Complex Coefficient Nonrecursive Digital Filter Design Using a Least-Squares Method

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Abstract—**In this correspondence, a method for the design of complex-coefficient nonrecursive filters is described. In this method, a mean-square error is formulated in a quadratic form. The filter coefficients are obtained by solving a system of linear equations using computationally robust and efficient algorithms. Examples are presented to demonstrate the efficacy of our method.**

I. Introduction

Least-squares design of nonrecursive digital filters has been dealt with quite extensively in the last two decades [1]–[5]. In all these methods, an error function is formulated in a quadratic form. In one method, the filter coefficients are contained in the eigenvector corresponding to the smallest eigenvalue of an associated real, symmetric, and positive-definite matrix [1]. This is popularly known as the eigenfilter method. In another method, the filter coefficients are obtained by solving a system of linear equations using either matrix inversion or other efficient algorithms [2]–[5].

It is only in the recent past that the design of complex coefficient nonrecursive digital filters has gained significant importance [6], [7]. In both [6] and [7], the eigenfilter method has been used to design complex coefficient filters. In [6], the complex-coefficient design problem is converted to an equivalent real-coefficient design problem. In [7], however, such a conversion is circumvented, and consequently, the dimension of the associated matrix is half that of the matrix involved in the design presented in [6].

In this correspondence, we extend the least-squares design presented in [4] and [5] to the design of 1-D and 2-D complex-coefficient filters. The approach is based on formulating the mean-square error between a desired complex-valued response and the complex-coefficient filter. The filter coefficients are obtained by solving a system of linear equations. By way of analysis and examples, it is shown that our method is not only significantly more computationally efficient but also yields lower peak and group delay errors than the eigenfilter method.

II. Problem Formulation for the Design of 1-D Complex-Coefficient Filters

The frequency response of a nonrecursive filter with N taps specified by a complex-valued impulse response $h(n)$ is given by

$$H(e^{\jmath\omega}) = \sum_{n=0}^{N-1} h(n)e^{-\jmath n\omega} = \mathbf{h}^T\mathbf{e}(\omega) \tag{1}$$

where

$$\mathbf{h} = [h(0)h(1)h(2)\cdots h(N-1)]^T$$

and

$$\mathbf{e}(\omega) = \left[1 e^{-\jmath\omega} e^{-\jmath 2\omega}\cdots e^{-\jmath(N-1)\omega}\right]^T$$

Manuscript received August 19, 1994; revised September 26, 1995. The associate editor coordinating the review of this paper and approving it for publication was Dr. Kamal Premaratne.

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Publisher Item Identifier S 1053-587X(96)02405-X.

The phase response of the filter is given by

$$\phi(\omega) = -\tan^{-1}\left(\frac{\mathbf{a}^T\mathbf{s}(\omega) - \mathbf{b}^T\mathbf{c}(\omega)}{\mathbf{a}^T\mathbf{c}(\omega) + \mathbf{b}^T\mathbf{s}(\omega)}\right) \tag{2}$$

where

$$\mathbf{a} = \mathrm{Re}(\mathbf{h}),$$
$$\mathbf{b} = \mathrm{Im}(\mathbf{h}),$$
$$\mathbf{c}(\omega) = [1\cos(\omega)\cos(2\omega)\cdots\cos((N-1)\omega)]^T$$

and

$$\mathbf{s}(\omega) = [0\sin(\omega)\sin(2\omega)\cdots\sin((N-1)\omega)]^T.$$

The group delay is given by

$$\tau(\omega) = -\frac{d}{d\omega}\phi(\omega) = \frac{k(\omega)}{l(\omega)}\cos^2(\phi(\omega)) \tag{3}$$

where

$$\begin{aligned} k(\omega) &= (\mathbf{a}^T\mathbf{c}(\omega) + \mathbf{b}^T\mathbf{s}(\omega))(\mathbf{a}^T\hat{\mathbf{c}}(\omega) + \mathbf{b}^T\hat{\mathbf{s}}(\omega)) \\ &\quad + (\mathbf{a}^T\mathbf{s}(\omega) - \mathbf{b}^T\mathbf{c}(\omega))(\mathbf{a}^T\hat{\mathbf{s}}(\omega) - \mathbf{b}^T\hat{\mathbf{c}}(\omega)) \\ l(\omega) &= (\mathbf{a}^T\mathbf{c}(\omega) + \mathbf{b}^T\mathbf{s}(\omega))^2 \end{aligned}$$

and

$$\hat{\mathbf{c}}(\omega) = [0\cos(\omega)\,2\cos(2\omega)\cdots(N-1)\cos((N-1)\omega)]^T$$
$$\hat{\mathbf{s}}(\omega) = [0\sin(\omega)\,2\sin(2\omega)\cdots(N-1)\sin((N-1)\omega)]^T.$$

If $D(\omega)$ is the desired complex-valued response, then the weighted mean-square error between $D(\omega)$ and $H(e^{\jmath\omega})$ can be expressed as

$$\begin{aligned} E_m &= \int_P W_P(\omega)|D(\omega) - H(e^{\jmath\omega})|^2 d\omega \\ &\quad + \int_S W_S(\omega)|H(e^{\jmath\omega})|^2 d\omega \\ &= \int_P W_P(\omega)|D(\omega) - \mathbf{h}^T\mathbf{e}(\omega)|^2 d\omega \\ &\quad + \int_S W_S(\omega)|\mathbf{h}^T\mathbf{e}(\omega)|^2 d\omega \end{aligned} \tag{4}$$

where

P	passband,
S	stopband,
$W_P(\omega)$	frequency dependent weighting function in the passband,
$W_S(\omega)$	frequency dependent weighting function in the stopband.

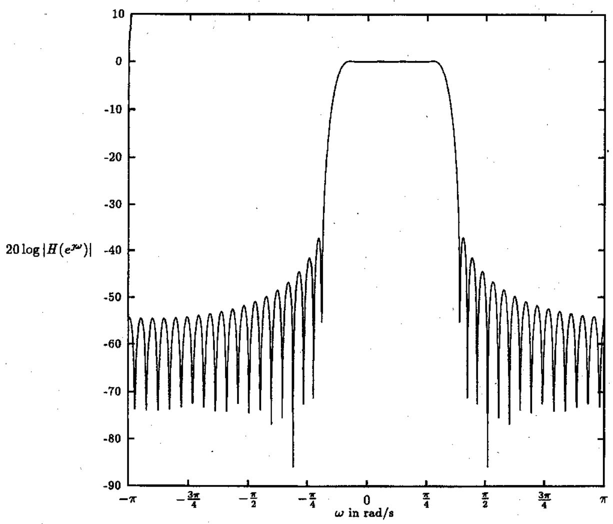
By minimizing the error function E_m with respect to the filter coefficients, the required filter can be designed. In minimizing E_m, we set $\frac{\partial E_m}{\partial h(n)} = 0$ for $n = 0, 1, \ldots, N-1$ to obtain a system of linear equations $\mathbf{Qh} = \mathbf{d}$, where

$$Q(n,m) = \int_P W_P(\omega)e^{\jmath(n-m)\omega}d\omega + \int_S W_S(\omega)e^{\jmath(n-m)\omega}d\omega \tag{5}$$

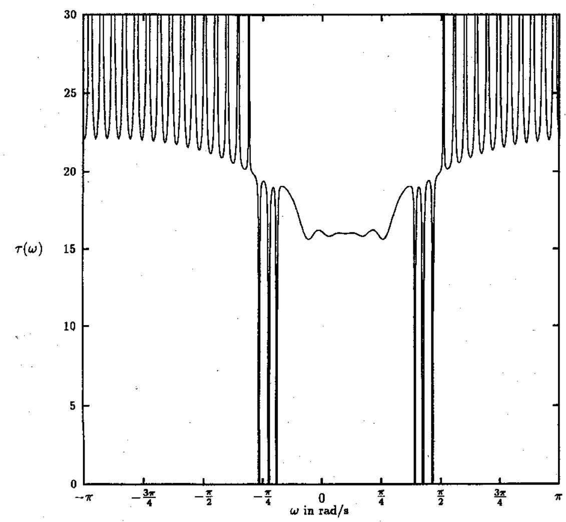
for $0 \le n, m \le N-1$ and

$$d(n) = \int_P W_P(\omega)D(\omega)e^{\jmath n\omega}d\omega. \tag{6}$$

It can be seen from (5) that $\mathbf{Q}$ is an $N \times N$ Hermitian Toeplitz matrix. Consequently, only the first row (or column) of $\mathbf{Q}$ needs to be calculated, and the system of linear equations can be solved efficiently using the Levinson algorithm [8].



(a)



(b)

Fig. 1. Frequency response of the filter of Example 1: (a) Magnitude response; (b) group delay response.

Example 1: We consider the design of a filter meeting the following specifications:

$$D(\omega) = \begin{cases} e^{-j16\omega} & \omega \in P = [-0.1\pi, 0.3\pi] \\ 0 & \omega \in S = [-\pi, -0.18\pi] \cup [0.38\pi, \pi]. \end{cases}$$

We have chosen $W_P(\omega) = 1$ and $W_S(\omega) = 2$. The magnitude response of the designed filter is shown in Fig. 1(a), whereas the group delay is shown in Fig. 1(b).

III. Design of 2-D Complex-Coefficient Filters

The frequency response of a 2-D $N_1 \times N_2$ nonrecursive filter with complex coefficients is given by

$$H(e^{j\omega_1}, e^{j\omega_2}) = \sum_{n=0}^{N_1-1}\sum_{m=0}^{N_2-1} h(n,m)e^{-jn\omega_1}e^{-jm\omega_2}. \tag{7}$$

For the sake of simplicity, we shall choose $N_1 = N_2 = N$. Let us now define

$$\mathbf{h} = \left[\mathbf{h}_0^T\,\mathbf{h}_1^T \cdots \mathbf{h}_{N-1}^T\right]^T$$

and

$$\mathbf{e}(\omega_1,\omega_2) = \left[\mathbf{e}_0^T(\omega_1,\omega_2)\,\mathbf{e}_1^T(\omega_1,\omega_2)\cdots\mathbf{e}_{N-1}^T(\omega_1,\omega_2)\right]^T$$

where

$$\mathbf{h}_n = [h(n,0)h(n,1)\cdots h(n,N-1)]^T$$

and

$$\mathbf{e}_n(\omega_1,\omega_2) = \left[e^{-jn\omega_1}e^{-jn\omega_1}e^{-j\omega_2}\cdots e^{-jn\omega_1}e^{-j(N-1)\omega_2}\right]^T$$

for $0 \le n \le N-1$. The phase response of the filter is given by

$$\phi(\omega_1,\omega_2) = -\tan^{-1}\left(\frac{\mathbf{a}^T\mathbf{s}(\omega_1,\omega_2) - \mathbf{b}^T\mathbf{c}(\omega_1,\omega_2)}{\mathbf{a}^T\mathbf{c}(\omega_1,\omega_2) + \mathbf{b}^T\mathbf{s}(\omega_1,\omega_2)}\right) \tag{8}$$

where $\mathbf{a}$ and $\mathbf{b}$ are the real and imaginary parts of the 2-D complex impulse response, and

$$\mathbf{c}(\omega_1,\omega_2) = \left[\mathbf{c}_0^T(\omega_1,\omega_2)\,\mathbf{c}_1^T(\omega_1,\omega_2)\cdots\mathbf{c}_{N-1}^T(\omega_1,\omega_2)\right]^T$$
$$\mathbf{s}(\omega_1,\omega_2) = \left[\mathbf{s}_0^T(\omega_1,\omega_2)\,\mathbf{s}_1^T(\omega_1,\omega_2)\cdots\mathbf{s}_{N-1}^T(\omega_1,\omega_2)\right]^T$$

where

$$\begin{aligned}&\mathbf{c}_n(\omega_1,\omega_2)\\ &\quad= [\cos(n\omega_1)\cos(n\omega_1+\omega_2)\cdots\cos(n\omega_1+(N-1)\omega_2)]^T\\ &\mathbf{s}_n(\omega_1,\omega_2)\\ &\quad= [\sin(n\omega_1)\sin(n\omega_1+\omega_2)\cdots\sin(n\omega_1+(N-1)\omega_2)]^T\end{aligned}$$

for $0 \le n \le N-1$. The group delay function is given by

$$\begin{aligned}\tau_i(\omega_1,\omega_2) &= -\frac{d}{d\omega_i}\phi(\omega_1,\omega_2)\\ &= \frac{k(\omega_1,\omega_2)}{l(\omega_1,\omega_2)}\cos^2(\phi(\omega_1,\omega_2))\end{aligned} \tag{9}$$

for $i = 1, 2$, and

$$\begin{aligned}k(\omega_1,\omega_2) &= (\mathbf{a}^T\mathbf{c}(\omega_1,\omega_2) + \mathbf{b}^T\mathbf{s}(\omega_1,\omega_2))\\ &\quad\times(\mathbf{a}^T\hat{\mathbf{c}}(\omega_1,\omega_2) + \mathbf{b}^T\hat{\mathbf{s}}(\omega_1,\omega_2))\\ &\quad+(\mathbf{a}^T\mathbf{s}(\omega_1,\omega_2) - \mathbf{b}^T\mathbf{c}(\omega_1,\omega_2))\\ &\quad\times(\mathbf{a}^T\hat{\mathbf{s}}(\omega_1,\omega_2) - \mathbf{b}^T\hat{\mathbf{c}}(\omega_1,\omega_2))\\ l(\omega_1,\omega_2) &= (\mathbf{a}^T\mathbf{c}(\omega_1,\omega_2) + \mathbf{b}^T\mathbf{s}(\omega_1,\omega_2))^2.\end{aligned}$$

When $i = 1$, we have

$$\hat{\mathbf{c}}(\omega_1,\omega_2) = \left[0\,\mathbf{c}_1^T(\omega_1,\omega_2)\cdots(N-1)\mathbf{c}_{N-1}^T(\omega_1,\omega_2)\right]^T$$
$$\hat{\mathbf{s}}(\omega_1,\omega_2) = \left[0\,\mathbf{s}_1^T(\omega_1,\omega_2)\cdots(N-1)\mathbf{s}_{N-1}^T(\omega_1,\omega_2)\right]^T.$$

On the other hand, when $i = 2$, we have

$$\hat{\mathbf{c}}(\omega_1,\omega_2) = \left[\hat{\mathbf{c}}_0^T(\omega_1,\omega_2)\,\hat{\mathbf{c}}_1^T(\omega_1,\omega_2)\cdots\hat{\mathbf{c}}_{N-1}^T(\omega_1,\omega_2)\right]^T$$
$$\mathbf{s}(\omega_1,\omega_2) = \left[\hat{\mathbf{s}}_0^T(\omega_1,\omega_2)\,\hat{\mathbf{s}}_1^T(\omega_1,\omega_2)\cdots\hat{\mathbf{s}}_{N-1}^T(\omega_1,\omega_2)\right]^T$$

where

$$\begin{aligned}&\hat{\mathbf{c}}_n(\omega_1,\omega_2)\\ &\quad= [0\cos(n\omega_1+\omega_2)\cdots(N-1)\cos(n\omega_1+(N-1)\omega_2)]^T\\ &\hat{\mathbf{s}}_n(\omega_1,\omega_2)\\ &\quad= [0\sin(n\omega_1+\omega_2)\cdots(N-1)\sin(n\omega_1+(N-1)\omega_2)]^T\end{aligned}$$

for $0 \le n \le N-1$.

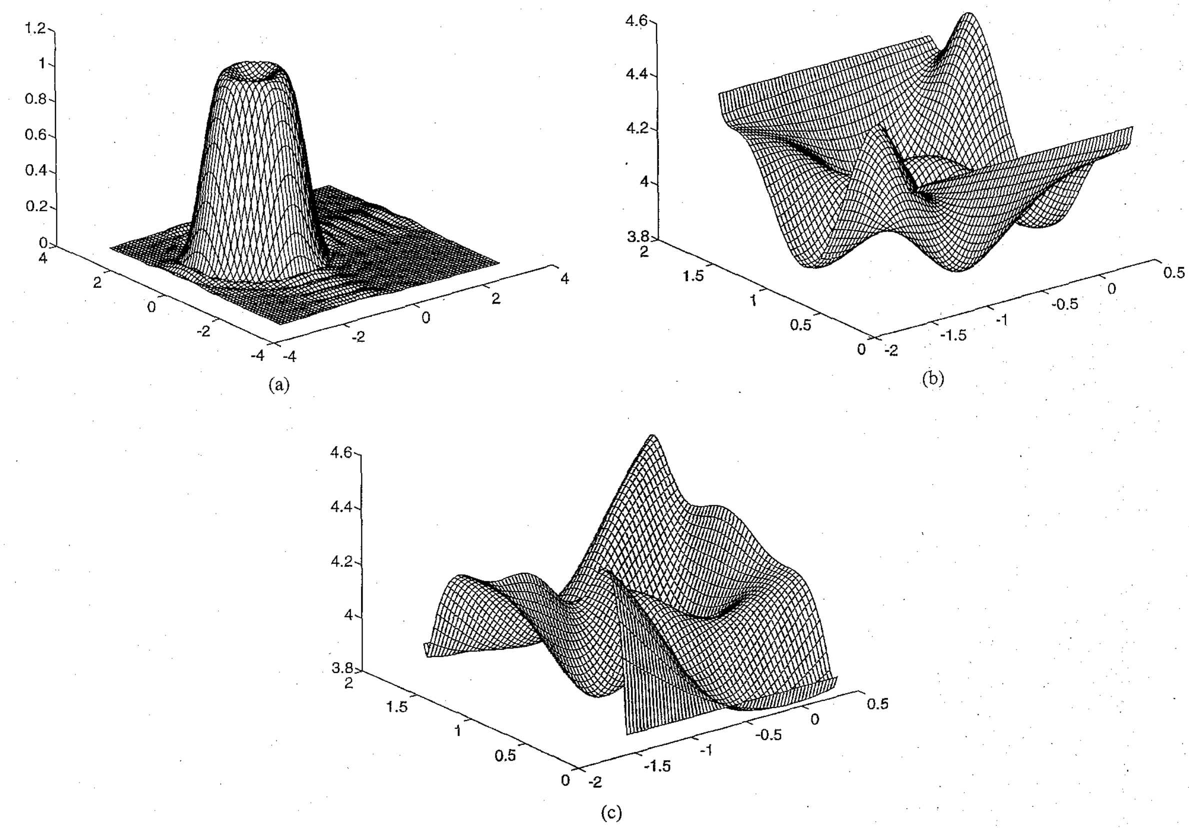


Fig. 2. Frequency response of the 2-D filter of Example 3: (a) Magnitude response; (b) group delay response along the ω_1 direction; (c) group delay response along the ω_2 direction.

If $D(\omega_1, \omega_2)$ is the desired 2-D complex-valued response, then the weighted mean-square error between it and $H(e^{j\omega_1}, e^{j\omega_2})$ can be formulated as

$$\begin{aligned} E_m &= \iint_P W_P(\omega_1,\omega_2)|D(\omega_1,\omega_2) - H(e^{j\omega_1, j\omega_2})|^2 d\omega_1 d\omega_2 \\ &\quad + \iint_S W_S(\omega_1,\omega_2)|H(e^{j\omega_1, j\omega_2})|^2 d\omega_1 d\omega_2 \\ &= \iint_P W_P(\omega_1,\omega_2)|D(\omega_1,\omega_2) - \mathbf{h}^T\mathbf{e}(\omega_1,\omega_2)|^2 d\omega_1 d\omega_2 \\ &\quad + \iint_S W_S(\omega_1,\omega_2)|\mathbf{h}^T\mathbf{e}(\omega_1,\omega_2)|^2 d\omega_1 d\omega_2 \end{aligned} \tag{10}$$

where P and S are as defined earlier. The functions $W_P(\omega_1, \omega_2)$ and $W_S(\omega_1, \omega_2)$ are 2-D frequency-dependent weighting functions in the passband and stopband, respectively.

By minimizing the error function E_m given in (10) with respect to the 2-D filter coefficients, the required filter can be designed. Again, in minimizing E_m, we set $\frac{\partial E_m}{\partial h(n,m)} = 0$ for $0 \leq n, m \leq N-1$ to obtain a system of linear equations $\mathbf{Qh} = \mathbf{d}$, where

$$\begin{aligned} \mathbf{Q} &= \iint_P W_P(\omega_1,\omega_2)\mathbf{e}^*(\omega_1,\omega_2)\mathbf{e}^T(\omega_1,\omega_2) d\omega_1 d\omega_2 \\ &\quad + \iint_S W_S(\omega_1,\omega_2)\mathbf{e}^*(\omega_1,\omega_2)\mathbf{e}^T(\omega_1,\omega_2) d\omega_1 d\omega_2 \end{aligned} \tag{11}$$

$$\mathbf{d} = \iint_P W_P(\omega_1,\omega_2) D(\omega_1,\omega_2)\mathbf{e}^*(\omega_1,\omega_2) d\omega_1 d\omega_2 \tag{12}$$

It can be seen that $\mathbf{Q}$ is a Hermitian block-Toeplitz matrix. Thus, only $N \times (N^2 - \frac{N-1}{2})$ (for odd N) elements have to be computed. The system of linear equations involving a Hermitian block-Toeplitz matrix can be solved efficiently using a modified version of the Levinson algorithm [8].

Example 2: We consider the design of an 11×11 filter having the response

$$D(\omega_1,\omega_2) = \begin{cases} e^{-j4\omega_1}e^{-j4\omega_2} & \sqrt{(\omega_1+0.2\pi)^2+(\omega_2-0.3\pi)^2} \leq 0.3\pi \\ 0 & \sqrt{(\omega_1+0.2\pi)^2+(\omega_2-0.3\pi)^2} \geq 0.5\pi. \end{cases}$$

In this example, we choose $W_P(\omega_1, \omega_2) = W_S(\omega_1, \omega_2) = 1$. Fig. 2(a) shows the magnitude response of the designed filter. The group delay of the filter along ω_1 and ω_2 directions are shown in Figs. 2(a) and (b), respectively.

IV. Performance Comparison

We shall compare our method with the eigenfilter method of [7] in terms of the peak and group delay errors and computational complexity. Tables I and II show a comparison of the two methods with respect to the peak and group delay error for Examples 1 and 2, respectively. Although the differences are negligible, our method yields better results.

For the comparison in terms of computational complexity, let us consider the 1-D filter design. In our method, $\mathbf{Q}$ is Hermitian Toeplitz and positive definite. Thus, only the first row (or column) of $\mathbf{Q}$ needs to be evaluated. Moreover, efficient and robust algorithms for solving a system of linear equations having a computational complexity of $\mathcal{O}(N^2)$ can be used. Our method does not depend on any reference frequency and consequently there is no need for any normalization.

TABLE I
COMPARISON OF THE PEAK AND GROUP DELAY ERROR FOR EXAMPLE 1

	Magnitude of the complex peak error in the passband	Magnitude of the complex peak error in the stopband	Group delay error in the passband
Our method	0.05048	0.04068	1.23
Method of [16]	0.05054	0.04121	1.25

TABLE II
COMPARISON OF THE PEAK ERRORS AND GROUP DELAY VARIATIONS FOR EXAMPLE 2

	Magnitude of the complex peak error in the passband	Magnitude of the complex peak error in the stopband	Group delay variation in the passband
Our method	0.08673	0.09632	ω_1: 3.81-4.45 ω_2: 3.81-4.45
Method of [16]	0.08774	0.09671	ω_1: 3.77-4.45 ω_2: 3.78-4.44

In addition, the filter coefficients are obtained by solving a system of linear equations just once.

For the eigenfilter method, the mean-square error for the 1-D filter design is formulated as

$$E_e = \mathbf{h}^H \mathbf{Q}_e \mathbf{h} \tag{13}$$

where $\mathbf{Q}_e$ is an $N \times N$ Hermitian positive-definite matrix. As a consequence, $N(N+1)/2$ elements of $\mathbf{Q}_e$ have to be evaluated. The coefficients of the filter are obtained as the complex eigenvector corresponding to the smallest eigenvalue of $\mathbf{Q}_e$. The smallest eigenvalue is computed by using the iterative inverse power method. This method requires solving a system of linear equations several times before obtaining the required eigenvector. Furthermore, $\mathbf{Q}_e$ is not Toeplitz, and consequently, the computational complexity for solving a system of equations involving $\mathbf{Q}_e$ is $\mathcal{O}(N^3)$. In addition, since the method is iterative, the results are only approximate. The mean-square error E_e is dependent on a reference frequency, and as a consequence, so are the filter coefficients. Moreover, the eigenvector of the smallest eigenvalue has to be normalized with respect to the reference frequency to obtain the filter coefficients. For Example 1, the saving in the number of floating point operations (flops) in our method is approximately 85% compared with the eigenfilter approach.

Let us now consider the 2-D filter design. As mentioned earlier, it is evident from Example 2 that our method yields better results. In our method, $\mathbf{Q}$ is Hermitian block-Toeplitz. Consequently, only $N \times (N^2 - \frac{N-1}{2})$ (for odd N) elements have to be computed, and the system of linear equations involving a Hermitian block-Toeplitz matrix can be solved efficiently using a modified version of the Levinson algorithm [8].

The mean-square error using the 2-D eigenfilter formulation is again given by (13), where $\mathbf{Q}_e$ is Hermitian and positive-definite. Here, $N^2(N^2+1)/2$ entries have to be evaluated. All the comments regarding the 1-D eigenfilter formulation apply equally well to the 2-D eigenfilter formulation. The saving in flops for Example 2 using our method is approximately 94%.

V. CONCLUSION

A method for the design of 1-D complex-coefficient nonrecursive filters has been described. In this method, the error between the desired complex-valued response and the frequency response of a filter with a complex-valued impulse response is formulated in a quadratic form. The filter coefficients are obtained by solving a system of linear equations that involves a Hermitian Toeplitz and positive-definite matrix. Consequently, the system of linear equations can be solved by using the computationally efficient and robust Levinson algorithm. The method for the design of 1-D filters has been extended to the design of 2-D complex-coefficient filters. Here again, the filter coefficients are obtained by solving a system of linear equations involving a Hermitian block-Toeplitz matrix. Such a system of equations can also be solved efficiently by a modified version of the Levinson algorithm. Examples have been presented to demonstrate that our method is superior to the eigenfilter method.

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