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Design of two-dimensional linear-phase nonrecursive filters using a least-squares criterion

Sunder S. Kidambi[a,*], Ravi P. Ramachandran[b]

[a] *Analog Devices, 181 Ballardvale Street, Wilmington, MA 01887, USA*
[b] *Caip Center, Department of Electrical Engineering, Rutgers University, Piscataway, NJ 08855-1390, USA*

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Abstract

A method is described which can be used to design two-dimensional nonrecursive linear-phase filters. The approach is based on formulating the mean-square error between the amplitude responses of the practical and ideal digital filters as a quadratic function. The coefficients of the filters are obtained by solving a set of linear equations. This method leads to a lower mean-square error and is computationally more efficient than the eigenfilter method.

Zusammenfassung

Es wird eine Methode beschrieben, die zum Entwurf von zweidimensionalen nichtrekursiven linearphasigen Filtern benutzt werden kann. Die Lösung basiert auf der Formulierung des mittleren quadratischem Fehlers zwischen den Betragsfrequenzgängen der praktischen und der idealen digitalen Filter als quadratische Funktion. Man erhält die Koeffizienten der Filter durch die Lösung eines linearen Gleichungssystems. Diese Methode führt auf einen geringeren mittleren quadratischem Fehler und ist recheneffizienter als die Eigenfilter-Methode.

Résumé

Une methode est décrite, qui peut être utilisée pour concevoir des filtres à phase linéaire nonrécursifs bi-dimensionnels. Cette approche est basée sur la formulation comme fonction quadratique de l'erreur quadratique moyenne entre la réponse en amplitude des filtres numériques idéal et pratique. Les coefficients du filtre sont obtenus en resolvant un système d'équations linéaires. Cette méthode mene à une erreur quadratique moyenne plus faible, et est plus efficace en terme de calculs que la methode du filtre aux valeurs propres.

Keywords: Two-dimensional; Digital filters; Nonrecursive; Least-squares

* Corresponding author.

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1. Introduction

The use of two-dimensional (2-D) nonrecursive filters is motivated by their inherent stability and their rendering of linear phase by the imposition of coefficient symmetry. This coefficient symmetry that achieves linear phase is important for image processing applications and for simplifying design and implementational complexity [2]. Additional coefficient symmetry can be imposed to further alleviate the design and computational effort [3]. Design approaches that are extensions of the approaches used for one-dimensional (1-D) filters include the use of windows and the frequency sampling technique [3]. The frequency transformation method starts with a 1-D linear phase filter designed by a 1-D technique and transforms it into a 2-D linear-phase filter [3]. The transformation function is the Fourier transform of a 2-D zero-phase sequence. A well known example was introduced by McClellan [4]. Although designs based on the Chebyshev approximation problem exist [3], the methodology is not a simple extension of the Remez exchange alogrithm. The eigenfilter approach proposed in [11] for the design of 1-D filters has recently been extended to the design of 2-D filters in [5]. In this method, an error function based on the difference between a desired response and the amplitude response of the practical filter is formulated. The desired response is equal to a scaled version of the ideal response where the scaling factor depends on the amplitude response of the designed filter at an arbitrary 2-D frequency. This is done to set up the error function in a quadratic form in order that the filter coefficients are found by computing the eigenvector corresponding to the smallest eigenvalue of a real, symmetric and positive-definite matrix.

In this paper, the least-squares approach for the design of 1-D nonrecursive linear-phase filters described in [6–8] is extended to the design of 2-D nonrecursive linear-phase filters. The procedure involves formulating the error between the practical and ideal responses as a quadratic function. The explicit inclusion of the ideal amplitude response in the error function leads to a more meaningful formulation than the eigenfilter method and does not necessitate the use of a reference frequency. The coefficients of the filter are obtained by solving a system of linear equations. By way of some design examples, our method is compared with the eigenfilter approach in terms of several performance measures and it is shown that our method is superior to the eigenfilter approach.

2. Two-dimensional quadrantally-symmetric nonrecursive filters

A 2-D nonrecursive filter with N_1 by N_2 taps can be represented by the transfer function

$$H(z_1,z_2) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} h(n_1,n_2) z_1^{-n_1} z_2^{-n_2}, \tag{1}$$

where $h(n_1,n_2)$ is the impulse response of the filter. By incorporating the linear-phase symmetry constraints, the frequency response of the 2-D filter is given by

$$H(e^{j\omega_1},e^{j\omega_2}) = M(\omega_1,\omega_2) e^{-j(\frac{N_1-1}{2})\omega_1} e^{-j(\frac{N_2-1}{2})\omega_2}, \tag{2}$$

where $M(\omega_1,\omega_2)$ is the amplitude response. For the case when the filter has quadrantal symmetry, the following condition holds:

$$h(n_1,n_2) = h(N_1-1-n_1,n_2) = h(n_1,N_2-1-n_2) \tag{3}$$

for $0 \leqslant n_1 \leqslant N_1 - 1$ and $0 \leqslant n_2 \leqslant N_2 - 1$. When N_1 and N_2 are both odd, from Eq. (3) we have

$$\begin{aligned} h\left(\frac{N_1-1}{2} - k_1, \frac{N_2-1}{2} - k_2\right) &= h\left(\frac{N_1-1}{2} - k_1, \frac{N_2-1}{2} + k_2\right) \\ &= h\left(\frac{N_1-1}{2} + k_1, \frac{N_2-1}{2} - k_2\right) \\ &= h\left(\frac{N_1-1}{2} + k_1, \frac{N_2-1}{2} + k_2\right) \end{aligned} \tag{4}$$

for $1 \leqslant k_1 \leqslant (N_1 - 1)/2$, $1 \leqslant k_2 \leqslant (N_2 - 1)/2$. Consequently, the amplitude response is given by

$$M(\omega_1, \omega_2) = \sum_{n_1=0}^{\frac{N_1-1}{2}} \sum_{n_2=0}^{\frac{N_2-1}{2}} a(n_1, n_2) \cos(n_1 \omega_1) \cos(n_2 \omega_2). \tag{5}$$

The coefficients $a(n_1, n_2)$ are related to the filter coefficients $h(n_1, n_2)$ by

$$a(0,0) = h\left(\frac{N_1-1}{2}, \frac{N_2-1}{2}\right),$$

$$a(0, n_2) = 2h\left(\frac{N_1-1}{2}, \frac{N_2-1}{2} - n_2\right) \quad \text{for } n_2 = 1, 2, \ldots, \frac{N_2-1}{2},$$

$$a(n_1, 0) = 2h\left(\frac{N_1-1}{2} - n_1, \frac{N_2-1}{2}\right) \quad \text{for } n_1 = 1, 2, \ldots, \frac{N_1-1}{2},$$

$$a(n_1, n_2) = 4h\left(\frac{N_1-1}{2} - n_1, \frac{N_2-1}{2} - n_2\right) \quad \text{for } n_1 = 1, 2, \ldots, \frac{N_1-1}{2}, \; n_2 = 1, 2, \ldots, \frac{N_2-1}{2}.$$

Other cases of symmetry for different N_1 and N_2 can be obtained in a straightforward manner [1]. The ideal linear-phase frequency response can be written as

$$H_I(e^{j\omega_1}, e^{j\omega_2}) = D(\omega_1, \omega_2) e^{-jQ_1\omega_1} e^{-jQ_2\omega_2}. \tag{6}$$

By comparing (2) and (6), we note that a 2-D nonrecursive filter can be designed whose amplitude response approximates any arbitrary desired characteristic $D(\omega_1, \omega_2)$.

3. Error function minimization

The mean-square error between $D(\omega_1, \omega_2)$ and $M(\omega_1, \omega_2)$ can be expressed as

$$E_{\text{mse}} = \alpha \iint_P [D(\omega_1, \omega_2) - M(\omega_1, \omega_2)]^2 \, d\omega_1 \, d\omega_2 + \beta \iint_S M^2(\omega_1, \omega_2) \, d\omega_1 \, d\omega_2, \tag{7}$$

where P is the passband and S is the stopband in the (ω_1, ω_2) plane. As can be noted from Eq. (7), $D(\omega_1, \omega_2)$ is zero in the stopband. The quantities α and β reflect the relative emphasis given to the passband and stopband, respectively. By minimizing the error function with respect to the filter coefficients, the required filter can be designed.

For the case when N_1 and N_2 are odd, let the amplitude response be given by

$$M(\omega_1, \omega_2) = \boldsymbol{a}^{\mathrm{T}} \boldsymbol{c}(\omega_1, \omega_2), \tag{8}$$

where

$$\boldsymbol{a} = \begin{bmatrix} a(0,0) \\ a(0,1) \\ \vdots \\ a\left(0, \frac{N_2-1}{2}\right) \\ a(1,0) \\ a(1,1) \\ \vdots \\ a\left(1, \frac{N_2-1}{2}\right) \\ \vdots \\ a\left(\frac{N_1-1}{2}, 0\right) \\ a\left(\frac{N_1-1}{2}, 1\right) \\ \vdots \\ a\left(\frac{N_1-1}{2}, \frac{N_2-1}{2}\right) \end{bmatrix}, \quad \boldsymbol{c}(\omega_1,\omega_2) = \begin{bmatrix} 1 \\ \cos(\omega_2) \\ \vdots \\ \cos\left[\left(\frac{N_2-1}{2}\right)\omega_2\right] \\ \cos(\omega_1) \\ \cos(\omega_1)\cos(\omega_2) \\ \vdots \\ \cos(\omega_1)\cos\left[\left(\frac{N_2-1}{2}\right)\omega_2\right] \\ \vdots \\ \cos\left[\left(\frac{N_1-1}{2}\right)\omega_1\right] \\ \cos\left[\left(\frac{N_1-1}{2}\right)\omega_1\right]\cos(\omega_2) \\ \vdots \\ \cos\left[\left(\frac{N_1-1}{2}\right)\omega_1\right]\cos\left[\left(\frac{N_2-1}{2}\right)\omega_2\right] \end{bmatrix}. \tag{9}$$

The mean-square error given by (7) can be rewritten as

$$\begin{aligned} E_{\mathrm{mse}} = \alpha \iint_P & [D^2(\omega_1,\omega_2) - 2D(\omega_1,\omega_2)\boldsymbol{a}^{\mathrm{T}}\boldsymbol{c}(\omega_1,\omega_2) + \boldsymbol{a}^{\mathrm{T}}\boldsymbol{c}(\omega_1,\omega_2)\boldsymbol{c}^{\mathrm{T}}(\omega_1,\omega_2)\boldsymbol{a}]\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2 \\ & + \beta \iint_S \boldsymbol{a}^{\mathrm{T}}\boldsymbol{c}(\omega_1,\omega_2)\boldsymbol{c}^{\mathrm{T}}(\omega_1,\omega_2)\boldsymbol{a}\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2. \end{aligned} \tag{10}$$

In minimizing E_{mse}, we set $\partial E_{\mathrm{mse}}/\partial a(i,j) = 0$, for $0 \leqslant i \leqslant (N_1-1)/2$, $0 \leqslant j \leqslant (N_2-1)/2$, to obtain a system of linear equations $(\alpha\boldsymbol{Q} + \beta\boldsymbol{R})\boldsymbol{a} = \alpha\boldsymbol{d}$ where

$$\boldsymbol{Q} = \iint_P \boldsymbol{c}(\omega_1,\omega_2)\boldsymbol{c}^{\mathrm{T}}(\omega_1,\omega_2)\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2, \tag{11}$$

$$\boldsymbol{R} = \iint_S \boldsymbol{c}(\omega_1,\omega_2)\boldsymbol{c}^{\mathrm{T}}(\omega_1,\omega_2)\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2, \tag{12}$$

$$\boldsymbol{d} = \iint_P D(\omega_1,\omega_2)\boldsymbol{c}(\omega_1,\omega_2)\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2. \tag{13}$$

It can be noted from the above equations that $\boldsymbol{Q}$ and $\boldsymbol{R}$ are positive-definite, real and symmetric matrices. Consequently, the system of linear equations can be solved by a computationally efficient method that avoids matrix inversion [10].

4. Design examples

In this section, we provide two design examples in which the entries of $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{d}$ are obtained either in closed form or by numerical integration. In the first example, these entries are obtained in closed form while in the second, the entries are obtained by numerical integration. For the sake of comparison, the examples chosen here are the same as those presented in [5]. For all the designs, $\alpha = \beta = 1$ and $N = N_1 = N_2$.

Example 1 (*Fan filter*). For this filter the desired amplitude response is given by

$$D(\omega_1, \omega_2) = \begin{cases} 1 & P\colon\ 0 \leqslant \omega_1 \leqslant \pi,\ \omega_1 \leqslant \omega_2 \leqslant \pi, \\ 0 & S\colon\ \omega_a \leqslant \omega_1 \leqslant \pi,\ 0 \leqslant \omega_2 \leqslant \pi - \omega_a. \end{cases}$$

Here, N has been chosen to be equal to 23 and $\omega_a = 0.16\pi$. The magnitude response of the designed filter is shown in Fig. 1.

Example 2 (*Circular lowpass filter*). In this example, the desired amplitude response is given by

$$D(\omega_1, \omega_2) = \begin{cases} 1 & P\colon\ 0 \leqslant \sqrt{\omega_1^2 + \omega_2^2} \leqslant \omega_p, \\ 0 & S\colon\ \omega_a \leqslant \sqrt{\omega_1^2 + \omega_2^2} \leqslant \pi. \end{cases}$$

Here, $\omega_p = 0.5\pi$, $\omega_a = 0.7\pi$ and $N = 25$. Fig. 2 shows the magnitude response of the designed filter.

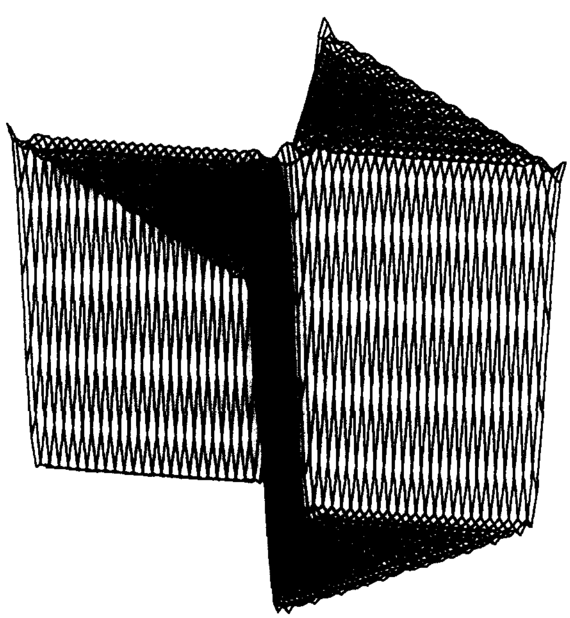

Fig. 1. Magnitude response of a 23×23 fan filter with $\omega_a = 0.16\pi$.

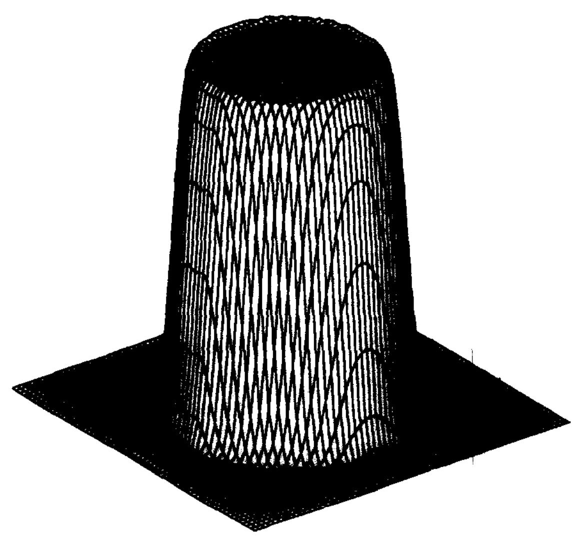

Fig. 2. Magnitude response of a 25×25 circular lowpass filter with $\omega_p = 0.5\pi$ and $\omega_a = 0.7\pi$.

5. Performance results

In this section, we compare our design method with the eigenfilter approach from three points of view, namely, the saving in the number of floating point operations (flops), the mean-square error E_{mse}, and the peak error, both in the passband and the stopband. The passband peak error E_{p} and the stopband peak error E_{s} are defined, respectively, as

$$E_{\mathrm{p}} = \max_{(\omega_1,\omega_2)\in P} |D(\omega_1,\omega_2) - M(\omega_1,\omega_2)|, \tag{14}$$

$$E_{\mathrm{s}} = \max_{(\omega_1,\omega_2)\in S} |M(\omega_1,\omega_2)|. \tag{15}$$

For the sake of comparison, the examples in [5] were designed using our method. A comparison of the two methods with respect to the number of flops is shown in Table 1 and that with respect to E_{mse} is shown in Table 2. It must be mentioned that the entries for the eigenfilter method in Table 1 have been normalized relative to the number of flops in our method. Table 3 shows a comparison of the peak error, both in the passband and stopband for the two methods. As can be seen from the tables, our method is computationally more efficient and yields a lower mean-square error as compared to the eigenfilter method. A detailed analysis of the computational complexity and error measure of the two methods can be found in [9].

Table 1
Comparison of the two methods with respect to the number of floating point operations

Examples	Floating point operations (flops) (normalized relative to our method)		Saving in flops (%)
	Our method	Eigenfilter method	
Rectangular filter	1	4.3845	77.19
Fullband filter	1	3.0071	66.75
Fan filter	1	2.0027	50.07
Conic filter	1	1.4435	30.72
Circular filter	1	1.2435	19.58
Minimum energy filter	1	1.0059	0.59

Table 2
Comparison of the two methods with respect to the mean-square error

Examples	Mean-square error (E_{mse})	
	Our method	Eigenfilter method
Rectangular filter	1.600×10^{-5}	2.121×10^{-5}
Fullband filter	3.461×10^{-4}	5.132×10^{-4}
Fan filter	7.000×10^{-3}	7.000×10^{-3}
Conic filter	5.819×10^{-4}	1.113×10^{-3}
Circular filter	1.745×10^{-3}	1.745×10^{-3}
Minimum energy filter	4.788×10^{-6}	4.799×10^{-6}

Table 3
Comparison of the two methods with respect to the peak error

Examples	Peak error in the passband (E_{p})		Peak error in the stopband (E_{s})	
	Our method	Eigenfilter method	Our method	Eigenfilter method
Rectangular filter	2.399×10^{-2}	2.154×10^{-2}	8.084×10^{-3}	8.144×10^{-3}
Fullband filter	4.179×10^{-2}	4.455×10^{-2}	—	—
Fan filter	1.013×10^{-1}	1.005×10^{-1}	1.339×10^{-1}	1.348×10^{-1}
Conic filter	3.823×10^{-2}	2.457×10^{-2}	1.355×10^{-2}	1.326×10^{-2}
Circular filter	2.216×10^{-2}	2.191×10^{-2}	1.794×10^{-2}	1.808×10^{-2}
Minimum energy filter	—	—	8.716×10^{-3}	8.735×10^{-3}

6. Conclusions

In this paper, a method to design 2-D nonrecursive linear-phase filters has been presented. In this method, we explicitly minimize the absolute mean-square error between the ideal and actual frequency responses. This leads to a closed form solution for the filter coefficients in terms of a system of linear equations. The filter coefficients are found in a noniterative and computationally simple manner. It has been shown that the filters designed using our method have a lower mean-square error as compared to those designed using the eigenfilter method. Moreover, the computational complexity in our method is significantly lower than that in the eigenfilter method.

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