Journal of the IETE
Vol 40, Nos 5&6, September-December 1994, pp 283-286

Least-Squares Design of Linear-Phase FIR Half-Band Filters

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Two approaches are described which can be used to design half-band filters for multirate applications. The approaches are based on the method of formulating the weighted mean-square error between the amplitude responses of the practical and ideal filters as a quadratic function. The filter coefficients are obtained by solving a set of linear equations. This method yields filters that are optimal in the least-squares sense. Design examples are provided.

Indexing terms : FIR half band filter, Least square design.

A linear phase half-band filter is an efficient tool to increase or decrease the sampling rate by a factor of 2 and is used in multirate digital filtering with nearly a two-fold reduction in the computational burden as compared to symmetric linear-phase finite impulse response (FIR) filters [1]. Half-band filters have been designed using the well known McClellan-Parks algorithm [2-3], the linear-programming techniques [4] and the eigenfilter method [5]. The first two methods yield filters that are optimal in the minimax sense. In the eigenfilter method, an error function between the desired and practical amplitude responses is formulated in a quadratic form. The desired amplitude response is equal to the amplitude response of the designed filter at an arbitrary reference frequency. The coefficients of the filter are obtained as the eigenvector corresponding to the smallest eigenvalue of a real, symmetric and positive-definite matrix.

The least-squares method was first proposed in [6] for the design of lowpass filters and involves the solution of a linear system of equations to obtain the filter coefficients. The method advanced in [6] has been used to design first-order differentiators in [7] and higher-order differentiators in [8]. Later, in [9], this method was generalized to accommodate various types of linear-phase nonrecursive filters including those with time domain constraints. The motivation of this method is to formulate an error function that directly minimizes the weighted mean-square error by explicitly including the ideal amplitude characteristic and obtain the filter coefficients with low computational complexity. It has been shown that the weighted mean-square error and computational complexity achieved by this least-squares method is lower than those resulting from the eigenfilter approach. In this paper, we present two approaches based on the method in [9] to design half-band filters that are optimal in the least-squares sense.

HALF-BAND FILTERS

Consider a nonrecursive digital filter with N taps represented by an impulse response $h(n)$ for $0 \le n \le N-1$. For the case of a linear-phase filter having a symmetric impulse response, we have $h(n) = h(N-1-n)$. Half-band filters belong to the class of linear-phase filters with a symmetric impulse response. In addition, N is required to be odd. The ideal amplitude response $D(\omega)$ of half-band filters is that of a lowpass filter as given by

$$D(\omega) = \begin{cases} 1 & \omega \in P \\ 0 & \omega \in S \end{cases} \tag{1}$$

where P is the passband and S is the stopband. Furthermore, the frequency response of a half-band filter is symmetric with respect to the half-band frequency $\pi/2$ in that

$$H(e^{j\omega}) + H(e^{j(\pi-\omega)}) = 1 \tag{2}$$

As a consequence of this symmetry,

$$\omega_p + \omega_a = \pi \tag{3}$$

and

$$\delta_p = \delta_a \tag{4}$$

where ω_p and ω_a are the passband edge and stopband edge frequencies, respectively, and δ_p and δ_a are the maximum passband and stopband ripples, respectively. In addition, the impulse response satisfies

$$h(n) = \begin{cases} 0 & n \text{ odd} \neq \frac{N-1}{2} \\ 0.5 & n = \frac{N-1}{2} \end{cases} \tag{5}$$

Note that we force $(N-1)/2$ to be odd since, if it is even, $h(0) = 0$ and $h(n)$ has a support starting at $n = 1$. The frequency response of a half-band filter can be written as

$$H(e^{j\omega}) = M(\omega)\, e^{-j\omega(N-1)/2} \tag{6}$$

where

$$M(\omega) = a(0) + \sum_{n=1,3...}^{(N-1)/2} a(n)\cos(n\omega)$$

$$= a(0) + M_1(\omega) \tag{7}$$

Paper No 195-A; received 1993 July 7;

where $$M_1(\omega) = \sum_{n=1,3....}^{(N-1)/2} a(n)\cos(n\omega) \qquad (8)$$

$a(0) = h[(N-1)/2] = 0.5$ and $a(n) = 2h[(N-1)/2 - n]$ for odd $n > 0$. Below, we shall present two approaches to design half-band filters.

Approach 1

The objective function E_{mse} that reflects the weighted mean-square difference between the ideal amplitude response, $D(\omega)$, and the amplitude response of the filter, $M(\omega)$, is expressed as [9]

$$E_{mse} = \frac{\alpha}{\pi}E_p + \frac{\beta}{\pi}E_s \qquad (9)$$

where

$$E_p = \int_P W_P(\omega)[D(\omega) - M(\omega)]^2\, d\omega \qquad (10)$$

and

$$E_s = \int_S W_S(\omega)[D(\omega) - M(\omega)]^2\, d\omega \qquad (11)$$

The quantities α and β reflect the relative emphasis given to the passband and stopband, respectively. On the other hand, $W_P(\omega)$ and $W_S(\omega)$ are nonnegative frequency domain weighting functions for the passband and stopband that can be used to emphasize certain frequencies over others.

By virtue of (4), $\alpha = \beta$ and $W_P(\omega) = W_S(\pi - \omega)$ for $0 \le \omega \le \omega_p$. Also, due to the frequency response symmetry as given by (2), it is enough to minimize either E_p or E_s. Given the decomposition of $M(\omega)$ as the sum of $a(0) = 0.5$ and $M_1(\omega)$, we determine the coefficients $a(i)$ for $i = 1,3, \ldots, (N-1)/2$ such that $M_1(\omega)$ approximates a modified ideal response $D_1(\omega)$ given by

$$D_1(\omega) = \begin{cases} 0.5 & \omega \in P \\ -0.5 & \omega \in S \end{cases} \qquad (12)$$

For the case of half-band filters we have $M_1(\omega) = \mathbf{a}^T\mathbf{c}(\omega)$, where

$$\mathbf{a} = \left[a(1)\; a(3) \ldots .\, a\left(\frac{N-1}{2}\right)\right]^T \qquad (13)$$

and

$$\mathbf{c}(\omega) = \left[\cos\omega \;\; \cos3\omega \;\ldots .\cos\left(\frac{N-1}{2}\,\omega\right)\right]^T \qquad (14)$$

In terms of $D_1(\omega)$ and $M_1(\omega)$, we have

$$E_p = \int_P W_P(\omega)[D_1(\omega) - M_1(\omega)]^2\, d\omega \qquad (15)$$

$$E_s = \int_S W_S(\omega)[D_1(\omega) - M_1(\omega)]^2\, d\omega \qquad (16)$$

Filter coefficients that are optimal in a weighted least-squares sense are obtained by minimizing either E_p or E_s.

Choosing the minimization of E_s, we set $\frac{\partial E_s}{\partial a(i)} = 0$, for $i = 1, 3, \ldots, (N-1)/2$, to obtain a system of linear equations $\mathbf{Ra} = \mathbf{d}$, where

$$\mathbf{R} = \int_s W_S(\omega)\, \mathbf{c}(\omega)\, \mathbf{c}^T(\omega)\, d\omega \qquad (17)$$

and

$$\mathbf{d} = \int_s W_S(\omega)\, D_1(\omega)\, \mathbf{c}(\omega)\, d\omega \qquad (18)$$

It can be noted that $\mathbf{R}$ is a real, symmetric and positive-definite matrix and thus, a unique solution is guaranteed. In addition, the system of linear equations can be solved by a computationally efficient method, like the Cholesky decomposition, that avoids matrix inversion. The filter coefficients $h(n)$ are easily found from $\mathbf{a}$.

***Example* 1**

A half-band linear-phase filter with $N = 31$, $\omega_s = 0.55\pi$ and $W(\omega) = 1$ is designed. Figure 1 shows the magnitude response of the designed filter.

Approach 2

In this method, $H(z)$ is designed by a two-stage process [3]. First, a one-band prototype linear-phase filter $G(z)$ with an even number of taps equal to $J = (N+1)/2$ is designed. Its passband is $P = [0, 2\omega_p]$ and its transition band extends from $2\omega_p$ to π (there is no stopband). The ideal response is $D(\omega) = 1$ in P. Since J is even, $G(z)$ has a zero at $\omega = \pi$. Given $G(z)$, $H(z)$ is defined as

$$H(z) = \frac{G(z^2) + z^{-\left(\frac{N-1}{2}\right)}}{2} \qquad (19)$$

Then,

$$h(n) = \begin{cases} 0.5g\left(\frac{n}{2}\right) & n \text{ even} \\ 0 & n \text{ odd} \ne \frac{N-1}{2} \\ 0.5 & n = \frac{N-1}{2} \end{cases} \qquad (20)$$

where $g(n)$ is the impulse response of $G(z)$.

The frequency response of $G(z)$ is

$$G(e^{j\omega}) = K(\omega)\, e^{-j\omega(J-1)/2} \qquad (21)$$

where

$$K(\omega) = \sum_{n=1}^{J/2} b(n)\cos(n - 1/2)\,\omega \qquad (22)$$

and $b(n) = 2g(J/2 - n)$ for $1 \le n \le J/2$. The mean-square error

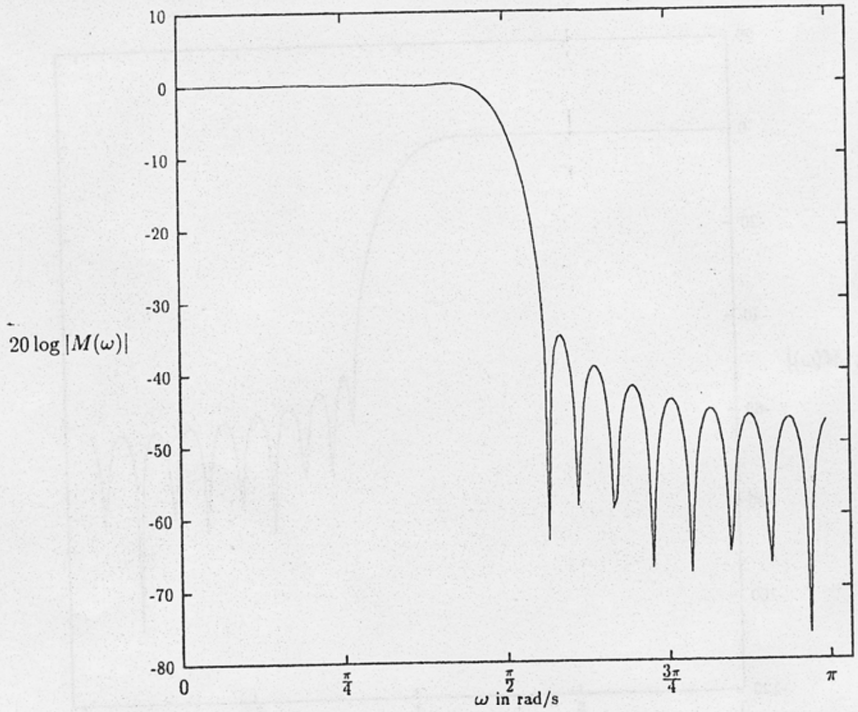

Fig 1 Magnitude response of a 31-tap linear-phase FIR half-band filter (example 1)

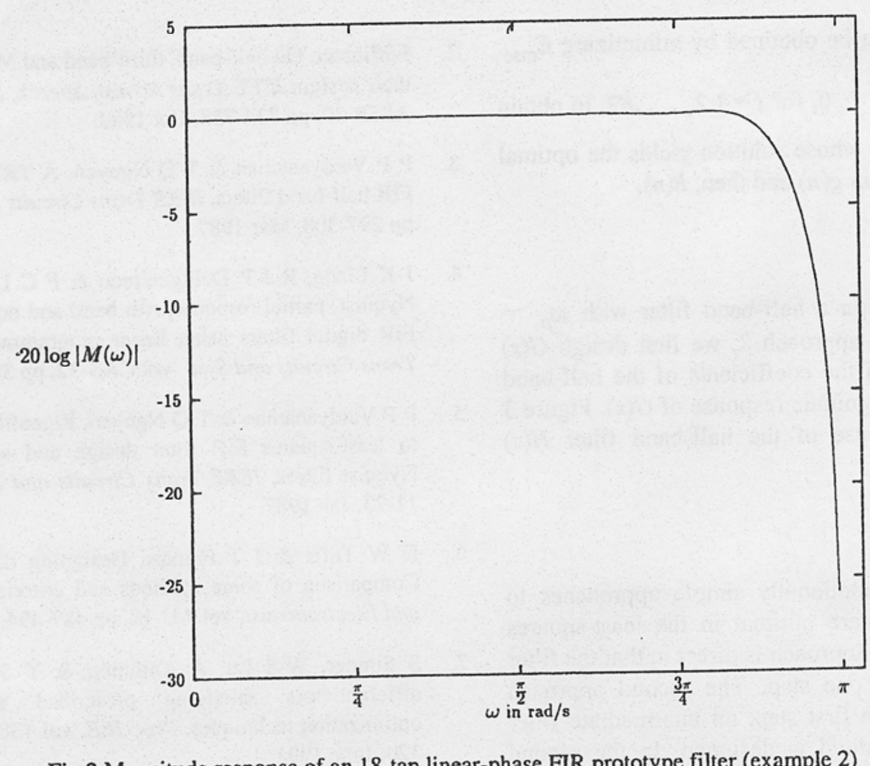

Fig 2 Magnitude response of an 18-tap linear-phase FIR prototype filter (example 2)

$$E_{\text{mse}} = \frac{1}{\pi} \int_P [D(\omega) - K(\omega)]^2 \, d\omega \tag{23}$$

The amplitude response, $K(\omega)$, can be expressed as $K(\omega) = \mathbf{b}^T \mathbf{c}(\omega)$, where

$$\mathbf{b} = [b(1)\ \ b(2)\ \ \ldots\ b(J/2)]^T \tag{24}$$

and

$$\mathbf{c}(\omega) = \left[\cos\frac{1}{2}\omega\ \cos\frac{3}{2}\omega\ \ \ldots \cos\left(\frac{N-1}{2}\omega\right)\right]^T \tag{25}$$

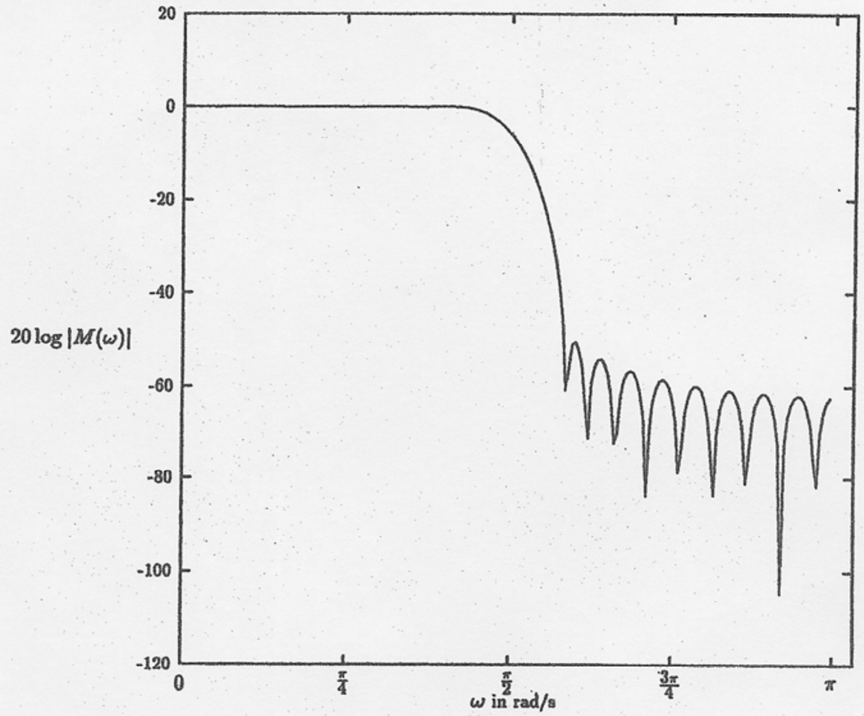

Fig 3 Magnitude response of a 35-tap linear-phase FIR half-band filter obtained from the prototype (example 3)

The filter coefficients can be obtained by minimizing E_{mse}. Consequently, we set $\frac{\partial E_{mse}}{\partial b(i)} = 0$, for $i = 1,2, \ldots, J/2$, to obtain a system of linear equations whose solution yields the optimal vector $\boldsymbol{b}$. From $b(n)$, we obtain $g(n)$ and then, $h(n)$.

Example 2

In this example, we design a half-band filter with $\omega_p = 0.4225\pi$ and $N = 35$. Using approach 2, we first design $G(z)$ with $J = 18$ and then obtain the coefficients of the half-band filter. Figure 2 shows the magnitude response of $G(z)$. Figure 3 shows the magnitude response of the half-band filter $H(z)$ obtained from $G(z)$.

CONCLUSIONS

In this paper, two computationally simple approaches to design half-band filters that are optimal in the least-squares sense are discussed. The first approach is direct in that the filter coefficients are obtained in one step. The second approach consists of two stages. In the first step, an intermediate filter based on a least-squares method is designed. In the second step, the filter obtained in the first step is transformed to yield the half-band filter.

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