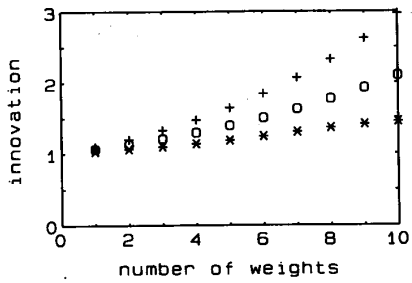

Fig. 3. Normalized mean-squared innovation $J'(\infty)/\sigma^2$ as a function of the number of weights in the adaptive filter for $\theta = 0.8$, 0.9, and 0.95 (indicated by crosses, circles and stars, respectively). The data are calculated from the average over 200 independent runs of the adaptive mechanism.

statistically independent, stationary, zero-mean, white-noise sources. The normalized mean-squared innovation $J'(\infty)/\sigma^2$ is plotted in Fig. 3 as a function of the number of parameters for three different values of θ. We observe that a larger number of weights increases the mean-squared innovation error, as is to be expected: a larger number of fluctuating parameters yields larger fluctuations in the innovation. For $\theta \to 1$, i.e., operating with a long memory, we have $J'(\infty) \to \sigma^2$, i.e., a perfect performance of the algorithm. The conclusion is that the Laguerre-domain adaptive filter is expected to perform well for a small number of weights and a large θ. This is exactly the problem we addressed in the introduction.

VI. DISCUSSION

We developed an adaptive filter starting by considering windowed versions of the input and reference signals. Windowing of the data is required since in adaptive filtering it is assumed that the correlation between input and reference signal is (slowly) time varying. An exponential window was chosen, and consequently, the signals were described as Laguerre series.

Our starting point differs from the usual approach in system identification using Laguerre functions [5]–[12]. There one commonly starts by a description of the model by a Laguerre filter, whereas we started by transforming the windowed signals to the Laguerre domain. This provides a local analysis of the input signal. We did not actually use the information provided by the local analysis in the present study. As a consequence of this and the fact that the model sets of the introduced adaptive filter and the more commonly used Laguerre filter are equal (Section IV), the differences between these two approaches should be sought in aspects of implementation.

There are two degrees of freedom in the Laguerre domain adaptive filter. The first one is the initial filtering stage F_0. Taking $F_0(z) = z\sqrt{1-\theta}/(z-\theta)$ is equal to taking a truncated Laguerre filter. Other choices for F_0 can be interpreted as prefiltering the input data x and subsequently using a truncated Laguerre filter. In this way, one can shape the spectrum of the input signal on the basis of *a priori* information in order to optimally exploit the properties of a truncated LDAF with a minimal number of coefficients, e.g., if it is known that we are approximating a process with a high-frequency falloff, one can already suppress these parts of the input signal by F_0.

The second degree of freedom is the parameter θ, which is in essence a scale factor determining how long the memory of the filter is taken. Although each stable linear causal time-invariant operator can be approximated by a Laguerre filter, the discount factor θ must be carefully chosen in order to obtain a minimal number of meaningful coefficients in the adaptive filter. In the case that the unknown process has a broadband spectrum, one can resort to multiscale adaptive filtering [12].

Current research includes further software simulations of the LDAF. Hardware implementation of the proposed adaptive filter is considered as well.

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Least-Squares Design of Higher Order Nonrecursive Differentiators

S. Sunder and Ravi P. Ramachandran

***Abstract*— A method is described that can be used to design nonrecursive linear-phase higher order differentiators that can perform differentiation over any frequency range. The method is based on formulating the absolute mean-square error between the amplitude responses of the practical and ideal differentiator as a quadratic function. The coefficients of the differentiators are obtained by solving a set of linear equations. This method leads to a lower mean-square error and is computationally more efficient than both the eigenfilter method and the method based on the Remez exchange algorithm. Design of differentiators based on minimization of the relative mean-square error is also carried out. Finally, our method is extended to the design of frequency selective higher order differentiators.**

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I. Introduction

Higher-order differentiators yield samples of the higher order (greater than one) derivatives of a band-limited continuous time signal. Higher order differentiators have been designed as nonrecursive linear-phase filters that approximate the ideal frequency characteristic that varies a as power of frequency with frequency. The design approaches used are extensions of those used for first-order differentiators [1]–[3]. In [1], the minimax method based on the Remez exchange algorithm [4] has been extended to incorporate the parameters involved in the design of higher order differentiators. The eigenfilter method [5] has been extended to the design of higher order differentiators in [2] by formulating an error function in a quadratic form. The error function involves the square of the difference between the desired amplitude response and the amplitude response of the designed filter. In this method, the desired amplitude response is equal to the amplitude response of the designed filter at an arbitrary reference frequency as opposed to being equal to the ideal amplitude characteristic. The filter coefficients are found by computing the eigenvector corresponding to the smallest eigenvalue of a positive-definite symmetric matrix. In [3], the Fourier-series method in conjunction with accuracy constraints [6] has been extended to the design of higher order differentiators. In this techinque, differentiators are designed by imposing magnitude and derivative constraints at a particular frequency. Consequently, these differentiators are accurate only in the neighbourhood of the frequency at which these constraints are imposed. In this correspondence, the least-squares approach described in [7] is extended to the design of higher order differentiators. The procedure involves formulating the absolute mean-square error between the practical and ideal differentiator as a quadratic function. The coefficients of the differentiators are obtained by solving a system of linear equations. The explicit inclusion of the ideal amplitude response in the error function leads to a more meaningful formulation than the eigenfilter method and does not necessitate the use of a reference frequency. The motivation of our approach is to do the following:

a) Achieve a direct route that explicitly considers the ideal amplitude response in the design procedure.
b) Offer a closed-form solution for the filter coefficients.
c) Devise a noniterative method to obtain this solution with a low computational complexity.

Comparisons of the differentiators designed using our method with that designed using the eigenfilter approach in terms of several performance measures reveal the superiority of our method to the eigenfilter method. The design of differentiators based on the relative mean-square error is also considered. By way of an example, it is shown that the error variation of differentiators designed by minimizing the relative mean-square error is lower than those designed in [3].

II. Higher Order Differentiators

Consider a nonrecursive digital filter with N taps having an impulse response $h(n)$ ($n = 0$ to $N-1$). For the case of a linear-phase filter having a symmetric impulse response, we have $h(n) = k(N-1-n)$. Consequently, the frequency response is $H(e^{j\omega})M(\omega)e^{-j\omega(N-1)/2}$, where

$$M(\omega) = \begin{cases} \sum_{n=0}^{(N-1)/2} a(n)\cos n\omega & N \text{ odd} \\ \sum_{n=1}^{N/2} a(n)\cos(n-1/2)\omega & N \text{ even} \end{cases} \tag{1}$$

If N is odd, $a(0) = h[(N-1)/2]$ and $a(n) = 2h[(N-1)/2-n]$ for $1 < n < (N-1)/2$. If N is even, $a(n) = 2h(N/2-n)$ for $1 \le n \le N/2$.

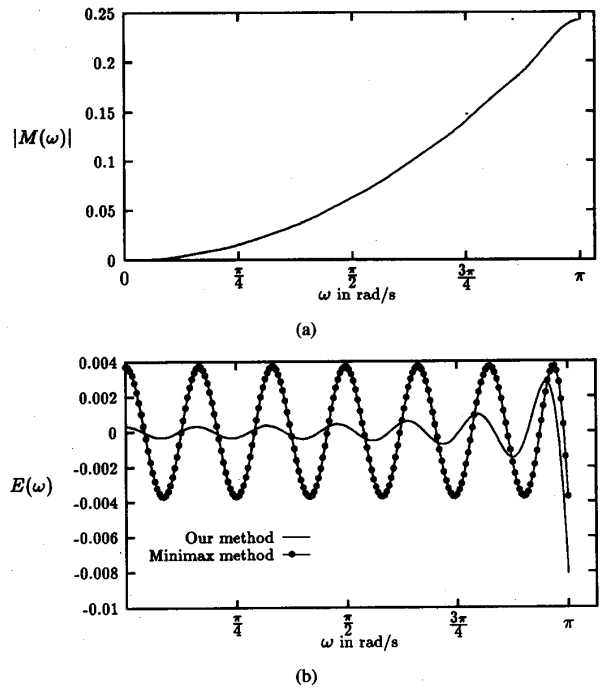

Fig. 1. (a) Amplitude response of a second-order differentiator with $N = 25$ and $\omega_p = \pi$; (b) variation of the error function with respect to ω.

On the other hand, for the case of an antisymmetric impulse response, we have $h(n) = -h(N-1-n)$. Therefore, $H(e^{j\omega}) = M(\omega)e^{j(\pi/2-\omega(N-1)/2)}$ where

$$M(\omega) = \begin{cases} \sum_{n=1}^{(N-1)/2} b(n)\sin n\omega & N \text{ odd} \\ \sum_{n=1}^{N/2} b(n)\sin(n-1/2)\omega & N \text{ even} \end{cases} \tag{2}$$

If N is odd, $b(n) = 2h[(N-1)/2-n]$ for $1 \le n \le (N-1)/2$. If N is even, $b(n) = 2h(N/2-n)$ for $1 \le n \le N/2$.

An ideal kth-order differentiator has a frequency response $H_I(e^{j\omega}) = D(\omega)e^{jk\pi/2}$, where $D(\omega) = (\omega/2\pi)^k$ for $0 \le \omega \le \omega_p \le \pi$. The upper passband edge frequency is ωp. For an even-order differentiator (k is even), it can be seen that $H_I(e^{j\omega})$ is a real-valued function. Hence, only a nonrecursive filter with a symmetrical impulse response can be used for the design of even-order differentiators. It should be noted that a full-band differentiator ($\omega_p = \pi$) can be designed only when N is odd. When N is even, it is required that $\omega_p < \pi$. Similarly, for an odd-order differentiator (k is odd), the fact that $H_I(e^{j\omega})$ is purely imaginary mandates the design of a nonrecursive filter with an antisymmetrical impulse response. For this case, a full-band differentiator can be designed only when N is even.

III. Error Function Minimization

The mean-square difference between $D(\omega)$ and $M(\omega)$ with respect to the differentiator passband can be expressed as

$$E_{\text{mse}} = \frac{1}{\pi}\int_0^{\omega_p} E^2(\omega)d\omega \tag{3}$$

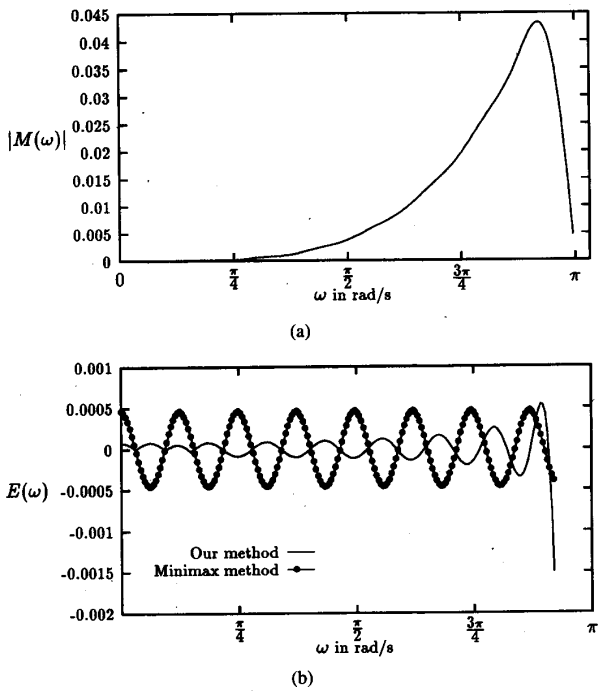

Fig. 2. (a) Amplitude response of a fourth-order differentiator with $N = 32$ and $\omega_p = 0.92\pi$; (b) variation of the error function with respect to ω.

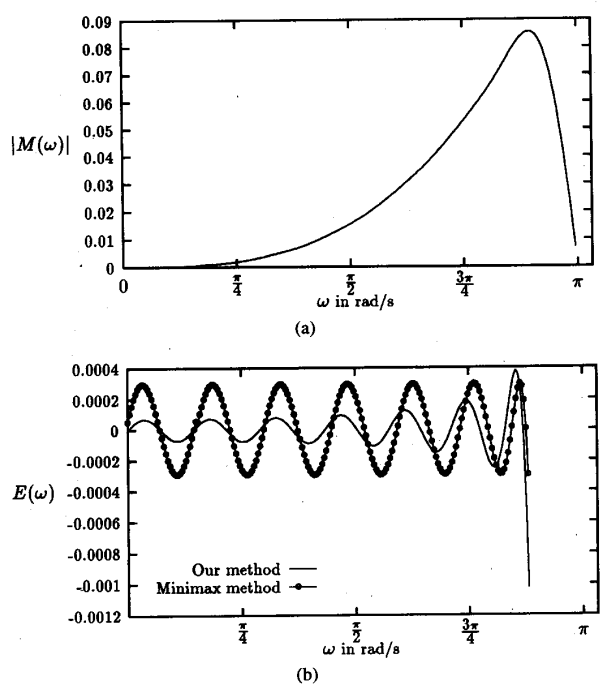

Fig. 3. Amplitude response of a third-order differentiator with $N = 27$ and $\omega_p = 0.88\pi$; (b) variation of the error function with respect to ω.

where $E(\omega) = D(\omega) - M(\omega)$. By minimizing the error function E_{mse} with respect to the filter coefficients, the required differentiator can be designed. For the case of even-order differentiators, we have

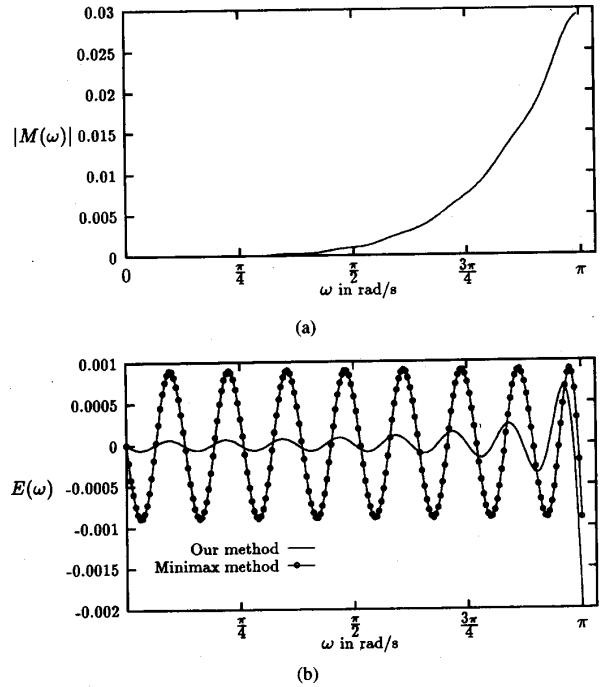

Fig. 4. Amplitude response of a fifth-order differentiator with $N = 32$ and $\omega_p = \pi$; (b) variation of the error function with respect to ω.

$M(\omega) = \mathbf{a}^T\mathbf{c}(\omega)$, where

$$\mathbf{a} = \begin{cases} [a(0)\, a(1) \cdots a((N-1)/2)]^T & N \text{ odd} \\ [a(1)\, a(2) \cdots a(N/2)]^T & N \text{ even} \end{cases} \tag{4}$$

$$\mathbf{c}(\omega) = \begin{cases} \left[1 \cos\omega \cdots \cos\left(\frac{N-1}{2}\omega\right)\right]^T & N \text{ odd} \\ \left[\cos\frac{1}{2}\omega \cdots \cos\frac{3}{2}\omega \cdots \cos\left(\frac{N-1}{2}\omega\right)\right]^T & N \text{ even} \end{cases} \tag{5}$$

In minimizing E_{mse}, we set $\frac{\partial E_{\text{mse}}}{\partial a(i)} = 0$ to obtain a system of linear equations $\mathbf{Qa} = \mathbf{d}$, where $\mathbf{Q} = \int_0^{\omega_p} \mathbf{c}(\omega)\mathbf{c}^T(\omega)d\omega$ and $\mathbf{d} = \int_0^{\omega_p} D(\omega)\mathbf{c}(\omega)d\omega$. It can be noted that $\mathbf{Q}$ is a positive-definite (unless $\omega_p = 0$) real symmetric matrix, and thus, a unique solution is guaranteed. Consequently, the system of linear equations can be solved by a computationally efficient method like the Cholesky decomposition, which avoids matrix inversion. The entries of $\mathbf{Q}$ and $\mathbf{d}$ can be computed by evaluating the respective integrals in closed form. Similarly, for odd k, $M(\omega) = \mathbf{b}^T\mathbf{s}(\omega)$, where

$$\mathbf{b} = \begin{cases} [b(1)\, b(2) \cdots b((N-1)/2)]^T & N \text{ odd} \\ [b(1)\, b(2) \cdots b(N/2)]^T & N \text{ even} \end{cases} \tag{6}$$

$$\mathbf{s}(\omega) = \begin{cases} \left[\sin\omega \sin 2\omega \cdots \sin\left(\frac{N-1}{2}\omega\right)\right]^T & N \text{ odd} \\ \left[\sin\frac{1}{2}\omega \cdots \sin\frac{3}{2}\omega \cdots \sin\left(\frac{N-1}{2}\omega\right)\right]^T & N \text{ even} \end{cases} \tag{7}$$

The resulting system of equations is given by $\mathbf{Qb} = \mathbf{d}$, where $\mathbf{Q} = \int_0^{\omega_p} \mathbf{s}(\omega)\mathbf{s}^T(\omega)d\omega$ and $\mathbf{d} = \int_0^{\omega_p} d(\omega)\mathbf{s}(\omega)d\omega$. As in the case of even-order differentiators, $\mathbf{Q}$ is a positive-definite real symmetric matrix. The entries of $\mathbf{Q}$ and $\mathbf{d}$ can be calculated by evaluating the respective integrals in closed form.

TABLE I
COMPARISON OF THE THREE METHODS WITH RESPECT TO THE MEAN-SQUARE AND PEAK ERRORS

Examples	Mean-square error (E_{mse})			Peak error (E_{peak})		
	Our method	Eigenfilter method	Minimax method	Our method	Eigenfilter method	Minimax method
1	8.732e-07	8.799e-07	6.872e-06	8.101e-03	8.105e-03	3.724e-03
2	4.095e-04	4.096e-04	4.099e-04	1.504e-03	1.483e-03	4.626e-04
3	1.817e-03	1.818e-03	1.817e-03	1.022e-03	1.010e-03	2.967e-04
4	1.774e-04	1.775e-04	1.783e-04	1.975e-03	1.962e-03	8.973e-04

IV. DESIGN EXAMPLES

Design examples are provided to demonstrate the design of both odd- and even-order differentiators. The examples include filters with an even and odd number of taps and with $\omega_p = \pi$ and $\omega_p \neq \pi$. Figs. 1 to 4 show plots of the magnitude response and the corresponding variation of the error function $E(\omega)$ with frequency for the differentiators designed using our method and the minimax method. The error curves for the eigenfilter method practically coincide with those obtained using our method.

Example 1: For the first example, we design a second-order ($k = 2$) full-band ($\omega_p = \pi$) differentiator with 25 taps ($N = 25$). The magnitude response and the variation of the error function with frequency are shown in Fig. 1.

Example 2: Fig. 2 shows the plots for a 32-tap fourth-order differentiator with a passband edge frequency $\omega_p = 0.92\pi$.

Example 3: In Fig. 3, the magnitude response and the error function variation of a 27-tap third-order differentiator with a passband edge $\omega_p = 0.88\pi$ is illustrated.

Example 4: For the last example in Fig. 4, a fifth-order, full-band differentiator with 32 taps is designed.

V. PERFORMANCE RESULTS

In this section, we compare our design method with the minimax technique and the eigenfilter approach from three points of view, namely, the number of floating point operations (flops), the passband mean-square error E_{mse}, and the passband peak error $E_{\text{peak}} = \max_{0 \le \omega \le \omega_p} |D(\omega) - M(\omega)|$. A comparison of the three methods with respect to E_{mse} and E_{peak} is shown in Table I and that with respect to the number of flops is shown in Table II for the examples in the previous section. It must be mentioned that the entries in Table II have been normalized relative to the number of flops in our method. The reference frequencies for all the designs using the eigenfilter method have been chosen to be equal to half the corresponding passband edge frequencies as in [2].

A. Error Measure

Our method formulates a better error measure than the eigenfilter method in that we explicitly minimize the mean-square error between the ideal and the practical amplitude responses. In contrast, the eigenfilter method does not take the ideal response into account but rather the frequency response of the practical filter at an arbitrary frequency. In fact, the filter that is designed depends on the reference frequency. The performance comparisons show that our method leads to a lower mean-square error than the eigenfilter method, although the differences are small. The error criteria for our method and the minimax method are different in the sense that in the former, we find the filter coefficients that minimize E_{mse}, whereas the latter determines the filter coefficients that minimize E_{peak}. As a consequence, our method leads to a lower E_{mse}, whereas the minimax method guarantees a lower E_{peak}.

B. Computational Complexity

For our method, the filter parameters are obtained by solving a system of linear equations involving a positive-definite matrix $\mathbf{Q}$. A real symmetric positive-definite matrix can be decomposed as $\mathbf{Q} = \mathbf{L}\mathbf{L}^T$, where $\mathbf{L}$ is a real lower triangular matrix [8].

Consequently, the system of linear equations can be written as $\mathbf{L}\mathbf{L}^T\mathbf{a} = \mathbf{d}$. By letting $\mathbf{v} = \mathbf{L}^T\mathbf{a}$, we get $\mathbf{d} = \mathbf{L}\mathbf{v}$. Given $\mathbf{L}$ and $\mathbf{d}$, we can obtain $\mathbf{v}$ by recursively solving a set of linear equations. Then, we can again recursively find the vector $\mathbf{a}$ for a given $\mathbf{v}$ and $\mathbf{L}$. Let T_{chol} be the time taken for the decomposition of $\mathbf{Q}$. Then, the total time required to obtain the solution is $T_o = T_{chol} + T_{eqt}$, where $T_{eqt} = (T_a + T_m)N_E(N_E - 1) + 2N_E T_d$ is the time required to solve the system of equations given $\mathbf{L}$, where N_E is the dimension of the system. Here, T_a, T_m, and T_d are, respectively, the time required for one real addition, multiplication, and division.

In the eigenfilter approach, the error is formulated as $E_{\text{mse}} = \mathbf{a}^T\mathbf{P}\mathbf{a}$, where $\mathbf{P}$ is a real, symmetric, and positive-definite matrix. The coefficients of the differentiators are obtained as the eigenvector that corresponds to the smallest eigenvalue of $\mathbf{P}$. In order to compute the smallest eigenvalue and its corresponding eigenvector, generally, an iterative inverse power method is used. At the $(k+1)$st iteration, a vector $\mathbf{X}_{k+1}$ is computed from the previous iterate $\mathbf{x}_k$ as

$$\mathbf{Y}_{k+1} = \mathbf{P}^{-1}\mathbf{x}_k \tag{8}$$

$$\mathbf{x}_{k+1} = \mathbf{Y}_{k+1}/||\mathbf{Y}_{k+1}|| \tag{9}$$

where $||\mathbf{Y}_{k+1}||$ denotes the L_2 norm of $\mathbf{Y}_{k+1}$. If $||x_{k+1} - \mathbf{x}_k|| \le \epsilon$ (typically, ϵ is about 10^{-6}), then $\mathbf{x}_{k+1}$ is a good approximation of the eigenvector corresponding to the smallest eigenvalue. We can rewrite (8) as $\mathbf{x}_k = \mathbf{P}\mathbf{y}_{k+1}$. Using the technique described above for solving a system of linear equations, we can obtain $\mathbf{Y}_{k+1}$ and, subsequently, $\mathbf{x}_{k+1}$.

It can now be seen that the eigenfilter method requires solving a system of linear equations several times before obtaining the eigenvector corresponding to the smallest eigenvalue. On the contrary, our approach requires solving a system of linear equations only once. If M is the number of iterations required in the eigenfilter method, then the total time taken for obtaining the filter coefficients using the eigenfilter method is $T_e = T_{chol} + M(T_{eqt} + T_{nd})$, where T_{nd} is the time taken to obtain $\mathbf{x}_{k+1}$ from $\mathbf{Y}_{k+1}$ using (9). The value of M increases as the ratio λ_2/λ_1, where λ_1 is the smallest eigenvalue, and λ_2 is the next smallest eigenvalue, decreases. If the ratio is too small, it may not even be possible to evaluate the smallest eigenvalue and its corresponding eigenvector using the inverse power method (the inverse power method may not converge).

The other aspect that influences the computational complexity is in finding the entries of $\mathbf{Q}$ and $\mathbf{d}$ for our method and those of $\mathbf{P}$ for the eigenfilter approach. In general, the number of multiplications, additions, and trigonometric function evaluations is more for the eigenfilter method.

TABLE II
COMPARISON OF THE THREE METHODS WITH RESPECT TO THE NUMBER OF FLOATING POINT OPERATIONS

Examples	Floating point operations (flops) (Normalized relative to our method)		
	Our method	Eigenfilter method	Minimax method
1	1	6.84	9.22
2	1	4.24	5.36
3	1	4.23	6.66
4	1	7.58	12.33

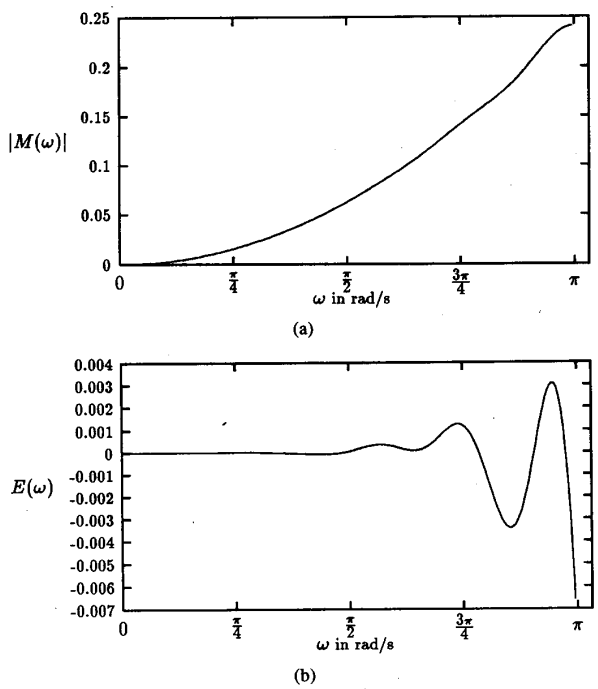

Fig. 5. (a) Amplitude response of a second-order differentiator with $N = 25$ and $\omega_p = \pi$ designed using the relative error miniinization; (b) variation of the error function with respect to ω.

From a computational point of view, our method is considerably more efficient than the minimax method. The minimax method uses the iterative Remez exchange algorithm, which takes up the major computational burden. It must be mentioned that in designing the differentiators using the minimax method, the procedure forwarded in [1] has been used. More efficient techniques to design nonrecursive filters using the Remez exchange algorithm have been advanced in [9]. However, at this stage, we have not extended the method advanced in [9] to the design of higher order differentiators.

VI. RELATIVE ERROR MEASURE

In some applications, it may be necessary to assign different weights at different frequencies in the passband [10]. In such cases, we consider the relative mean-square error given by

$$E_{\text{rel}} = \frac{1}{\pi}\int_0^{\omega_p} [(E(\omega)/W(\omega)]^2 d\omega \tag{10}$$

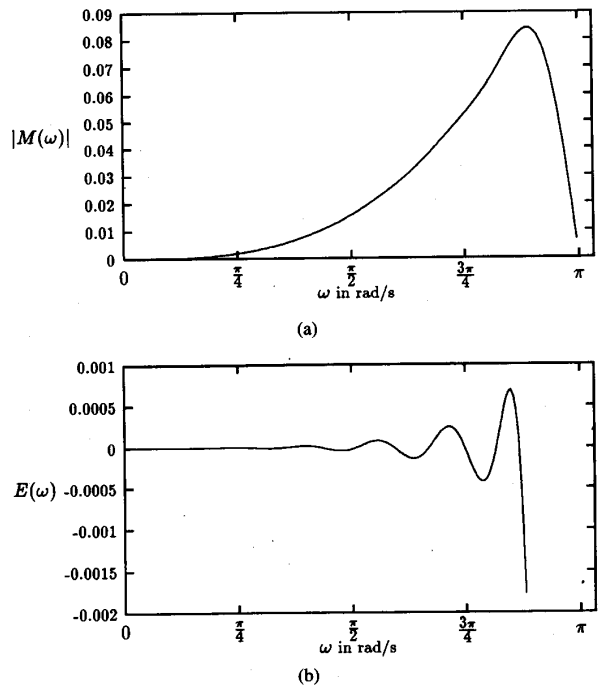

Fig. 6. (a) Amplitude response of a third-order differentiator with $N = 27$ and $\omega_p = 0.88\pi$ designed using the relative error minimization; (b) variation of the error function with respect to ω.

where $W(\omega) = (w/2\pi)^k$. It can be noted that E_{rel} is a general form of E_{mse} in that an extra weighting function $W(\omega)$ is included. It is, however, not possible to minimize the relative mean-square error for higher order differentiators ($k > 1$) since the integral does not exist due to a discontinuity in the integrand at $\omega = 0$. The above problem can be overcome by defining the weighting function as $W(\omega) = ((\omega + \epsilon)/2\pi)^k$, where ϵ is a small number (say 10^{-4}).

In minimizing E_{rel}, we set $\frac{\partial E_{\text{rel}}}{\partial a(i)} = 0$ to obtain a system of linear equations $\mathbf{Qa} = \mathbf{d}$ where $\mathbf{Q} = \int_0^{\omega_p} \frac{\mathbf{c}(\omega)\mathbf{c}^T(\omega)}{W^2(\omega)} d\omega$ and $\mathbf{d} = \int_0^{\omega_p} \frac{D(\omega)\mathbf{c}(\omega)}{W^2(\omega)} d\omega$. As can be observed from the above equations, the entries of $\mathbf{Q}$ and $\mathbf{d}$ are obtained by numerical integration. Consequently, the computational complexity for designing differentiators by minimizing E_{rel} is more than the case of minimizing E_{mse}. Similar expressions can be obtained for odd-order differentiators.

As design examples of differentiators designed by minimizing E_{rel}, we redesign the differentiators of Examples 1 and 3. Amplitude responses and variations of the error function with frequency are shown in Figs. 5 and 6. As can be observed in Figs. 5 and 6, variations of $E(\omega)$ in most of the passband is less for differentiators designed by minimizing E_{rel} as compared with those designed by minimizing E_{mse}. In fact, at low frequencies, the differentiators designed by minimizing E_{rel} is very close to the ideal response. It must be mentioned that for the differentiators of Examples 1 and 3, the number of flops required for the design using E_{rel} is 369 and 208 times that needed using E_{mse}, respectively. By extensive experimentation, it has been observed that if ϵ is decreased below 10^{-4}, $\mathbf{Q}$ becomes ill conditioned and is no longer positive definite.

A relative error, which is defined as $E_{\text{rel}}(\omega) = [D(\omega) - M(\omega)]/D(\omega)$, is used to evaluate the performance of differentiators designed in [3]. Here, coefficients of differentiators are obtained by

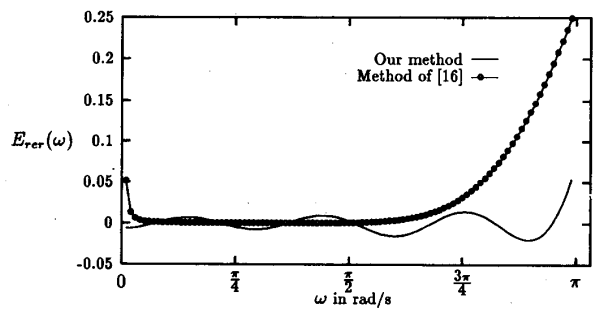

Fig. 7. Variation of the relative error of a 15-tap second-order differentiator with $\omega_p = \pi$.

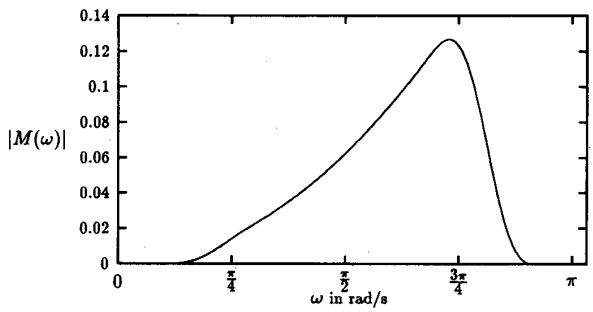

Fig. 8. Amplitude response of a second-order band-pass differentiator with $N = 31$ with passband $P = [0.3\pi, 0.7\pi]$ and stopband $S = [0, 0.1\pi] \cup [0.9\pi, \pi]$.

imposing magnitude and derivative constraints at zero frequency. Consequently, these differentiators are accurate only in the neighborhood of zero frequency. Moreover, the coefficients are not obtained by explicitly minimizing the relative error. In comparing this design method with that of ours, we have chosen the second-order differentiator example of [3], where $N = 15$ and $\omega_p = \pi$. In Fig. 7, the variation of the relative error with frequency is shown for the differentiator designed using our method and the method of [3]. As can be seen, the variation for our method is much less as compared with that for the method of [3].

VII. Extension to Other Types of Frequency Selective Filters

Our method can easily be extended to the design of filters that perform differentiation over any particular frequency range. In this case, the ideal frequency response is given by $H_I(e^{j\omega}) = D(\omega)e^{jk\pi/2}$, where $D(\omega) = (\omega/2\pi)^k$ in the passband (P) and zero in the stopband (S). The error function E_{mse} now consists of two terms: one reflecting the passband error and the other reflecting the stopband error. The error function is given by

$$E_{\text{mse}} = \frac{\alpha}{\pi}\int_P E_P^2(\omega)d\omega + \frac{\beta}{\pi}\int_S E_S^2(\omega)d\omega \tag{11}$$

where $E_P(\omega) = D(\omega) - M(\omega), E_S(\omega) = M(\omega)$, and α and β are the passband and stopband weights, respectively. Minimizing E_{mse}, with respect to the filter coefficients, again results in a system of linear equations that can be solved by the Cholesky decomposition. For the case of filters having a symmetrical impulse response, the system of equations is $[\alpha\mathbf{Q}+\beta\mathbf{R}]\mathbf{a} = \alpha\mathbf{d}$, where $\mathbf{Q} = \int_P \mathbf{c}(\omega)\mathbf{c}^T(\omega)d\omega, \mathbf{R} = \int_S \mathbf{c}(\omega)\mathbf{c}^T(\omega)d\omega$, and $\mathbf{d} = \int_P D(\omega)\mathbf{c}(\omega)d\omega$. A similar development exists for filters with an antisymmetrical impulse response.

Low-pass, high-pass, band-pass, and band-stop differentiators can be designed by assigning appropriate passbands and stopbands. Fig. 8 shows the amplitude response of a 31-tap, second-order bandpass differentiator designed by minimizing E_{mse} with weights $\alpha = \beta = 0.5$. The passband region is $0.3\pi \le \omega \le 0.7\pi$, whereas two stopband regions are $0 \le \omega \le 0.1\pi$ and $0.9\pi \le \omega \le \pi$.

VIII. Conclusions

In this paper, a method to design nonrecursive linear-phase higher order digital differentiators has been presented. In this method, the absolute mean-square error between the ideal and actual frequency responses is explicitly minimized. This leads to a closed-form solution for the filter coefficients in terms of a system of linear equations. The filter coefficients are found in a noniterative and computationally simple manner. The absolute mean-square error and the ccmputational complexity achieved by our method is lower than those achieved by the eigenfilter and minimax methods. Design of differentiators by minimizing the relative mean-square error has also been carried out. It has been shown that the error variation in most of the passband is less when minimizing the relative error measure compared with the absolute error measure. However, the tradeoff is in the greater computational complexity required for the relative error measure. Finally, it has been shown that various types of frequency selective (low-pass, high-pass, band-pass, and band-stop) differentiators satisfying a wide range of specifications can be designed.

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