

$$H_{FLP8}(z) = \frac{(z+1)^8}{34z^8 + 21z^7 + 13z^6 + 8z^5 + 5z^4 + 3z^3 + 2z^2 + z + 1} \quad ..(5)$$

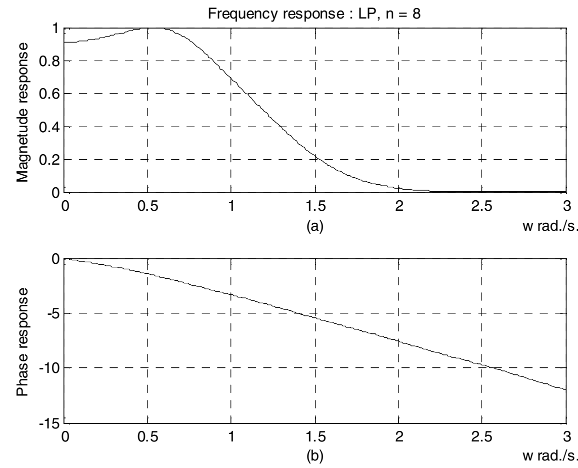


Figure 1(a): Magnitude (b) phase responses of a lowpass filter

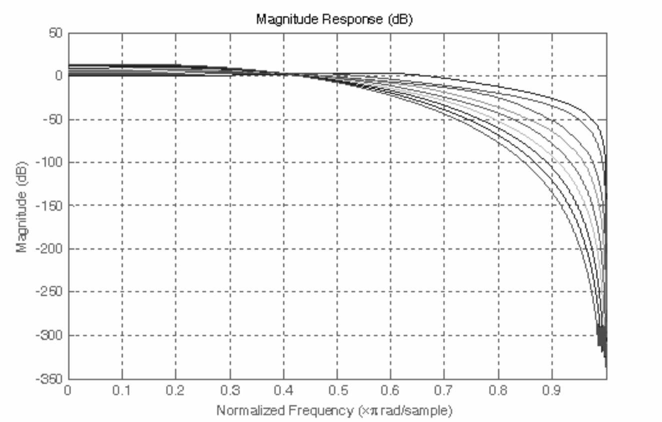


Figure 2(a) Magnitude responses of different lowpass filters

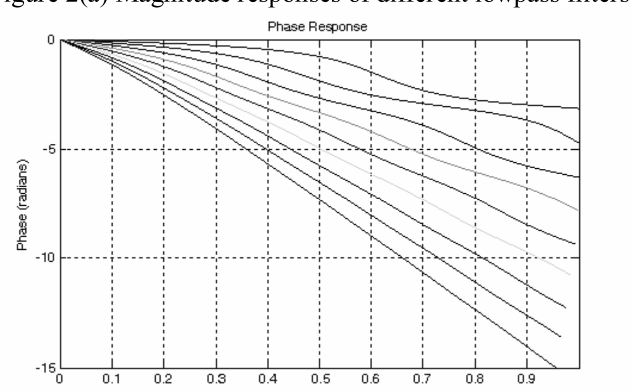


Figure 2(b) Phase responses of different lowpass filters

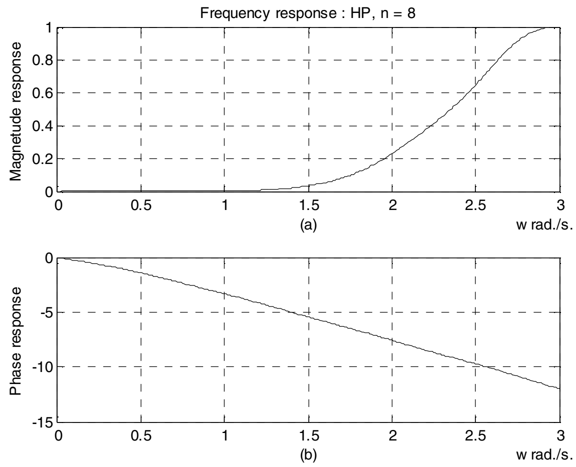
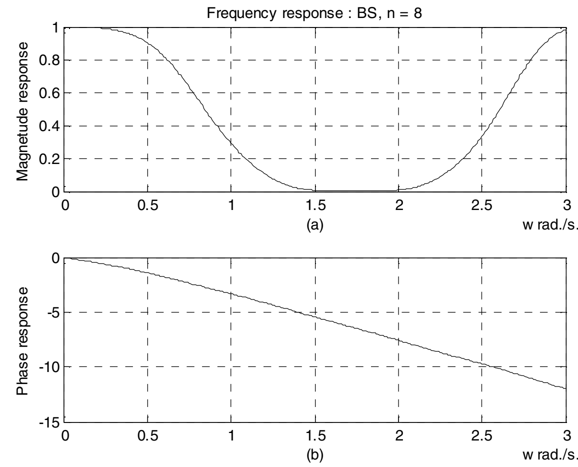
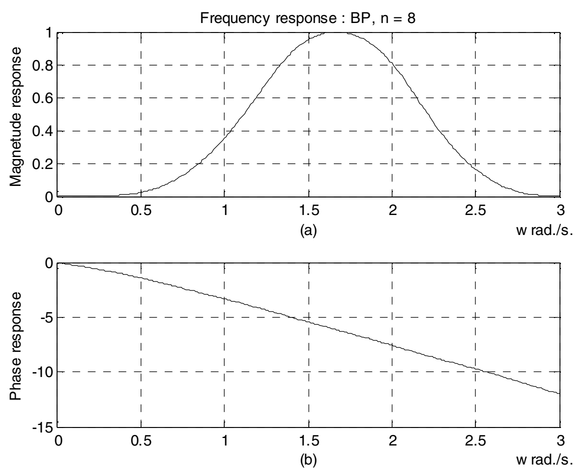


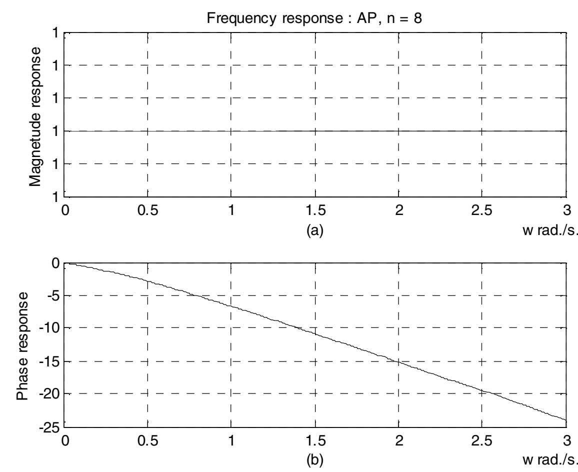
Figure 3(a) and (b): Magnitude and phase responses of the highpass filter



Figures.5(a) and (b) Magnitude and phase responses of the band-elimination filter



Figuress.4(a) and (b) Magnitude and phase responses of the bandpass filter



Figures.6(a) and (b) Magnitude and phase responses of the allpass filter

Figure 2(a) gives the magnitude responses of different orders of such lowpass filters and Fig. 2(b) gives the corresponding phase responses.

Figures.3, 4, 5 and 6 show the magnitude and phase responses for 8^{th} order highpass, bandpass, band-elimination and allpass filters obtained by starting from the 8^{th} order lowpass. In all cases,. the denominator is the same as that of (5). The numerator for the highpass filter is $(z - 1)^8$, that for the bandpass filter is $(z^2 - 1)^4$; and that for the band-elimination filter is $(z^2 + 0.34z + 1)^4$. For the allpass filter, the numerator is :

$$(34 + 21z + 13z^2 + 8z^3 + 5z^4 + 3z^5 + 2z^6 + z^7 + z^8)$$

III. SUMMARY AND DISCUSSION

In this paper, it has been established that when a one-dimensional (1-D) z-domain polynomial satisfies the condition that the coefficient of z^i is greater than the coefficient of z^{i-1}, its roots are always contained within the unit circle. Consequently if this type of polynomial is the denominator of a 1-D causal transfer function, the resulting filter is guaranteed to be stable. As a particular example, we have associated Fibonacci numbers with the various coefficients and different types of filters have been obtained. It is noted in particular that the phase response of these filters closely approximates linear phase characteristics particularly for higher orders. This avoids the need to optimize the phase response to achieve linearity. A simple design approach does result for achieving s good magnitude and phase response. However, it remains to be investigated how an approximately linear phase characteristic is naturally achieved. Also, the generation, design and study of stable IIR filters using other types of number sequences is possible with this approach.

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