GENERATION OF TRANSFER FUNCTIONS YIELDING MONOTONIC FREQUENCY RESPONSES BY THE INTEGRATION OF BUTTERWORTH POLYNOMIALS

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Abstract

Starting from two lower order Butterworth polynomials, 1-D transfer functions are obtained by suitably integrating them and associating the same to the denominator of a transfer function. These functions guarantee monotonic response in the frequency domain. By suitable association of the polynomials obtained by integration, higher order functions yielding monotonic responses are obtained. These can be readily used to obtain 1-D and 2-D discrete filters yielding such responses.

1. Introduction

Recently, considerable attention has been paid towards the generation of monotonic magnitude - frequency responses of transfer functions. [1-5]. In these investigations, the starting point will be filters originating from Butterworth, Papoulis, Filanovsky, Bessel-Thomson filters or their suitable combinations. However, an alternative method [6] has been proposed where one can start from a known polynomial which remains positive throughout the frequency range, integrate it and associate to the denominator polynomial of a transfer function. This also results in monotonic responses and the numerator coefficient has to be adjusted so that the response is unity at zero frequency.

In this paper, we start with a Butterworth polynomial and employ the techniques of [6] and generate transfer functions which give monotonic magnitude-frequency responses.

2. Generation of the required transfer functions.

In this method, we start with the denominator of the square of the magnitude of the Butterworth filter of order 'q', given by

$$g_q(x) = x^q + 1 \qquad (1a)$$

with $x = \omega^2$ (1b)

This polynomial can be integrated a number of times (say k) with respect to x and is associated to the denominator of the derived transfer function. Accordingly, we have

$$g_{qk}(x) = \frac{q!}{(q+k)!} x^{(q+k)} + \sum_{i=1}^{k} \frac{x^i}{i!} + 1 \qquad (2)$$

This permits to obtain the magnitude response of the corresponding monotonic transfer function as :

$$\left|T_{qk}\left(\omega^2\right)\right| = \frac{1}{\sqrt{g_{qk}(\omega^2)}} \qquad (3)$$

with (1b) substituted appropriately. The quantity added in (2) is the constant of integration and ensures that the response is unity at zero frequency. It is also noted that 'q' and 'k' are quite independent of each other and the total order of the resulting transfer function in s (= jω) will be (q + k). Table I gives the various transfer functions $T_{qk}(s) = N_{qk}(s)/D_{qk}(s)$ up to k =5 for q = 1. They are obtained by the factorization of $\left|T_{qk}\left(-s^2\right)\right|^2$ and the rejection of its right hand side poles. It also gives the numerator coefficient for unit response at zero frequency, and ω_c, the cutoff frequency (that is, the response is -3 dB at this point)..

k,q	$D_{qk}(s)$	$N_{qk}(s)$	ω_c
1,0	(s+1)	1	1
1,1	(s^2+2.1974s+1.4142)	1.4142	0.8556
1,2	(s+1.2634) (s^2+2.2982s+1.9388)	2.4495	0.8360
1,3	(s^2+2.3622s+2.5194) (s^2+2.7108s+1.9446)	4.89338	0.8330
1,4	(s + 1.4767)(s^2+2.4074s + 3.1376)(s^2+2.8334s + 2.3645)	10.9554	0.8326
1,5	(s^2+2.4414s + 3.7838) (s^2+2.9228s + 2.8296) (s^2+3.1202s + 2.5061)	26.8319	0.8306

Table I: The transfer functions generated when q = 1 and k up to 5.

Fig.1 gives the various magnitude responses for the transfer functions given in Table I.

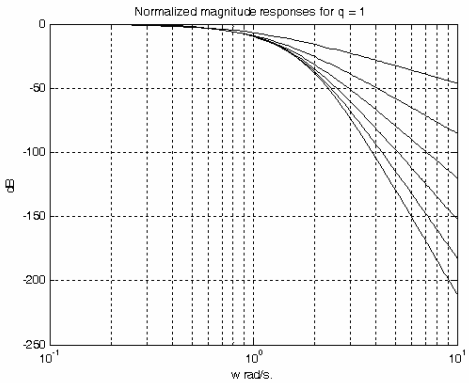


Fig.1: The various magnitude responses of the transfer functions listed in Table I.

It is readily seen that the cut-off frequency point does not vary much as the order increases for the same value of q and that all the $-3\,\mathrm{dB}$ frequency points are clustered around the same value {Fig. 1}.

We can start with the same transfer function as in (1) with a value of q = 2 and carry out the integrations. Following the same procedure, the various transfer functions are obtained and given in Table II up to k=5.

Table II The transfer functions generated when q = 2 and k up to 5			
K,q	$D_{qk}(s)$	$N_{qk}(s)$	ω_c
2,0	$(s^2+1.414s+1)$	1	1
2,1	(s+0.9043) $(s^2+1.7358s+1.9153)$	1.732	0.9043
2,2	$(s^2+1.9442s+2.8205)$ $(s^2+2.0778s+1.5542)$	4.3836	0.8480
2,3	(s+1.1982) $(s^2+2.1008s+3.7151)$ $(s^2+2.2496s+1.74)$	7.7455	0.8351
2,4	$(s^2+2.227s+4.5979)$ $(s^2+2.3748s+2.3113)$ $(s^2+2.6058s+1.7854)$	18.9737	0.8329
2,5	$(s^2+2.333s+5.4452)$ $(s^2+2.4672s+2.9182)$ $(s^2+2.7574s+2.2017)$ (s+1.4249)	49.8507	0.8326

The magnitude responses of the transfer functions listed in Table II are illustrated by Fig. 2

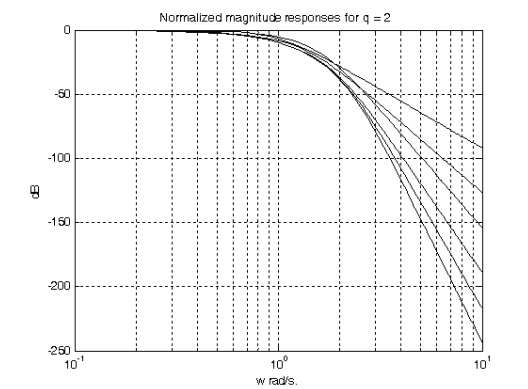


Fig.2: The magnitude responses for the transfer functions listed in Table II.

We can now combine selected factors from Tables I and II such that, when they are associated with the denominator of a transfer function, monotonic responses result.

F1	(s+0.9043)
F2	(s+1)
F3	(s+1.1982)
F4	(s+1.2634)
F5	(s+1.4249)
F6	(s+1.4767)

Table III: First order transfer functions extracted from Tables I and II

S_{11}	$(s^2+2.1974s+1.4142)$
S_{12}	$(s^2+2.2982s+1.9388)$
S_{13}	$(s^2+2.3622s+2.5194)$
S_{14}	$(s^2+2.7018s+1.9446)$
S_{15}	$(s^2+2.8334s+2.3645)$
S_{16}	$(s^2+2.9228s+2.8296)$
S_{17}	$(s^2+3.1202s+2.5061)$

Table IV: Second-order transfer functions extracted from Table I and guaranteed to result in monotonic response

S_{21}	$(s^2+1.414s+1)$
S_{22}	$(s^2+2.0778s+1.5542)$
S_{23}	$(s^2+2.2496s+1.74)$
S_{24}	$(s^2+2.3748s+2.3113)$
S_{25}	$(s^2+2.6058s+1.7854)$
S_{26}	$(s^2+2.4672s+2.9182)$
S_{27}	$(s^2+2.7574s+2.2017)$

Table V: Second-order transfer functions extracted from those in Table III and guaranteed to result in monotonic response

Tables III, IV and V give those factors, which when associated with the denominator of the transfer function, definitely yield monotonic frequency responses. Any type of combination is possible, giving the order of the transfer

function up to 34, starting from the first order. If we increase the value of k in Tables I and II, more of such factors will be available. In Tables IV and V, it is readily observed that the roots of s-functions are such that the magnitude of the real part is greater than or equal to that of the imaginary part. For a second-order system, it is readily shown that this is both necessary and sufficient. However, for higher order systems, this is only sufficient and suitable second order functions can be added as factors to the denominator so that monotonic response is maintained.

Fig. 3 shows a magnitude response generated by the above technique. The transfer function chosen is given by

$$T_{ck}(s) = \frac{3.8717}{(s+0.9043)(s^2+2.7018s+1.9446)(s^2+2.7574s+2.2017)} \quad (4)$$

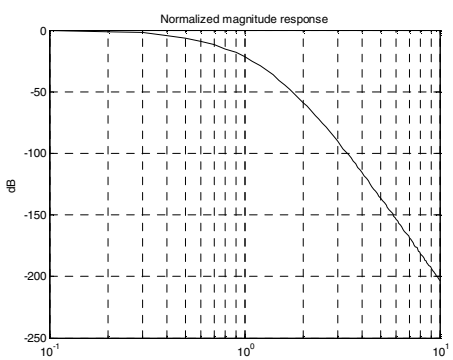


Fig.3: Magnitude response of the transfer function given by (4)

3. Generation of 2-D filters

The 2-D filters can now be generated starting from the above 1-D filters. Several approaches are possible. However, for the purposes of illustration, the structure shown in Fig.4 is considered.

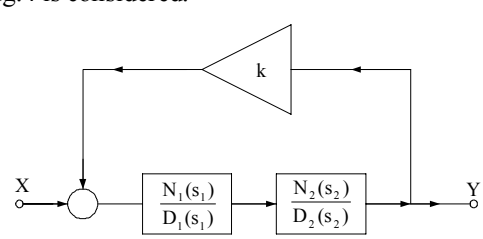


Fig.4: The structure considered for the 2-D filter

Analysis yields the overall transfer function as

$$T(s_1,s_2) = \frac{N_1(s_1).N_2(s_2)}{D_1(s_1).D_2(s_2) + kN_1(s_1)N_2(s_2)} \quad ..(5)$$

It is required that the denominator should necessarily be a VSHP in order that the designed filter is stable. Any order of s_1 and s_2 can be chosen, however we will use equal order to maintain symmetry. As a numerical example, the following transfer functions have been chosen:

$$H_i(s_i) = \frac{1}{s_i + 1.4249}, i = 1,2 \quad ..(6)$$

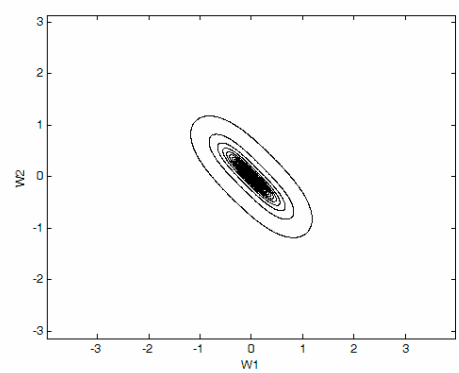
The overall transfer function becomes

$$T(s_1,s_2) = \frac{1}{s_1s_2 + 1.4249s_1 + 1.4249s_2 + 2.03034 + k} \quad ..(7)$$

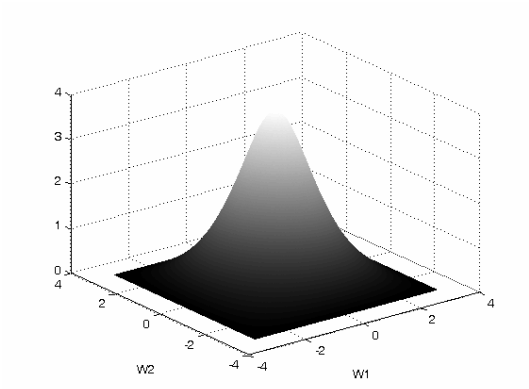
In order that the denominator shall be a VSHP, it can be shown that is required that k > -2.03034. Now, the Generalized Bilinear Transformation (GBT) given by [7]

$$s_i = k_i \frac{z_i - a_i}{z_i + 1}, i = 1,2,\ 0 \le a_i \le 1 \quad ..(8)$$

giving a low-pass filter. Other filters can be obtained by using appropriate transformations. Fig.4(a) and 4(b) give the magnitude and contour characteristics for the case $k_1=k_2= 1$, $a_1 =a_2= 0.75$ and k = -1.5. By changing these variables, variable magnitude characteristics can be readily obtained as can be seen in Figs.5(a) and (b), when the generalized bilinear transformation is used with different parameters.



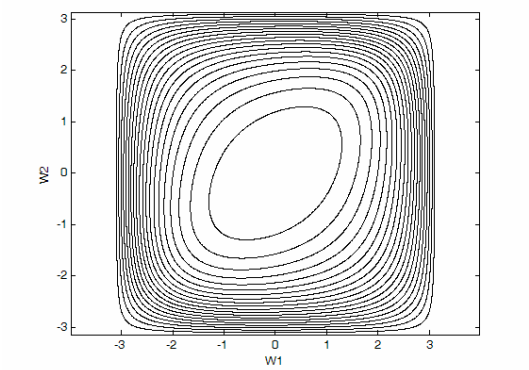
(a)



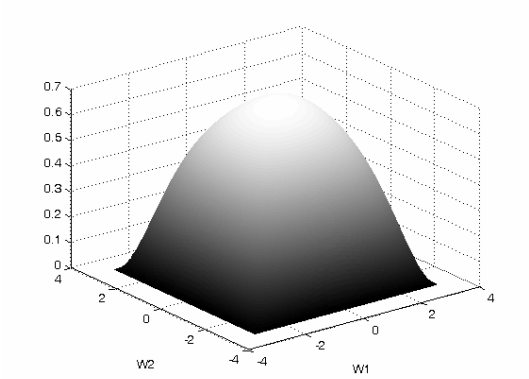
(b)

Fig.4(a): Contour plot for the case when $k_1=k_2$ 1, $a_1 = a_2 =0.75$, k=-1.5.

Fig.4(b): The corresponding 3-D plot.



(a)



(b)

Fig.5(a) and (b): The contour and 3-D plots when the familiar bilinear transformations are used with k = -1.

4. Conclusion

Starting from two lower order Butterworth polynomials in the frequency domain, a method to generate transfer functions yielding monotonic responses is presented. Specifically, these polynomials are successively integrated with respect to ω^2 and then associated with the denominator of the magnitude response of an all-pole transfer function, taking care to see that the response at zero frequency is made unity. Results up to five successive integrations have been given, it is readily seen that more integrations can be carried out. By the combination of appropriately chosen s-domain factors of lower denominator polynomials, higher order polynomials can be obtained, which when associated with the denominator of a transfer function, yield monotonic magnitude frequency responses. Even with five integrations, it is shown that the order of the transfer function could be as high as thirty-four or more and these definite yield monotonic responses. The corresponding discrete-time filters can be obtained. using generalized bilinear transformations with different parameters. By proper cascading or other combinations of 1D filters obtained by the proposed method two-dimensional digital filters can be obtained. It has to be ensured that they are always stable. Their 2-D responses can, but need not be monotonic.

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