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A unified and efficient least-squares design of linear-phase nonrecursive filters

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Abstract

A method is described which can be used to design a wide class of nonrecursive linear-phase filters including those with arbitrary magnitude specifications and with time-domain constraints. The approach is based on formulating the weighted mean-square error between the amplitude responses of the practical and ideal filters as a quadratic function. The filter coefficients are obtained by solving a set of linear equations. This method leads to a lower weighted mean-square error and is computationally more efficient than both the eigenfilter method and the method based on the Remez exchange algorithm. However, the main advantage of our method lies in its computational efficiency. Design examples of many different types of filters are provided.

Zusammenfassung

Es wird eine Methode beschrieben, die zum Entwurf einer großen Klasse von nichtrekursiven linearphasigen Filtern benutzt werden kann, wobei solche mit beliebigen Frequenzgang-Spezifikationen und mit Zeitbereichsbedingungen eingeschlossen sind. Der Entwurf basiert auf der Formulierung eines gewichteten mean-square Fehlers zwischen den Frequenzgängen der praktischen und der idealen Filter als eine quadratische Funktion. Die Filterkoeffizienten erhält man aus der Lösung eines linearen Gleichungssystems. Diese Methode führt zu einem geringeren gewichteten mittleren quadratischen Fehler und ist recheneffizienter als die Eigenfilter-Methode und die Methode basierend auf dem Remez-Algorithmus. Der Hauptvorteil unserer Methode liegt jedoch in der Recheneffizienz. Es werden Entwurfsbeispiele für viele verschiedene Typen von Filtern angeführt.

Résumé

La méthode décrite peut être utilisée pour décrire une large classe de filtres non-récursifs à phase non linéaire incluant ceux ayant des spécifications de magnitude arbitraires et avec des contraintes dans le domaine réel. L'approche est basée sur la formulation de l'erreur quadratique moyenne pondérée entre les amplitudes de réponse des filtres idéaux et pratiques en tant que fonction quadratique. Les coefficients du filtre sont obtenus en résolvant un ensemble d'équations linéaires. Cette méthode procure une plus faible erreur quadratique moyenne pondérée et son calcul est plus efficace que

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la méthode des filtres propres et la méthode basée sur l'algorithme d'échange de Remez. Néanmoins, le principal avantage de notre méthode réside dans son efficacité de calcul. De nombreux exemples de filtres de type différent sont décrit.

Key words. Least squares; Linear phase; Nonrecursive; Efficient; Filters

1. Introduction

The design of nonrecursive filters with linear-phase characteristics has been dealt with extensively over the past three decades. Over these years, the McClellan–Parks (MP) algorithm [4], used for the design of filters that are optimal in the minimax sense, and the least-squares methods have received considerable attention. The least-squares approach, first proposed in [19] for the design of low-pass filters, involves the solution of a linear system of equations to obtain the filter coefficients. This approach was later revisited in [20] resulting in the concept of eigenfilters. In the eigenfilter method, an error function between the desired and the practical amplitude responses is formulated in a quadratic form. The desired amplitude response is equal to the amplitude response of the designed filter at an arbitrary reference frequency. The coefficients of the filter are obtained as the eigenvector corresponding to the smallest eigenvalue of a real symmetric positive-definite matrix. This method has been used to design frequency-selective filters including those involving time-domain constraints, differentiators and Hilbert transformers [9,10].

The method advanced in [19] has been used to design first-order differentiators in [17] and higher-order differentiators in [18]. In this paper, this method is generalized to accommodate various types of linear-phase nonrecursive filters including those with time-domain constraints. The motivation of our approach is to formulate an error function that directly minimizes the weighted mean-square error by explicitly including the ideal amplitude characteristic and obtaining the filter coefficients with low computational complexity. This results in a more meaningful formulation than the eigenfilter method. Our design approach is for a class of least-square nonrecursive filters that includes the usual frequency-selective filters (low-pass, high-pass, bandpass and bandstop), differentiators, Hilbert transformers, maximally flat filters and interpolating filters. Minimum-energy filters are designed by minimizing the stop band energy subject to a constraint on the frequency response at the zero frequency. The criterion used here is slightly different from that proposed in [19] to obtain the prolate spheroidal wave sequence. In addition, Nyquist and partial response filters that involve time-domain constraints are designed. Our approach can also include linear-phase filters with arbitrary magnitude characteristics.

The outline of the paper is as follows. Section 2 gives background material on the types of linear-phase nonrecursive filters. In Section 3, the objective function is described together with the method of obtaining the optimal filter coefficients. Sections 4–11 depict the designs of various filters and finally in Section 12, we offer our conclusions.

2. Characterization of linear-phase nonrecursive filters

Nonrecursive linear-phase filters with N taps having an impulse response $h(n)$ ($n = 0$ to $N-1$) can be divided into four types [11] as described below. Type 1 and Type 2 filters include those with symmetrical impulse responses ($h(n) = h(N-1-n)$). Type 1 filters have an odd number of taps while Type 2 filters have an even number of taps. The frequency response of these filters can be expressed as $H(e^{j\omega}) = M(\omega)e^{-j(N-1)\omega/2}$, where

$$M(\omega) = \begin{cases} \sum_{n=0}^{(N-1)/2} b(n)\cos n\omega & \text{(Type 1)}, \\ \sum_{n=1}^{N/2} b(n)\cos(n-1/2)\omega & \text{(Type 2)}. \end{cases} \tag{1}$$

If N is odd, $b(0) = h[(N-1)/2]$ and $b(n) = 2h[(N-1)/2 - n]$ for $1 \leqslant n \leqslant (N-1)/2$. If N is even, $b(n) = 2h(N/2 - n)$ for $1 \leqslant n \leqslant N/2$.

On other hand, Type 3 and Type 4 filters have an antisymmetrical impulse response ($h(n) = -h(N-1-n)$). Type 3 filters have an odd number of taps while Type 4 filters have an even number of taps. Consequently, the frequency response of these filters can be expressed as $H(e^{j\omega}) = jM(\omega)e^{-j(N-1)\omega/2}$, where

$$M(\omega) = \begin{cases} \sum_{n=1}^{(N-1)/2} b(n)\sin n\omega & \text{(Type 3)}, \\ \sum_{n=1}^{N/2} b(n)\sin(n-1/2)\omega & \text{(Type 4)}. \end{cases} \tag{2}$$

If N is odd, $b(n) = 2h[(N-1)/2 - n]$ for $1 \leqslant n \leqslant (N-1)/2$. If N is even, $b(n) = 2h(N/2 - n)$ for $1 \leqslant n \leqslant N/2$.

3. Objective function

In this section, we introduce the objective function E_{mse} that reflects the weighted mean-square difference between the ideal amplitude response, $D(\omega)$, and the amplitude response of the filter, $M(\omega)$. The function is expressed as

$$E_{\text{mse}} = \frac{\alpha}{\pi}\int_P W_P(\omega)[D(\omega) - M(\omega)]^2\,\mathrm{d}\omega + \frac{\beta}{\pi}\int_S W_S(\omega)M^2(\omega)\,\mathrm{d}\omega, \tag{3}$$

where P is the passband and S is the stopband. The quantities α and β reflect the relative emphasis given to the passband and stopband, respectively. On the other hand, $W_P(\omega)$ and $W_S(\omega)$ are nonnegative frequency-domain weighting functions for the passband and stopband that can be used to emphasize certain frequencies over others.

For the case of Type 1 and Type 2 filters, we have $M(\omega) = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{c}(\omega)$, where

$$\boldsymbol{b} = \begin{cases} [b(0)\ b(1)\ \cdots\ b((N-1)/2)]^{\mathrm{T}} & \text{(Type 1)}, \\ [b(1)\ b(2)\ \cdots\ b(N/2)]^{\mathrm{T}} & \text{(Type 2)} \end{cases} \tag{4}$$

and

$$\boldsymbol{c}(\omega) = \begin{cases} [1\cos\omega\ \ldots\ \cos(\tfrac{1}{2}(N-1)\omega)]^{\mathrm{T}} & \text{(Type 1)}, \\ [\cos\tfrac{1}{2}\omega\ \cos\tfrac{3}{2}\omega\ \ldots\ \cos(\tfrac{1}{2}(N-1)\omega)]^{\mathrm{T}} & \text{(Type 2)}. \end{cases} \tag{5}$$

Rewriting Eq. (3), we get

$$E_{\text{mse}} = \frac{\alpha}{\pi}\int_P W_P(\omega)[D^2(\omega) - 2D(\omega)\boldsymbol{b}^{\mathrm{T}}\boldsymbol{c}(\omega) + \boldsymbol{b}^{\mathrm{T}}\boldsymbol{c}(\omega)\boldsymbol{c}^{\mathrm{T}}(\omega)\boldsymbol{b}]\,\mathrm{d}\omega + \frac{\beta}{\pi}\int_S W_S(\omega)\boldsymbol{b}^{\mathrm{T}}\boldsymbol{c}(\omega)\boldsymbol{c}^{\mathrm{T}}(\omega)\boldsymbol{b}\,\mathrm{d}\omega. \tag{6}$$

Filter coefficients that are optimal in a weighted least-squares sense are obtained by minimizing E_{mse}. Consequently, we set $\partial E_{\text{mse}}/\partial b(i) = 0$ to obtain a system of linear equations $[\alpha\boldsymbol{Q} + \beta\boldsymbol{R}]\boldsymbol{b} = \alpha\boldsymbol{d}$, where

$$\boldsymbol{Q} = \int_P W_P(\omega)\boldsymbol{c}(\omega)\boldsymbol{c}^{\mathrm{T}}(\omega)\,\mathrm{d}\omega, \tag{7}$$

$$\boldsymbol{R} = \int_S W_S(\omega)\boldsymbol{c}(\omega)\boldsymbol{c}^{\mathrm{T}}(\omega)\,\mathrm{d}\omega \tag{8}$$

and

$$\boldsymbol{d} = \int_P W_P(\omega)D(\omega)\boldsymbol{c}(\omega)\,\mathrm{d}\omega. \tag{9}$$

It can be noted that both $\boldsymbol{Q}$ and $\boldsymbol{R}$ are positive-definite real symmetric matrices. Thus, a unique solution is guaranteed for nonnegative values of α and β. In addition, the system of linear equations can be solved by a computationally efficient method, like the Cholesky decomposition, that avoids matrix inversion.

Similarly, for Type 3 and Type 4 filters, $M(\omega) = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{s}(\omega)$, where

$$\boldsymbol{b} = \begin{cases} [b(1)\ b(2)\ \ldots\ b((N-1)/2)]^{\mathrm{T}} & \text{(Type 3)}, \\ [b(1)\ b(2)\ \ldots\ b(N/2)]^{\mathrm{T}} & \text{(Type 4)} \end{cases} \tag{10}$$

and

$$s(\omega) = \begin{cases} [\sin\omega \ \sin 2\omega \ \ldots \ \sin(\tfrac{1}{2}(N-1)\omega)]^{\mathrm{T}} \ \text{(Type 3)}, \\ [\sin\tfrac{1}{2}\omega \ \sin\tfrac{3}{2}\omega \ \ldots \ \sin(\tfrac{1}{2}(N-1)\omega)]^{\mathrm{T}} \ \text{(Type 4)}. \end{cases} \tag{11}$$

The system of equations resulting from the minimization of the objective function is $[\alpha \boldsymbol{Q} + \beta \boldsymbol{R}]\boldsymbol{b} = \alpha \boldsymbol{d}$, where

$$\boldsymbol{Q} = \int_P W_P(\omega) s(\omega) s^{\mathrm{T}}(\omega)\, \mathrm{d}\omega, \tag{12}$$

$$\boldsymbol{R} = \int_S W_S(\omega) s(\omega) s^{\mathrm{T}}(\omega)\, \mathrm{d}\omega \tag{13}$$

and

$$\boldsymbol{d} = \int_P W_P(\omega) D(\omega) s(\omega)\, \mathrm{d}\omega. \tag{14}$$

Owing to the positive-definiteness of $\boldsymbol{Q}$ and $\boldsymbol{R}$, the system of linear equations can be solved by using the Cholesky decomposition.

In comparing our approach with the eigenfilter method, we consider two aspects, namely weighted mean-square error and computational complexity. Our approach formulates a better error measure than the eigenfilter method in that we explicitly minimize the weighted mean-square error between the ideal and the practical amplitude responses. In contrast, the eigenfilter method does not take the ideal response into account, but rather the frequency response of the practical filter at an arbitrary frequency. Consequently, the filter that is designed depends upon the reference frequency chosen. By considering the ideal response and not requiring a reference frequency, our approach will result in a filter with a lower weighted mean-square error than its eigenfilter counterpart.

The difference in the computational complexity of the two methods primarily lies in the algorithms used to obtain the filter coefficients. Our approach merely involves the solution of a system of linear equations. Consequently, it is noniterative and does not require any tests for convergence. For the eigenfilter method, the filter coefficients are obtained by computing the eigenvector corresponding to the smallest eigenvalue of a positive-definite symmetric matrix $\boldsymbol{P}$. Such an eigenvector is computed by using the inverse power method where a system of linear equations is solved M times. The value of M is almost always greater than one and is influenced by the smallest two eigenvalues of $\boldsymbol{P}$ (see Appendix A for more details). Thus, for finding the filter coefficients, our method is about M times faster than the eigenfilter method. Another aspect that influences the computational complexity is the evaluation of the entries of $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{d}$ for our method and $\boldsymbol{P}$ for the eigenfilter method. Although, for properly chosen weighting functions this does not make up the major computational burden, the number of multiplications, additions and trigonometric function evaluations is generally more for the eigenfilter method.

4. Common frequency-selective filters

In this section, we consider the design of some of the common frequency-selective filters for which $D(\omega)$ is 1 in P and 0 in S. For these filters, the ideal frequency response must be necessarily real and, consequently, only Type 1 and Type 2 filters are considered. For a low-pass filter, $P = [0, \omega_p]$ and $S = [\omega_s, \pi]$. In the case of a band-pass filter, $P = [\omega_{p1}, \omega_{p2}]$ and $S = [0, \omega_{s1}] \cup [\omega_{s2}, \pi]$.

EXAMPLE 1. A low-pass filter with $N = 29$, $\omega_p = 0.3\pi$, $\omega_s = 0.4\pi$, $\alpha = \beta = 0.5$ and $W_P(\omega) = W_S(\omega) = 1$ is considered. Fig. 1 shows the magnitude response of this filter. It must be mentioned that this example is the same as that given in [19].

EXAMPLE 2. Example 2 is a band-pass filter in which $N = 51$, $\omega_{s1} = 0.3\pi$, $\omega_{s2} = 0.8\pi$, $\omega_{p1} = 0.35\pi$, $\omega_{p2} = 0.7\pi$, $\alpha = 2/3$, $\beta = 1/3$ and $W_P(\omega) = W_S(\omega) = 1$. Fig. 2 shows the magnitude response of this filter.

We compare our design method with the eigenfilter approach and the MP algorithm based on three performance measures, namely (1) the number of floating point operations (flops), (2) the weighted mean-square error E_{mse}, and (3) the peak error

$E_{\text{peak}} = \max(A_1, A_2)$, where

$$A_1 = \max_P |W_P(\omega)[D(\omega) - M(\omega)]| \tag{15}$$

and

$$A_2 = \max_S |W_S(\omega) M(\omega)|. \tag{16}$$

A comparison of the three methods with respect to the number of flops is shown in Table 1 and that with respect to E_{mse} and E_{peak} is shown in Table 2. It must be mentioned that the entries in Table 1 have been normalized relative to the number of flops in our method. In the eigenfilter method, the reference frequency for the low-pass design is zero. For the band-pass design, the reference frequency is the center frequency in the passband.

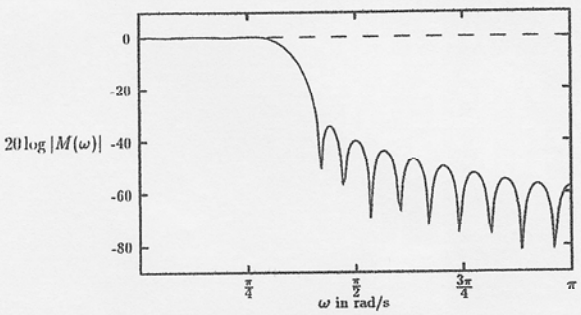

Fig. 1. Magnitude response of a low-pass filter with $\omega_p = 0.3\pi$, $\omega_s = 0.4\pi$, $\alpha = \beta = 0.5$, $W_P(\omega) = W_S(\omega) = 1$ and $N = 29$ (Example 1).

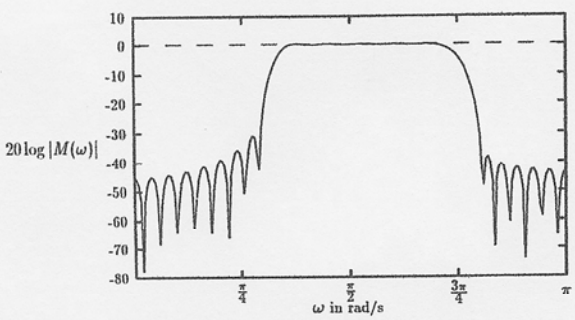

Fig. 2. Magnitude response of a band-pass filter with $\omega_{s1} = 0.3\pi$, $\omega_{s2} = 0.8\pi$, $\omega_{p1} = 0.35\pi$, $\omega_{p2} = 0.7\pi$, $\alpha = 2/3$, $\beta = 1/3$, $W_P(\omega) = W_S(\omega) = 1$ and $N = 51$ (Example 2).

It can be seen from Table 1 that our method is computationally more efficient than both the eigenfilter method and the MP algorithm. As has been mentioned in Section 3, the eigenfilter method is iterative and entails more function evaluations as compared to our method. For the MP algorithm, the computational complexity stems from the fact that it uses the iterative Remez exchange algorithm.

Since our method explicitly minimizes the weighted mean-square error, it guarantees the lowest E_{mse} for a given set of filter specifications. This is exemplified by the results in Table 2. However, the differences in the value of E_{mse} between our method and the eigenfilter approach are small. Since the MP algorithm minimizes the peak error, E_{peak} is the lowest for this approach.

Table 1
Comparison of the three methods with respect to the number of floating point operations

Examples	Floating point operations (flops) (normalized relative to our method)		
	Our method	Eigenfilter method	MP method
1	1	2.89	7.07
2	1	4.80	5.94

Table 2
Comparison of the three methods with respect to the mean-square and peak errors

Examples	Weighted mean-square error (E_{mse})			Peak error (E_{peak})		
	Our method	Eigenfilter method	MP method	Our method	Eigenfilter method	MP method
1	1.685×10^{-4}	1.763×10^{-4}	6.646×10^{-4}	7.665×10^{-2}	7.665×10^{-2}	3.100×10^{-2}
2	3.840×10^{-5}	3.946×10^{-5}	1.982×10^{-4}	9.330×10^{-2}	9.240×10^{-2}	3.760×10^{-2}

5. Differentiators

An ideal kth-order differentiator has a frequency response

$$H_I(e^{j\omega}) = \begin{cases} D(\omega)e^{jk\pi/2}, & \omega \in P, \\ 0, & \omega \in S, \end{cases} \tag{17}$$

where $D(\omega) = (\omega/2\pi)^k$. For even k, only Type 1 and Type 2 differentiators can be designed while for odd k one is restricted to the design of Type 3 and Type 4 filters.

There are two passband weights often considered, namely $W_P(\omega) = 1$ (absolute error measure) and $W_P(\omega) = 1/D^2(\omega)$ (relative error measure). It has been shown in [18] that for $k > 1$, the relative error measure is not feasible due to a discontinuity in the integrand. A detailed comparison of the design of higher-order differentiators using our method, the eigenfilter approach [10] and the MP algorithm [12] in terms of the three performance measures mentioned earlier has been carried out in [18]. It is shown that minimizing E_{mse} using our approach leads to a lower mean-square error and is computationally more efficient than the eigenfilter and minimax methods.

In [17], only the method of design of first-order differentiators has been presented. A comparison with the eigenfilter method [9] and the MP algorithm has not been carried out. Here, we present a performance evaluation using design Example 3.

EXAMPLE 3. In this example, we use the objective function in Eq. (3) to design a first-order differentiator with 31 taps. The passband is given by $P = [0, 0.9\pi]$ and $W_P(\omega) = 1$. There is no stopband. Fig. 3 shows the magnitude response of this filter while Table 3 gives the details of the comparison.

Design Example 3 reflects a performance comparison using an absolute error measure. Now, consider the use of a relative error measure. In this case, the entries of $\boldsymbol{Q}$ and $\boldsymbol{d}$ have to be evaluated using numerical integration. Similarly, numerical integration has to be used to calculate the entries of the matrix involved in the eigenfilter design. The difference in the computational complexity is due to the difference in the methods of obtaining the filter coefficients. Also, our method yields a lower relative mean-square error. By specifying a nonconstant weighting function for the MP algorithm, no extra complexity is involved. Consequently, it is the most efficient method for designing differentiators with a relative error measure.

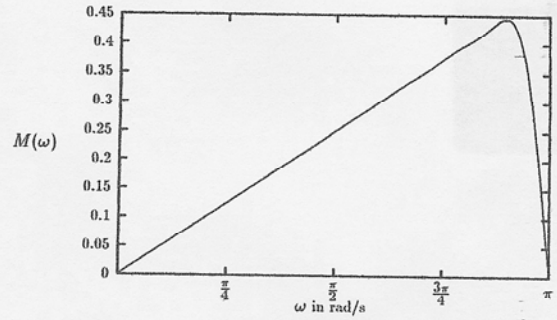

Fig. 3. Magnitude response of a first-order differentiator with passband given by $P = [0, 0.9\pi]$, $W_P(\omega) = 1$ and $N = 31$ (Example 3).

Table 3
Comparison of the three methods with respect to the three performance measures for the case a first-order differentiator (Example 3)

Performance measure	Our method	Eigenfilter method	MP method
Flops	1	3.44	12.82
E_{mse}	2.729×10^{-7}	2.739×10^{-7}	5.426×10^{-7}
E_{peak}	5.183×10^{-3}	5.175×10^{-3}	1.901×10^{-3}

6. Hilbert transformers

The ideal frequency response of a Hilbert transformer is given by $H_I(e^{j\omega}) = D(\omega)e^{j\pi/2}$ [11], where

$$D(\omega) = \begin{cases} -1, & \omega_1 \leqslant \omega \leqslant \omega_2, \\ 1, & -\omega_2 \leqslant \omega \leqslant -\omega_1. \end{cases} \tag{18}$$

Since the ideal frequency response is purely imaginary, only Type 3 and Type 4 filters can be designed. Below, we present a design example.

EXAMPLE 4. A 31-tap Hilbert transformer with $\omega_1 = 0.08\pi$ and $\omega_2 = 0.92\pi$ is designed. The

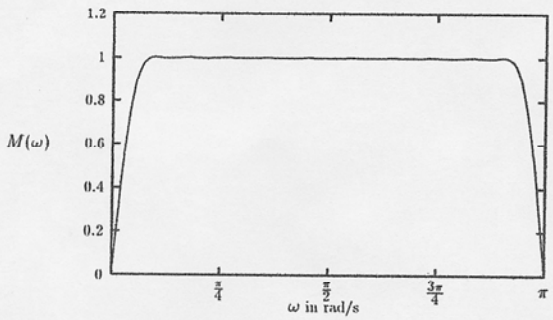

Fig. 4. Magnitude response of a 31-tap Hilbert transformer with $\omega_1 = 0.08\pi$, $\omega_2 = 0.92\pi$ and $W_P(\omega) = 1$ (Example 4).

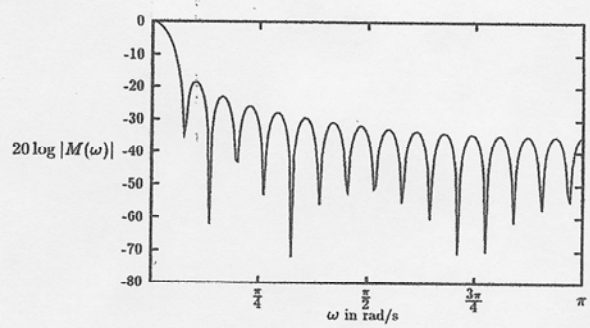

Fig. 5. Magnitude response of a 31-tap minimum-energy filter with $\omega_s = 0.05\pi$, $\alpha = \beta = 0.5$ and $W_P(\omega) = 1$ (Example 5).

Table 4
Comparison of the three methods with respect to the three performance measures for the case a Hilbert transformer (Example 4)

Performance measure	Our method	Eigenfilter method	MP method
Flops	1	2.77	10.74
E_{mse}	1.064×10^{-5}	1.110×10^{-5}	2.759×10^{-5}
E_{peak}	2.35×10^{-2}	2.38×10^{-2}	8.15×10^{-3}

passband weighting function is $W_P(\omega) = 1$. Fig. 4 shows the magnitude response of this filter while Table 4 gives the performance comparison.

At this point, we have demonstrated that our technique leads to a lower weighted mean-square error and is computationally more efficient than the eigenfilter and minimax approaches. In the sequel, we show that our approach can accommodate a wide variety of filters just like the eigenfilter method. Therefore, the advantages we offer come with no compromise in diversity.

7. Minimum-energy filters

A minimum-energy filter is one which yields the smallest stopband energy subject to a constraint that is imposed to avoid a zero solution vector. A prolate spheroidal wave sequence is obtained in [19] by using the constraint that the power gain is unity. The solution vector is the eigenvector corresponding to the smallest eigenvalue of a real, symmetric and positive-definite matrix. In fact, the eigenfilter method in [20] is a linear-algebraic generalization of this concept.

Since our objective function does not accommodate constraints, we transform the problem to that of an unconstrained minimization. The idea is to make the amplitude response at the zero frequency close to unity and minimize the stopband energy. Therefore, only Type 1 and Type 2 filters are considered. The objective function is modified as

$$E_{mse} = \frac{\alpha}{\pi}[1 - M(0)]^2 + \frac{\beta}{\pi}\int_P W_S(\omega) M^2(\omega)\,d\omega, \quad (19)$$

where $S = [\omega_s, \pi]$. Although the resulting filter is not strictly a prolate spheroidal wave sequence, suppression of the stopband energy is taken into account. Here again, the filter coefficients are obtained by solving a system of linear equations thereby ensuring a lower computational complexity than the eigenfilter method.

EXAMPLE 5. A minimum-energy filter in which $N = 31$, $\omega_s = 0.05\pi$, $\alpha = \beta = 0.5$ and $W_S(\omega) = 1$ is designed. Fig. 5 shows the magnitude response of this filter.

8. Nyquist and partial response filters

In bandwidth-efficient data transmission systems, low-pass Nyquist filters are used to avoid intersymbol interference. Cancellation of intersymbol interference implies that the output of a data receiver taken at symbol intervals depends only on

its corresponding transmitted symbol. To accomplish this, time-domain constraints in the form of zero crossings are imposed on the impulse response of the filter. Since these zero crossings are symmetric about a center coefficient, the filter has an odd number of taps. Hence, we consider Type 1 filters for the design. The time-domain constraints are specified as $h(n) = 0$ for $n - (N-1)/2 =$ a nonzero multiple of L, where L is the zero crossing or symbol interval. The frequency-domain specifications dictate a low-pass characteristic where the passband and stopband are given by $P = [0, \omega_p]$ and $S = [\omega_s, \pi]$, respectively, such that $\omega_p + \omega_s = 2\pi/L$. The ideal amplitude response is given by $D(\omega) = 1$ in P.

In addition to the eigenfilter method, techniques based on a linear programming approach [3, 14] and the MP algorithm [5, 15] give rise to minimax designs that satisfy the time-domain constraints exactly. Our method can also easily accommodate the time-domain constraints by taking only the terms in $M(\omega)$ that correspond to the nonzero impulse response coefficients to minimize E_{mse}. In fact, the dimension of the linear system of equations is now lower than what would be the case if the zero crossings were not imposed.

EXAMPLE 6. For this example, we design a Nyquist filter with 39 taps and a symbol interval $L = 4$. The passband and stopband edges are given by $\omega_p = 0.2125\pi$ and $\omega_s = 0.2875\pi$. Also, $\alpha = \beta = 1$ and $W_P(\omega) = W_S(\omega) = 1$. The magnitude response of the filter is shown in Fig. 6.

In communication systems, it is often desirable to split a linear-phase Nyquist filter into a transmitter/receiver pair that have identical magnitude responses. For this purpose, the Nyquist filter is split into minimum and maximum phase parts and hence must have zeros on the unit circle occurring as double zeros. This double-zero constraint is achieved by the approach in [14] and by two iterative approaches based on the eigenfilter method [8] and the MP algorithm [13]. The basic difference between the two iterative methods lies in the first step of designing an intermediate filter to achieve stopband control. Our method of designing minimum-energy filters can be used to obtain this intermediate filter. Then, the remaining steps as in

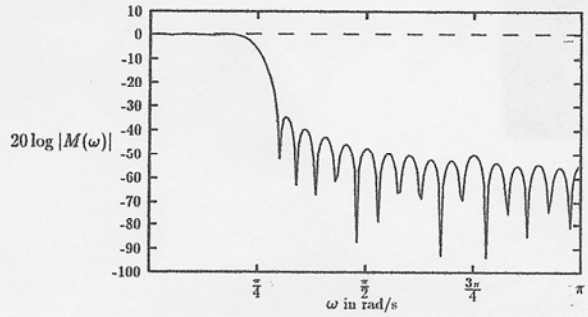

Fig. 6. Magnitude response of a Nyquist filter with $\omega_p = 0.2125\pi$, $\omega_s = 0.2875\pi$, $\alpha = \beta = 1$, $W_P(\omega) = W_S(\omega) = 1$, $N = 39$ and $L = 4$ (Example 6).

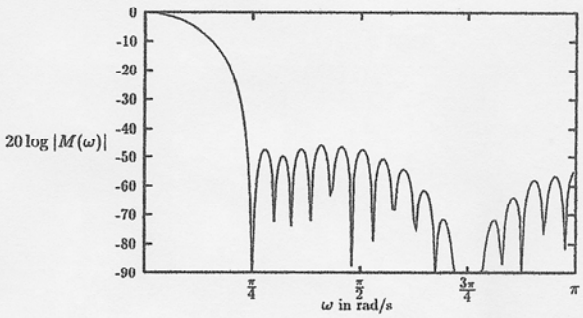

Fig. 7. Magnitude response of a 47-tap Class 2 partial-response filter (Example 7).

[8, 13] can be applied to get the Nyquist filter and its minimum/maximum phase parts.

Partial-response filters can be designed as a cascade of a Nyquist filter and a filter with transfer function equal to a polynomial in z^L that serves to introduce the controlled amount of intersymbol interference. Numerous examples of different polynomials in z^L are given in [2]. We present a design example.

EXAMPLE 7. A Class 2 partial response filter with 47 taps [2] is designed starting from the Nyquist filter in Example 5. The Nyquist filter is cascaded with the polynomial $1 + 2z^{-L} + z^{-2L}$. The magnitude response of the filter is shown in Fig. 7.

9. Maximally flat filters

Here, we demonstrate the design of a Type 1 maximally flat low-pass filter with an arbitrary

degree of flatness at $\omega = 0$. As in [20], we express the amplitude response as $M(\omega) = \boldsymbol{t}^{\mathrm{T}}\boldsymbol{v}(\omega)$, where

$$\boldsymbol{t} = [t(0)\ t(1)\ \cdots\ t((N-1)/2)]^{\mathrm{T}} \tag{20}$$

and

$$\boldsymbol{v}(\omega) = [1\ \sin^2\omega/2\ \sin^4\omega/2\ \ldots\ \sin^{N-1}\omega/2]^{\mathrm{T}}. \tag{21}$$

If $t(j) = 0$ for $j = 1$ to M, the first $2M + 1$ derivatives of $M(\omega)$ at $\omega = 0$ are equal to zero. Then, a filter with degree of flatness equal to $2M + 1$ is obtained. We impose this constraint and redefine the amplitude response as $M(\omega) = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{u}(\omega)$ where

$$\boldsymbol{a} = [t(0)\ t(M+1)\ t(M+2)\ \ldots\ t((N-1)/2)]^{\mathrm{T}} \tag{22}$$

and

$$\boldsymbol{u}(\omega) = [1\ \sin^{2(M+1)}\omega/2\ \sin^{2(M+2)}\omega/2\ \ldots\ \sin^{N-1}\omega/2]^{\mathrm{T}}. \tag{23}$$

The objective function as given by Eq. (3) is minimized with respect to $\boldsymbol{a}$. Again, a linear system of equations $[\alpha\boldsymbol{Q} + \beta\boldsymbol{R}]\boldsymbol{a} = \alpha\boldsymbol{d}$ results, where

$$\boldsymbol{Q} = \int_P W_P(\omega)\boldsymbol{u}(\omega)\boldsymbol{u}^{\mathrm{T}}(\omega)\,\mathrm{d}\omega, \tag{24}$$

$$\boldsymbol{R} = \int_S W_S(\omega)\boldsymbol{u}(\omega)\boldsymbol{u}^{\mathrm{T}}(\omega)\,\mathrm{d}\omega \tag{25}$$

and

$$\boldsymbol{d} = \int_P W_P(\omega)\boldsymbol{u}(\omega)\,\mathrm{d}\omega. \tag{26}$$

The impulse response of the filter is obtained from $\boldsymbol{a}$. It must be noted that the arbitrary flatness constraint can be imposed at any frequency ω_0 by expanding $M(\omega)$ in terms of $\sin^{2n}(\omega - \omega_0/2)$.

EXAMPLE 8. We design a 71-tap maximally flat filter with a degree of flatness equal to 3 imposed at zero frequency. The passband and stopband edges are $\omega_p = 0.2\pi$ and $\omega_s = 0.45\pi$, respectively. The weighting parameters are $\alpha = \beta = 0.5$ and $W_P(\omega) = W_S(\omega) = 1$. The magnitude response of the filter is shown in Fig. 8.

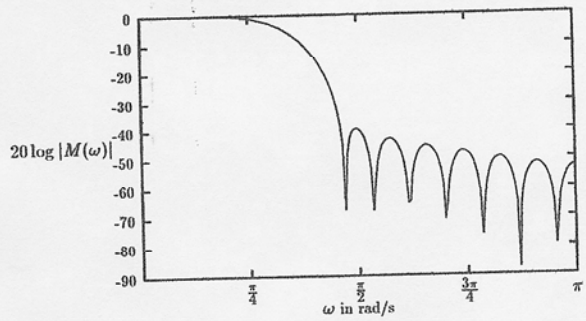

Fig. 8. Magnitude response of a 71-tap maximally flat low-pass filter with $\omega_p = 0.2\pi$, $\omega_s = 0.45\pi$, $\alpha = \beta = 0.5$ and $W_P(\omega) = W_S(\omega) = 1$. The degree of flatness imposed at zero frequency is equal to 3 (Example 8).

10. Interpolated FIR filters

The term interpolated FIR (IFIR) filter refers to the nonrecursive implementation of a linear-phase FIR filter with transfer function $H(z)$, as a cascade of two linear-phase filters with transfer functions $G(z^L)$ and $K(z)$. This requires about $1/L$ of the number of multiplications and additions of a conventional nonrecursive implementation of an equivalent FIR filter [6]. For the design of a low-pass $H(z)$, $D(\omega) = 1$ in $P = [0, \omega_p]$ and $S = [\omega_s, \pi]$ are specified. Consequently, $G(z)$ will have a passband $[0, L\omega_p]$ and a stopband $[L\omega_s, \pi]$. Although $G(z^L)$ has the proper band edges ω_p and ω_s, its additional passband images must be removed by a low-pass $K(z)$. It must be noted that $K(z)$ can have a wide transition band and hence a low order for a specified stopband attenuation. For the formulation, $G(z)$ must be designed to satisfy the requirements for $H(z)$ assuming that $K(z)$ is known. Design of IFIR filters based on the MP algorithm [6] and the eigenfilter method [1] exist. Here, we demonstrate the design of a Type 1 IFIR filter $H(z)$ using our method. Let $H(z)$, $G(z)$ and $K(z)$ have N, I and J taps, respectively. The amplitude response of $G(z)$, having an impulse response $g(n)$, is expressed as $\boldsymbol{r}^{\mathrm{T}}\boldsymbol{c}(\omega)$ where $\boldsymbol{r} = [r(0)\ r(1)\ \ldots\ r((I-1)/2)]^{\mathrm{T}}$, $r(0) = g((I-1)/2)$ and $r(n) = 2g((I-1)/2 - n)$ for $1 \leqslant n \leqslant (N-1)/2$.

In order to design $G(z)$, Eq. (3) has to be expressed in terms of $\boldsymbol{r}$ by transforming $\boldsymbol{b}$. Following the development in [1], we let $\boldsymbol{h}$ be the vector of

impulse response coefficients of $H(z)$ so that $\boldsymbol{b} = \boldsymbol{E}\boldsymbol{h}$, where $\boldsymbol{E}$ is an $(N-1)/2+1$ by N matrix given by

$$\boldsymbol{E} = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 1 & 0 & 1 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & 0 \\ 1 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{27}$$

If $\boldsymbol{g}_1$ is the vector of impulse response coefficients of $G(z^L)$, we can write $\boldsymbol{h} = \boldsymbol{K}\boldsymbol{g}_1$, where $\boldsymbol{K}$ is an N by $(I-1)L+1$ Toeplitz matrix whose first row is $[k(0)\ 0\ 0\ \ldots\ 0]$ and whose first column is $[k(0)\ k(1)\ \ldots\ k(J-1)\ 0\ \ldots\ 0]$. If $\boldsymbol{g}$ is the vector of impulse response coefficients of $G(z)$, we can write $\boldsymbol{g}_1 = \boldsymbol{S}\boldsymbol{g}$. The matrix S has a block structure,

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_1 \\ \boldsymbol{S}_2 \\ \vdots \\ \boldsymbol{S}_I \end{bmatrix}, \tag{28}$$

where the $\boldsymbol{S}_k (k = 1 \text{ to } I-1)$ are L by I matrices whose entries are all zero except $S_k(1,k) = 1$. The last block, $\boldsymbol{S}_I$ is an I element row vector whose entries are all zero except the last element, which equals 1. Finally, $\boldsymbol{g} = \boldsymbol{D}\boldsymbol{r}$, where D is an I by $(I-1)/2+1$ matrix given by

$$\boldsymbol{D} = \begin{bmatrix} 0 & 0 & \cdots & 0.5 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0.5 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 0.5 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0.5 \end{bmatrix}. \tag{29}$$

Thus, $\boldsymbol{b} = \boldsymbol{EKSDr} = \boldsymbol{Xr}$. By substituting $\boldsymbol{Xr}$ for $\boldsymbol{b}$, the objective function of Eq. (3) can be expressed as

$$E_{\text{mse}} = \frac{\alpha}{\pi}\int_P W_P(\omega)[1 - \boldsymbol{r}^{\mathrm{T}}\boldsymbol{X}^{\mathrm{T}}\boldsymbol{c}(\omega)]^2\,\mathrm{d}\omega + \frac{\beta}{\pi}\int_S W_S(\omega)[\boldsymbol{r}^{\mathrm{T}}\boldsymbol{X}^{\mathrm{T}}\boldsymbol{c}(\omega)]^2\,\mathrm{d}\omega. \tag{30}$$

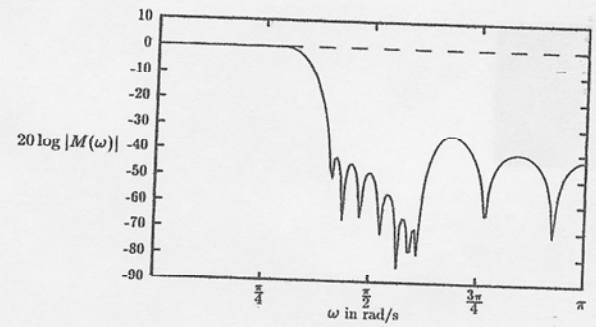

Fig. 9. Magnitude response of a 49-tap IFIR low-pass filter with $\omega_p = 0.3\pi$, $\omega_s = 0.4\pi$, $\alpha = \beta = 0.5$ and $W_P(\omega) = W_S(\omega) = 1$ (Example 9).

By minimizing E_{mse} with respect to $\boldsymbol{r}$, the filter coefficients for $G(z)$ are obtained by solving a system of linear equations. Subsequently, the coefficients for $H(z)$ are obtained.

EXAMPLE 9. A 49-tap IFIR low-pass filter with $\omega_p = 0.3\pi$, $\omega_s = 0.4\pi$, $\alpha = \beta = 0.5$ and $W_P(\omega) = W_S(\omega) = 1$ is designed. For $G(z)$ and $K(z)$, we choose $L = 2$, $I = 21$ and $J = 9$. The filter $K(z)$ is designed by our approach with a passband edge of 0.36π and a stopband edge of 0.55π. The weighting factors used are $\alpha = 1$, $\beta = 3$ and $W_P(\omega) = W_S(\omega) = 1$. The magnitude response of the IFIR filter is shown in Fig. 9.

11. Arbitrary magnitude specifications

In this section, we show how our approach can be used to design multiband filters having an arbitrary amplitude response in each band. Such filters have been designed using the eigenfilter approach [7] and the MP algorithm. Let M_P and M_S be the number of passbands and stopbands, respectively. The kth passband and lth stopband are specified as the interval $P_k = [\omega_p(k,1), \omega_p(k,2)]$ and $S_l = [\omega_s(l,1), \omega_s(l,2)]$, respectively. The objective function of Eq. (3) is generalized as

$$E_{\text{mse}} = \frac{1}{\pi}\sum_{k=1}^{Mr}\alpha_k\int_{P_k} W_{P_k}(\omega)[D_k(\omega) - M(\omega)]^2\,\mathrm{d}\omega + \frac{1}{\pi}\sum_{k=1}^{Ms}\beta_k\int_{S_k} W_{S_k}(\omega)M^2(\omega)\,\mathrm{d}\omega, \tag{31}$$

where $D_k(\omega)$ is the ideal response in P_k. The weighting functions in each passband P_k and each stopband S_k are given by $W_{P_k}(\omega)$ and $W_{S_k}(\omega)$, respectively. The only limitation is that $D_k(\omega)$ must be either purely real or purely imaginary to accommodate a linear-phase design. Consequently, Type 1 or 2 filters are used when $D_k(\omega)$ is real while Type 3 or 4 filters are used when $D_k(\omega)$ is imaginary. As in all the other cases, minimizing E_{mse} with respect to $\boldsymbol{b}$ results in a system of linear equations $(\boldsymbol{Q}+\boldsymbol{R})\boldsymbol{b}=\boldsymbol{d}$, where

$$\boldsymbol{Q}=\sum_{k=1}^{M_P}\alpha_k\int_{P_k}W_{P_k}(\omega)\boldsymbol{c}(\omega)\boldsymbol{c}^{\mathrm{T}}(\omega)\,\mathrm{d}\omega, \tag{32}$$

$$\boldsymbol{R}=\sum_{k=1}^{M_S}\beta_k\int_{S_k}W_{S_k}(\omega)\boldsymbol{c}(\omega)\boldsymbol{c}^{\mathrm{T}}(\omega)\,\mathrm{d}\omega \tag{33}$$

and

$$\boldsymbol{d}=\sum_{k=1}^{M_P}\alpha_k\int_{P_k}W_{P_k}(\omega)D_k(\omega)\boldsymbol{c}(\omega)\,\mathrm{d}\omega. \tag{34}$$

EXAMPLE 10. A 51-tap Type 1 filter with two passbands and one stopband is designed. The ideal responses in the passband are $D_1(\omega)=1+\cos 2\omega$ and $D_2(\omega)=1$. The passband edges are $\omega_p(1,1)=0$, $\omega_p(1,2)=0.5\pi$, $\omega_p(2,1)=0.9\pi$ and $\omega_p(2,2)=\pi$. The stopband edges are $\omega_s(1,1)=0.6\pi$ and $\omega_s(1,2)=0.8\pi$. The weighting factors are $\alpha_1=2/5$, $\alpha_2=2/5$ and $\beta_1=1/5$. All of the weighting functions $W_{P_k}(\omega)$ and $W_{S_k}(\omega)$ are equal to 1. The magnitude response of the filter is shown in Fig. 10.

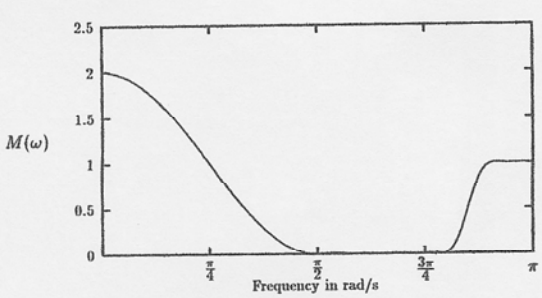

Fig. 10. Magnitude response of a filter with arbitrary magnitude specifications (Example 10).

12. Conclusions

In this paper, a least-squares approach to design a wide variety of linear-phase nonrecursive filters through a common framework is presented. Characteristics of this approach are that it (a) achieves a direct route that explicitly considers the ideal amplitude response in the design procedure, thereby resulting in filters that are optimal in the least-squares sense, (b) offers a closed-form solution for the filter coefficients, (c) is noniterative and computationally efficient in finding the filter coefficients and (d) is highly diverse in accommodating different filters including those with arbitrary magnitude responses and time-domain constraints. The weighted mean-square error and the computational complexity achieved by our method is lower than that achieved by the eigenfilter and minimax methods.

13. Appendix A

For our method, the filter parameters are obtained by solving a system of linear equations involving a real, symmetric and positive-definite matrix, which we denote as $\boldsymbol{S}$. It is well known that such a matrix can be decomposed as $\boldsymbol{S}=\boldsymbol{L}\boldsymbol{L}^{\mathrm{T}}$ where $\boldsymbol{L}$ is real and lower triangular [16]. Consequently, the system of linear equations can be written as $\boldsymbol{L}\boldsymbol{L}^{\mathrm{T}}\boldsymbol{b}=\boldsymbol{d}$. By letting $\boldsymbol{v}=\boldsymbol{L}^{\mathrm{T}}\boldsymbol{b}$, we obtain $\boldsymbol{d}=\boldsymbol{L}\boldsymbol{v}$. Given $\boldsymbol{L}$ and $\boldsymbol{d}$ we can obtain $\boldsymbol{v}$ by recursively solving a set of linear equations. Let l_{ij} be the element in the ith row and jth column of $\boldsymbol{L}$. It can be shown that

$$v(n)=\frac{1}{l_{nn}}\left\{d(n)-\sum_{j=0}^{n-1}l_{nj}v(j)\right\} \tag{A.1}$$

for $n=0,1,\ldots,N_E-1$, where N_E is the dimension of the system of equations. Since $\boldsymbol{S}$ is positive-definite, l_{nn} in the above equation are nonzero. We first solve for $v(0)$ and then recursively obtain $v(n)$. A total of $N_E(N_E-1)/2$ multiplications, N_E divisions, and $N_E(N_E-1)/2$ additions are required to compute $\boldsymbol{v}$. Similarly, we can recursively obtain $\boldsymbol{b}$ for a given $\boldsymbol{v}$ and $\boldsymbol{L}$. The total time required to

obtain the solution is

$$T_0 = T_{\text{chol}} + T_{\text{eqt}}. \tag{A.2}$$

Note that T_{chol} is the time taken for the decomposition of S and

$$T_{\text{eqt}} = (T_{\text{a}} + T_{\text{m}})N_E(N_E - 1) + 2N_E T_{\text{d}} \tag{A.3}$$

is the time required to obtain the solution given L, where T_{a}, T_{m} and T_{d} are the time required for one real addition, multiplication and division, respectively.

In the eigenfilter approach, the filter coefficients are obtained as the eigenvector corresponding to the smallest eigenvalue of a real, symmetric and positive-definite matrix P. In order to compute the smallest eigenvalue and its corresponding eigenvector, an iterative inverse power method is generally used [20]. At the $(k+1)$th iteration, a vector x_{k+1} is computed from the previous iterate x_k as

$$y_{k+1} = P^{-1} x_k, \tag{A.4}$$

$$x_{k+1} = y_{k+1}/\|y_{k+1}\|, \tag{A.5}$$

where $\|y_{k+1}\|$ denotes the L_2 norm of y_{k+1}. If $\|x_{k+1} - x_k\| \leqslant \varepsilon$ (typically $\varepsilon = 10^{-6}$), then x_{k+1} is a good approximation of the eigenvector corresponding to the smallest eigenvalue. Rewriting Eq. (A.4) as $x_k = Py_{k+1}$ and using the technique described above for solving a system of linear equations, we can obtain y_{k+1} and, subsequently, x_{k+1}.

It can now be seen that the eigenfilter method requires solving a system of linear equations several times before obtaining the eigenvector corresponding to the smallest eigenvalue. On the contrary, our approach requires solving a system of linear equations only once. If M is the number of iterations required in the eigenfilter method, then the total time taken for obtaining the filter coefficients using the eigenfilter method is

$$T_{\text{e}} = T_{\text{chol}} + M(T_{\text{eqt}} + T_{\text{nd}}), \tag{A.6}$$

where T_{nd} is the time taken to obtain x_{k+1} from y_{k+1} using Eq. (A.5). The value of M increases as the ratio λ_2/λ_1, where λ_1 is the smallest eigenvalue and λ_2 is the next smallest eigenvalue, decreases. If the ratio is too small, it may not even be possible to evaluate the smallest eigenvalue and its corresponding eigenvector using the inverse power method. In other words, the inverse power method may not converge.

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