Journal of

CIRCUITS, SYSTEMS, AND COMPUTERS

Volume 13 • Number 5 • October 2004

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Journal of Circuits, Systems, and Computers
Vol. 13, No. 5 (2004) 1085–1103

AN ALTERNATIVE APPROACH FOR OBTAINING 2D DISCRETE FILTERS BY CASCADING SECTIONS EACH HAVING UNITY DEGREE IN EACH VARIABLE

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Revised 17 September 2003

The properties of a 2D discrete transfer function with degree of each variable being unity are discussed. The coefficients of the denominator polynomial contain a parameter k (having real values) whose bounds are determined by stability considerations. These bounds are obtained by testing the overall polynomial at only four points $z_1 = \pm 1$ and $z_2 = \pm 1$. Suitable numerator polynomial can be associated to get the overall transfer function. Such structures can be cascaded so that the overall magnitude response can be changed by altering the response of one or more sections.

Keywords: 2D digital filters; variable magnitude characteristics.

1. Introduction

There has been much research done on two-dimensional (2D) IIR filter design.[1] The classical least-squares minimization technique whose objective function includes a penalty function term to ensure stability is formulated in Ref. 2. A minimax criterion that is solved by a linear programming approach is discussed in Ref. 3. The McClellan transformation can also be applied to the numerator and denominator of a one-dimensional (1D) filter.[1] Another method is to design an analog filter and apply the bilinear transformation.[4] The singular value decomposition has been applied in Ref. 5. Obtaining desired magnitude and group delay characteristics are discussed in Refs. 6–9.

A popular design method for variable digital filters is based on frequency transformations.[10] A more recent approach is to represent the transfer function as a polynomial of different frequencies corresponding to the various variables thereby making the frequency response variable.[11] This design technique is time consuming due to the many coefficients that need to be optimized and no guarantee of stability is achieved. The technique in Ref. 12 reduces the design complexity, guarantees stability and achieves approximately linear phase by decomposing the complex frequency response specifications into the product of two parts. The first part of this paper corresponds to the frequency responses of constant (not variable) 2D filters and the second part corresponds to the desired values of 1D polynomials which are relatively easier to approximate.

In 2D digital filter design, it is highly desirable to be able to adjust a single parameter so that the magnitude response characteristics can be changed and hence the corresponding contour plots can be altered. In this paper, we present an approach that involves adjustment of a single scalar parameter to achieve this objective in 2D IIR filters. Stability is guaranteed by deriving bounds on that scalar parameter.

One of the possible approaches is to use a multiplier k either in the feedback path or in the feed-forward path.[13] This multiplier can be adjusted, subject to the constraints imposed on it due to the stability considerations. As it has been shown in Ref. 13, the complexity of the determination of the limits of k increases with the degree of each of the variables. To a certain extent, this difficulty can be overcome by employing the graphical technique.[14]

An effective implementation of both approaches, which is described in this paper, is to design a 2D filter so that the degree of each variable is unity (the overall degree being two) and to cascade several such sections. The advantage of this method is that the stability of each section can be ensured independently. The overall response is the product of the responses of each individual sections.

2. The Basic Structures

Two basic structures (Structure A and Structure B) will be considered bere, depending on the location of the multiplier k (it should be noted that many other alternative realizations are possible). In both the structures,

$$\frac{V}{U} = H_d(z_1, z_2) = \frac{N_d(z_1, z_2)}{D_d(z_1, z_2)} \tag{1}$$

and $H_d(z_1, z_2)$ is the transfer function of the generating filter.

Structure A. In this structure, the multiplier k_1 (positive or negative) is in the feedback path, as shown in Fig. 1.

Analysis yields

$$\frac{Y_1}{X_1} = \frac{N_d(z_1, z_2)}{D_d(z_1, z_2) + k_1 N_d(z_1, z_2)}\,. \tag{2}$$

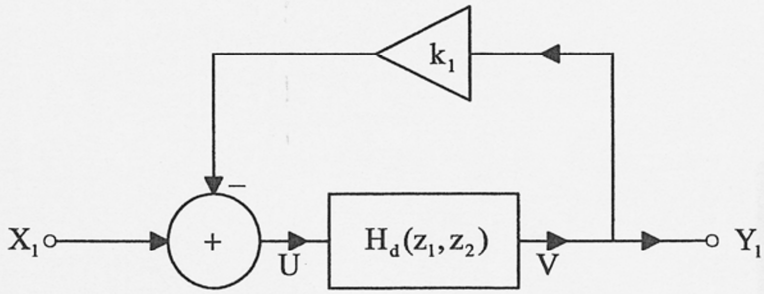


Fig. 1. Structure A: The multiplier k_1 in the feedback path.

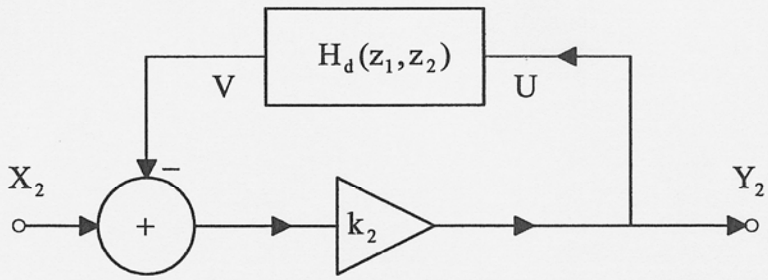


Fig. 2. The basic Structure B.

Structure B. In this structure, the multiplier k_2 (positive or negative) is in the forward path, as shown in Fig. 2.

It is required that $k_2 \neq 0$. Analysis yields

$$\frac{Y_2}{X_2} = \frac{k_2 D_d(z_1, z_2)}{D_d(z_1, z_2) + k_2 N_d(z_1, z_2)} = \frac{k_2 D_d(z_1, z_2)}{P_d(z_1, z_2)} . \tag{3}$$

A comparison of Eqs. (2) and (3) reveals the following:

(a) The denominator polynomial is the same for both the structures. Therefore, the limits of k_1 and k_2 from stability considerations will be the same, except that $k_2 \neq 0$.

(b) The zeroes of Eq. (1) are those of the starting function $H_d(z_1, z_2)$, while the zeroes of Eq. (2) are the poles of $H_d(z_1, z_2)$.

3. Stability Considerations

As has been mentioned earlier, the degree of each variable in the proposed 2D structures is always unity. In order to establish the stability conditions of each structure, the following two basic theorems are proven first.

Theorem 1. *Given*

$$D_d(z_1, z_2) = (a_{11} z_1 z_2 + a_{10} z_1 + a_{01} z_2 + a_{00}) , \tag{4}$$

the stability conditions for $1/D_d(z_1, z_2)$ *are given by:*

$$D_d(1,1) > 0\,, \tag{5a}$$

$$D_d(-1,1) < 0\,, \tag{5b}$$

$$D_d(1,-1) < 0\,, \tag{5c}$$

$$D_d(-1,-1) > 0\,. \tag{5d}$$

Proof. When the inverse bilinear transformations $z_i = (1+s_i)/(1-s_i)$, $i = 1, 2$, are applied to $D_d(z_1, z_2)$, the numerator of the corresponding s_1, s_2-domain function comes out to be

$$\begin{aligned} N_a(s_1, s_2) = (a_{11} - a_{10} - a_{01} + a_{00})s_1 s_2 + (a_{11} + a_{10} - a_{01} - a_{00})s_1 \\ + (a_{11} - a_{10} + a_{01} - a_{00})s_2 + (a_{11} + a_{10} + a_{01} + a_{00})\,. \end{aligned} \tag{6}$$

In order that $N_a(s_1, s_2)$ shall represent a VSHP (Very Strict Hurwitz Polynomial),[15] every coefficient shall be positive. This gives

$$D_d(1,1) = a_{11} + a_{10} + a_{01} + a_{00} > 0\,, \tag{7a}$$

$$(-1)D_d(-1,1) = a_{11} - a_{10} + a_{01} - a_{00} > 0\,, \tag{7b}$$

$$(-1)D_d(1,-1) = a_{11} + a_{10} - a_{01} - a_{00} > 0\,, \tag{7c}$$

$$D_d(-1,-1) = a_{11} - a_{10} - a_{01} + a_{00} > 0\,. \tag{7d}$$

Hence, the theorem follows. □

This theorem clearly shows that stability conditions need be tested at the points corresponding to $z_1 = \pm 1$ and $z_2 = \pm 1$ only and no other points need be tested on the unit bidisc.

A similar result has been derived in Ref. 16. However, the present interpretation helps us to obtain the bounds of any coefficient in either Structure A or Structure B in an easier fashion.

A set of general relationships involving the two structures is obtained in Theorem 2.

Theorem 2. *Let the generating filter's transfer function* $H_d(z_1, z_2)$ *be given by:*

$$H_d(z_1, z_2) = \frac{N(z_1, z_2)}{D(z_1, z_2)} = \frac{b_{11}z_1 z_2 + b_{10}z_1 + b_{01}z_2 + b_{00}}{c_{11}z_1 z_2 + c_{10}z_1 + c_{01}z_2 + c_{00}}\,. \tag{8}$$

Then, the bounds of k *(*k_1 *or* k_2*) for stability of structures A and B are obtained by:*

$$(c_{11} + c_{10} + c_{01} + c_{00}) + k(b_{11} + b_{10} + b_{01} + b_{00}) > 0\,, \tag{9a}$$

$$(c_{11} + c_{10} - c_{01} - c_{00}) + k(b_{11} + b_{10} - b_{01} - b_{00}) > 0\,, \tag{9b}$$

$$(c_{11} - c_{10} + c_{01} - c_{00}) + k(b_{11} - b_{10} + b_{01} - b_{00}) > 0\,, \tag{9c}$$

$$(c_{11} - c_{10} - c_{01} + c_{00}) + k(b_{11} - b_{10} - b_{01} + b_{00}) > 0\,. \tag{9d}$$

Proof. The proof is omitted for the sake of brevity, as it follows directly from Theorem 1. □

It should be noted that the order of each variable z_1 and z_2 is unity, whereas the total order of the denominator is two, given by the $z_1 z_2$ term. The first order terms are given by $c_{10}z_1$ and $c_{01}z_2$. Obviously, the constant term is given by c_{00}.

4. Classification and Realizations

Depending on where the variable multiplier coefficient k is placed in the overall denominator, the following classifications of the transfer functions are possible.

Type I. In this type, the quantity k occurs only in one of the terms. This type could be further sub-classified as follows:

(a) Type I(a), where k occurs only in the constant term,
(b) Type I(b), where k occurs only in the first-order or $(z_1 + z_2)$ terms,
(c) Type I(c), where k occurs only in the $z_1 z_2$ term.

Type II. In this type, the quantity k occurs in two of the terms. This type could be further classified as:

(a) Type II(a), where k occurs in the constant and the first-order terms,
(b) Type II(b), where k occurs in the constant and the $z_1 z_2$ terms,
(c) Type II(c), where k occurs in the first-order and the $z_1 z_2$ terms.

Type III. In this type, the quantity k occurs in all the three terms and hence only one possibility exists.

It is possible to further subdivide Types I(b), II(a) and II(c) in that k can occur only either in the z_1 or z_2 term only. This definitely could alter the symmetry of the response. Therefore, such cases are considered at appropriate places in the paper.

Having obtained the bounds of k (the actual value used may be positive or negative), it is required to obtain realizations without delay-free loops (this may not always be guaranteed with the structures considered). However, signal-flow graphs are given for each type considered above and these lead to possible delay-free loop implementations. A number of other possibilities may exist. In what follows, only the transfer functions obtained from Structure A are discussed. Similar treatment is possible for transfer functions derived from Structure B.

Throughout the paper, it is assumed that the starting filter $H_d(z_1, z_2)$ is stable and it does not contain any nonessential singularities of the second kind.

Type I(a). Before proceeding further in this case, we can consider that, in the denominator polynomial in Eq. (8), c_{11} can be taken to be unity without any loss of generality. Also, we have

$$b_{11} = b_{10} = b_{01} = 0 \quad \text{and} \quad b_{00} = 1\,. \tag{10}$$

Hence, the bounds of k can be obtained by Eq. (9) and are given by Theorem 3.

Theorem 3. *For the denominator of the transfer function of Type I(a), the bounds of k are obtained as follows:*

(i) $$\text{for } k \geq 0, 0 \leq k < \min\{|D(1,-1)|, |D(-1,1)|\} \tag{11a}$$

and

(ii) $$\text{for } k < 0, 0 \leq |k| < \min\{|D(1,1)|, |D(-1,-1)|\}\,. \tag{11b}$$

Proof.

(i) When k is non-negative, the inequalities $\{D(1,1)+k\} > 0$ and $\{D(-1,-1)+k\} > 0$ are always satisfied. It is also noted that $D(-1,1)$ and $D(1,-1)$ are negative and when k is added to these two quantities, the resulting inequalities $(-1)\{D(-1,1)+k\}$ and $(-1)\{D(1,-1)+k\}$ should become positive. Hence the relationship (11a) follows {in fact, this can be verified by the relationships (9b) and (9c) also}.

(ii) When k is negative, it is clear that $\{D(-1,1)+k\}$ and $\{D(1,-1)+k\}$ always remain negative. It is required to test only the inequalities $\{D(1,1)+k\}$ and $\{D(-1,-1)+k\}$ and they should be positive. Hence, the inequality (11b) follows. □

Some simplification results, if we write the corresponding denominator polynomial as follows:

$$D_{1a}(z_1, z_2) = (z_1 + a_1)(z_2 + a_2) + k_{1a}\,, \tag{12a}$$

where

$$k_{1a} = (k + c_{00} - a_1 a_2)\,, \tag{12b}$$

$$a_1 = c_{10}\,, \tag{12c}$$

and

$$a_2 = c_{01}\,. \tag{12d}$$

From stability considerations, it is required that

$$|a_1| < 1 \quad \text{and} \quad |a_2| < 1\,, \qquad \text{if } k_{1a} = 0\,. \tag{13}$$

The overall transfer function can be written as

$$H_{d1a}(z_1, z_2) = \frac{Y_{1a}(z_1, z_2)}{X_{1a}(z_1, z_2)} = \frac{K}{(z_1 + a_1)(z_2 + a_2) + k_{1a}}\,. \tag{14}$$

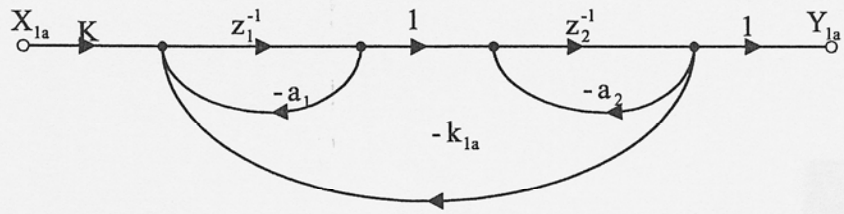


Fig. 3. Signal-flow graph for the realization of the transfer function $H_{d1a}(z_1, z_2)$ given in Eq. (14).

It is noted that K is a parameter which multiplies the overall transfer function and is different from the variable k. This is adopted throughout the paper.

Different signal-flow graphs which lead to realizations can be obtained. It should be noted that for realizability, these graphs should contain no delay-free loop. One such signal-flow graph is given in Fig. 3, as an example.

It corresponds to Structure A.

Type I(b). In this case also, c_{11} in Eq. (8) can be taken to be unity. The denominator polynomial can be written as:

$$D(z_1, z_2) + k_{1b}(b_{10}z_1 + b_{01}z_2)\,. \tag{15}$$

It is noted that $b_{11} = b_{00} = 0$. The bounds of k_{1b} can be obtained by Eq. (9). Simplification of the stability conditions results when we consider $b_{10} = b_{01}$. This is the case normally encountered, because of the symmetry considerations. In such a case, we can let $b_{10} = b_{01} = 1$, without any loss of generality. Under these conditions, Eq. (15) is rewritten as

$$D(z_1, z_2) + k_{1b}(z_1 + z_2)\,. \tag{16}$$

The bounds of k_{1b} are given by Theorem 4.

Theorem 4. *For the denominator of the transfer function given by Type I(b) of Eq.* (16), *the bounds are obtained as follows*:

(i) $$\textit{for } k \geq 0, 0 \leq |k_{1b}| \leq \frac{1}{2}\{D(-1,-1)\} \tag{17a}$$

and

(ii) $$\textit{for } k < 0, 0 \leq |k_{1b}| < \frac{1}{2}\{D(1,1)\}\,. \tag{17b}$$

Proof. It is readily observed that for the cases $z_1 = 1$, $z_2 = -1$ and $z_1 = -1$, $z_2 = 1$, conditions (9b) and (9c) are always satisfied. Therefore, only the cases $z_1 = z_2 = 1$ and $z_1 = z_2 = -1$ need be considered.

(i) When k_{1b} is non-negative, Eq. (9a) is always satisfied. Equation (9d) leads to Eq. (17a).
(ii) When k_{1b} is negative, Eq. (9d) is always satisfied. Equation (9a) leads to Eq. (17b). □

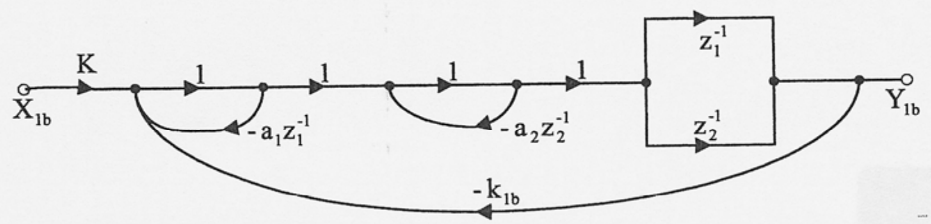


Fig. 4. A signal-flow graph for the realization of the transfer function $H_{d1b}(z_1, z_2)$ given in Eq. (18).

Some simplification in the realizations is possible, if one starts from the product-separable form for $D(z_1, z_2)$, as in the case of Type I(b). In such a case, the overall transfer function will be

$$H_{d1b}(z_1, z_2) = \frac{Y_{1b}(z_1, z_2)}{X_{1b}(z_1, z_2)} = \frac{K(z_1 + z_2)}{(z_1 + a_1)(z_2 + a_2) + k_{1b}(z_1 + z_2)} . \tag{18}$$

From stability considerations, it is required that

$$|a_1| < 1 \quad \text{and} \quad |a_2| < 1 , \qquad \text{if } k_{1b} = 0 . \tag{19}$$

A possible signal-flow graph which leads to a realization is given in Fig. 4.

As has been mentioned earlier, the constant k_{1b} may be associated with either z_1 or z_2 only. Under these circumstances, the transfer functions will have one of the following forms:

$$H_{d1b1}(z_1, z_2) = \frac{Y_{1b1}(z_1, z_2)}{X_{1b1}(z_1, z_2)} = \frac{Kz_1}{(z_1z_2 + c_{10}z_1 + c_{01}z_2 + c_{00}) + k_{1b1}z_1} \tag{20a}$$

or

$$H_{d1b2}(z_1, z_2) = \frac{Y_{1b2}(z_1, z_2)}{X_{1b2}(z_1, z_2)} = \frac{Kz_2}{(z_1z_2 + c_{10}z_1 + c_{01}z_2 + c_{00}) + k_{1b2}z_2} . \tag{20b}$$

In both cases, the bounds of stability for k_{1b1} or k_{1b2} are obtained by the application of Eq. (9).

Type I(c). In this case, the multiplier k is associated with the z_1z_2 term. Proceeding as before, the transfer function is obtained as:

$$H_{d1c}(z_1, z_2) = \frac{Y_{1c}(z_1, z_2)}{X_{1c}(z_1, z_2)} = \frac{Kz_1z_2}{D(z_1, z_2) + k_{1c}z_1z_2} , \tag{21a}$$

where

$$D(z_1, z_2) = z_1z_2 + c_{10}z_1 + c_{01}z_2 + c_{00} . \tag{21b}$$

The bounds of k_{1c} which ensure stability of H_{d1c} are given by Theorem 5. In this case, we have $b_{11} = 1$ and $b_{10} = b_{01} = b_{00} = 0$.

Theorem 5. *For the denominator of the transfer function given by Type I(c) of Eq. (21), the bounds of k_{1c} are obtained as follows:*

(i) $$\textit{for } k_{1c} \geq 0, 0 \leq |k_{1c}| < \infty \tag{22a}$$

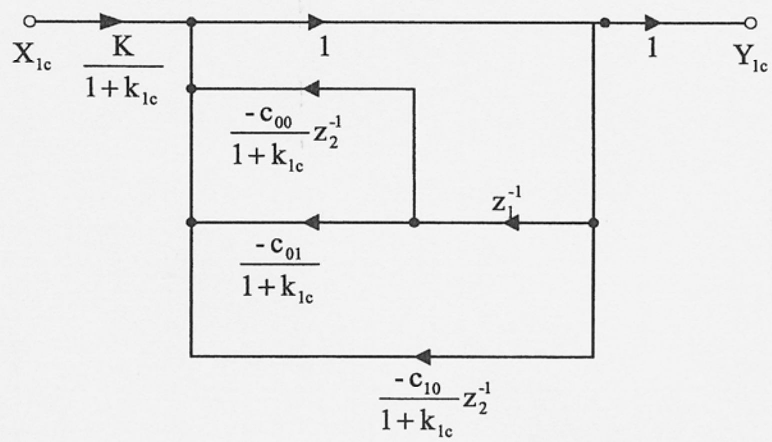


Fig. 5. Signal-flow graph for the realization of the transfer function $H_{d1c}(z_1, z_2)$ given in Eq. (21).

and

(ii) *for* $k_{1c} < 0, 0 \le |k_{1c}| < \min\{|D(1,1)|, |D(1,-1)|, |D(-1,1)|, |D(-1,-1)|\}$. (22b)

Proof.

(i) When k_{1c} is non-negative, the inequalities (9) are always satisfied. Hence, Eq. (22a) follows.
(ii) When k_{1c} is negative, all the inequalities need to be satisfied. Hence, Eq. (22b) follows. A signal-flow graph suitable for realization is given in Fig. 5. □

Type II(a). In this case, the quantity k_{2a} appears in both the constant and the first-order terms. The transfer function is:

$$H_{d2a}(z_1, z_2) = \frac{Y_{2a}}{X_{2a}} = \frac{K(b_{10}z_1 + b_{01}z_2 + b_{00})}{D(z_1, z_2) + k_{2a}(b_{10}z_1 + b_{01}z_2 + b_{00})}, \tag{23}$$

where $D(z_1, z_2)$ is given by Eq. (21b).

As the quantities b_{10}, b_{01} and b_{00} can be arbitrary, the bounds of k_{2a} to ensure stability are obtained by Eq. (9).

A signal-flow graph for the realization is given in Fig. 6.

Type II(b). In this case, the quantity k_{2b} is in both the constant and the z_1z_2 term. Accordingly, the transfer function is:

$$H_{d2b}(z_1, z_2) = \frac{Y_{2b}}{X_{2b}} = \frac{K(b_{11}z_1z_2 + b_{00})}{D(z_1, z_2) + k_{2b}(b_{11}z_1z_2 + b_{00})}, \tag{24}$$

where $D(z_1, z_2)$ is given by Eq. (21b).

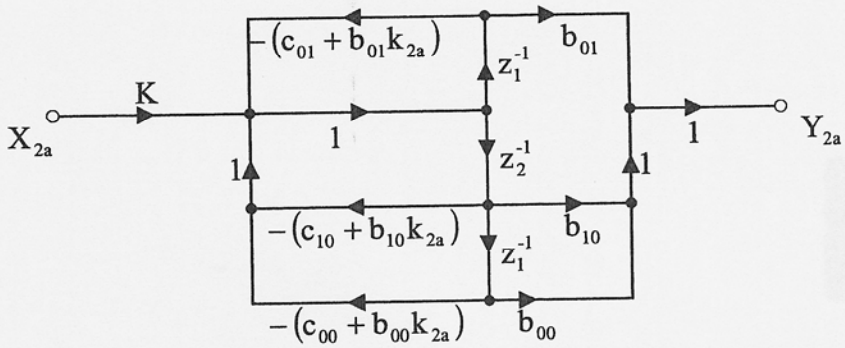


Fig. 6. Signal-flow graphs for the realization of the transfer function $H_{d2a}(z_1, z_2)$ given in Eq. (23).

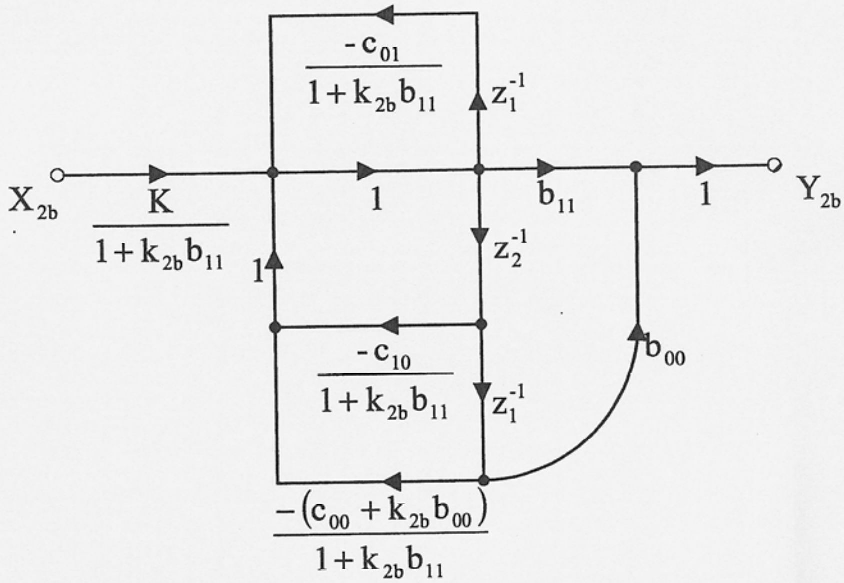


Fig. 7. A signal-flow graph for the realization of the transfer function $H_{d2b}(z_1, z_2)$ given in Eq. (24).

As the quantities b_{11} and b_{00} can be arbitrary, the bounds of k_{2b} to ensure stability are obtained by Eq. (9).

A signal-flow graph for the realization is given in Fig. 7.

Type II(c). In this case, the quantity k_{2c} appears in z_1z_2, z_1 and z_2 terms. Accordingly, the transfer function is:

$$H_{d2c}(z_1, z_2) = \frac{Y_{2c}}{X_{2c}} = \frac{K(b_{11}z_1z_2 + b_{10}z_1 + b_{01}z_2)}{D(z_1, z_2) + k_{2c}(b_{11}z_1z_2 + b_{10}z_1 + b_{01}z_2)}, \tag{25}$$

where $D(z_1, z_2)$ is given by Eq. (21b).

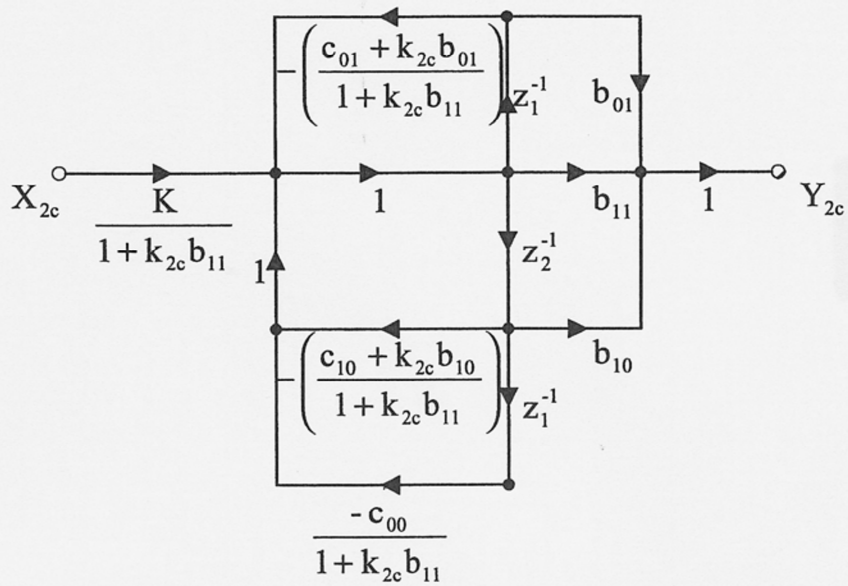


Fig. 8. Signal-flow graphs for the realization of the transfer function $H_{d2c}(z_1, z_2)$ given in Eq. (25).

As the quantities b_{11}, b_{10} and b_{01} could be arbitrary, the bounds of k_{2c} to ensure stability are obtained by Eq. (9).

A signal-flow graph for the realization is given in Fig. 8.

Type III. In this case, the quantity k_3 occurs in the z_1z_2, z_1, z_2 and the constant terms. Accordingly, the transfer function is:

$$H_{d3}(z_1, z_2) = \frac{Y_3}{X_3} = \frac{K(b_{11}z_1z_2 + b_{10}z_1 + b_{01}z_2 + b_{00})}{D(z_1, z_2) + k_3(b_{11}z_1z_2 + b_{10}z_1 + b_{01}z_2 + b_{00})}, \tag{26}$$

where $D(z_1, z_2)$ is given by Eq. (21b).

As the quantities b_{11}, b_{10}, b_{01} and b_{00} are arbitrary, the bounds of k_3 to ensure stability are obtained by Eq. (9).

A signal-flow graph for the realization is given in Fig. 9.

It can also be noted that different cases arise out of Type III by suitably equating the various constants b_{11}, b_{10}, b_{01} and b_{00} to zero.

Numerical Example 1. This example shows how the magnitude and the contour plots vary as the value of k is varied. For purposes of illustration, the transfer function considered is:

$$H_d(z_1, z_2) = \frac{1}{(z_1 - 0.5)(z_2 - 0.5) + k}. \tag{27}$$

It is noted that this is Type I(a) filter.

The bounds of k as determined from Theorem 3 are obtained as follows:

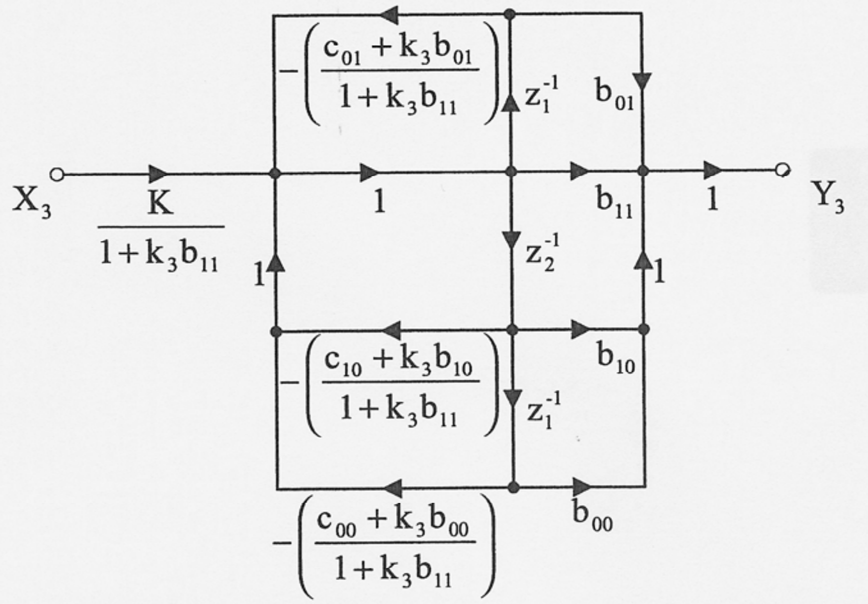
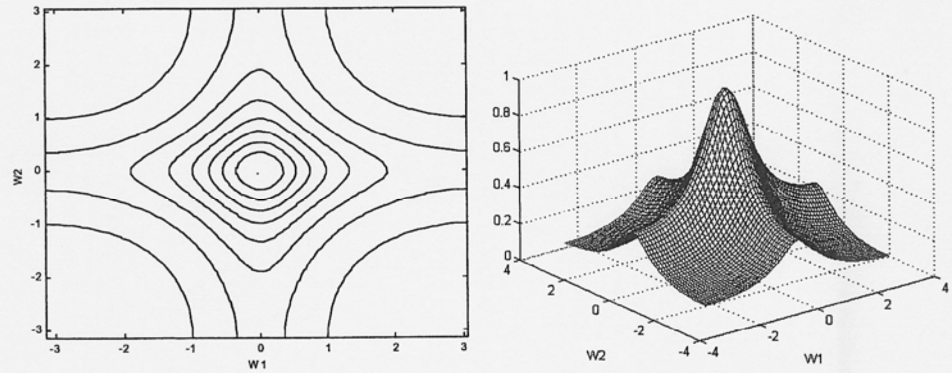


Fig. 9. Signal-flow graph for the realization of the transfer function $H_{d2c}(z_1, z_2)$ given in Eq. (26).



(a) (b)

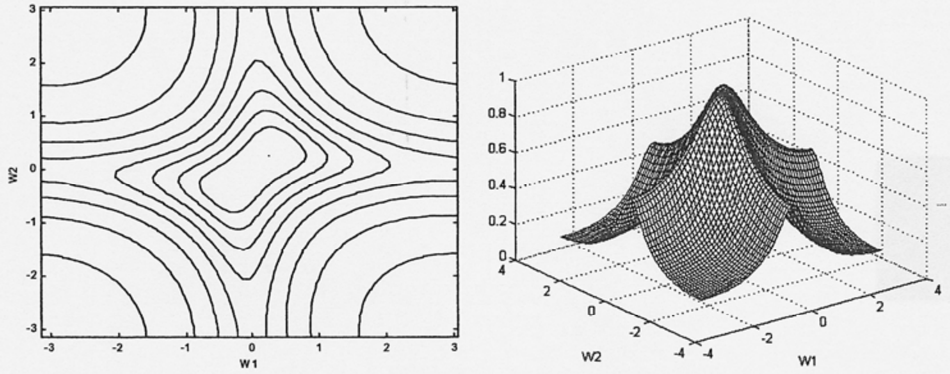
Fig. 10. (a) and (b) The contour and the magnitude plots of the transfer function of Eq. (27), when $k = 0$.

(a) for $k > 0$, we have $0 < k < 0.75$, from Eq. (11a)
(b) for $k < 0$, we have $0 < |k| < \min(|0.25|, |2.25|)$, or $0 < |k| < 0.25$.

Therefore, the bounds of stability will be

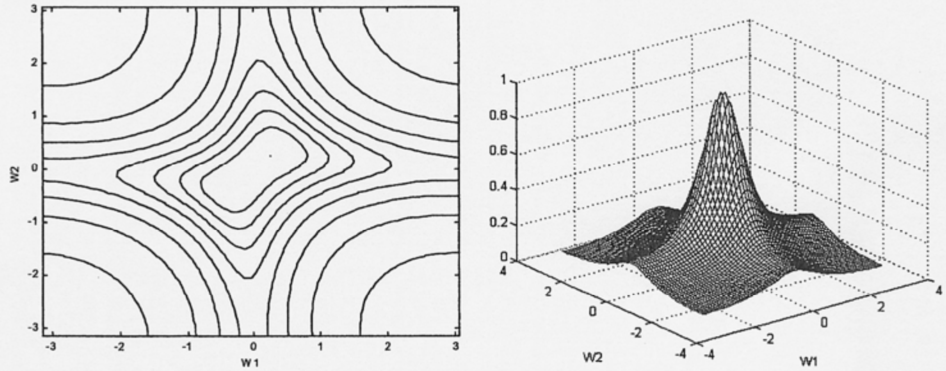
$$-0.25 < k < 0.75\,. \tag{28}$$

Figures 10(a) and 10(b), respectively, show the contour and the magnitude plots of Eq. (27), when $k = 0$. When k is made equal to 0.1, the contour and



(a) (b)

Fig. 11. (a) and (b) The contour and the magnitude plots of the transfer function of Eq. (27), when $k = 0.1$.



(a) (b)

Fig. 12. (a) and (b) The contour and the magnitude plots of the transfer function of Eq. (27), when $k = -0.1$.

the magnitude plots change and they are shown in Figs. 11(a) and 11(b), respectively. When k is changed to -0.1, the corresponding contour and the magnitude plots are shown in Figs. 12(a) and 12(b).

5. Bandwidth Considerations

Whenever a filter is designed, generally one would like to obtain information about the bandwidth of the passband. The cutoff frequencies can be obtained by the formula

$$|H_d(z_1, z_2)|\,|_{z_1=e^{j\omega_c 1}, z_2=e^{j\omega_c 2}} = \frac{1}{\sqrt{2}}, \tag{29}$$

where ω_{c1} and ω_{c2} are the cutoff frequencies in ω_1-axis and ω_2-axis, respectively. It can be shown that, for Type III,

$$\left[\frac{(B_{00} - B_{11}\omega_1\omega_2)^2 + (B_{10}\omega + B_{01}\omega)^2}{(A_{00} - A_{11}\omega_1\omega_2)^2 + (A_{10}\omega_1 + A_{01}\omega_2)^2}\right]^{1/2} = \frac{1}{(\sqrt{\beta})^{1/n}}, \tag{30}$$

where n = number of sections considered, A_{00}, A_{10}, A_{01} and A_{11} are given as in Eq. (9)

$$B_{00} = b_{11} + b_{10} + b_{01} + b_{00}, \tag{31a}$$

$$B_{10} = b_{11} + b_{10} - b_{01} - b_{00}, \tag{31b}$$

$$B_{01} = b_{11} - b_{10} + b_{01} - b_{00}, \tag{31c}$$

$$B_{11} = b_{11} - b_{10} - b_{01} + b_{00}; \tag{31d}$$

with $A_{00} = B_{00}$ so that the response at $\omega_1 = 0$ and $\omega_2 = 0$ is unity and $\beta = 2$ in order to determine the cut-off frequencies.

As indicated earlier, the other types can be obtained by suitably adjusting the constants a and b. Different frequency axes can be considered by putting

$$\omega_1 = \alpha\omega_2, \tag{32}$$

where α can be positive, negative or zero. We get

$$\begin{aligned}&\omega_2^4[\alpha^2(A_{11}^2 - \beta^2 B_{11}^2)]\\ &\quad + \omega_2^2\begin{bmatrix}(A_{01}^2 - \beta^2 B_{01}^2) + \alpha^2(A_{10}^2 - \beta^2 B_{10}^2)\\ + \alpha\{2A_{10}A_{01} - 2A_{11}A_{00} + \beta^2(-2B_{10}B_{01} + 2B_{11}B_{00})\}\end{bmatrix}\\ &\quad + (A_{00}^2 - \beta^2 B_{00}^2) = 0,\end{aligned} \tag{33}$$

from which the required ω_2 and hence ω_1 can be obtained.

For the Numerical Example 1 taken, we have $A_{11} = 2.25 + k$, $A_{10} = A_{01} = 0.75 - k$, $A_{00} = 0.25 + k$, $B_{11} = 1$, $B_{10} = B_{00} = 1$ and $A_{00} = B_{00}$. As can be readily seen, the cutoff contours vary as k is changed. Figure 13 shows the various cutoff contours as k is changed.

As can be seen, considerable variation in the bandwidth will be possible.

6. Other Types of Filters

In the previous section, it has been observed that the denominators of the transfer functions of the various filters are derived from either Structure A or Structure B. The numerators of these transfer functions are derived from either of these two structures only. Depending on the value of k chosen, a large number of possibilities exist.

However, it is possible to obtain a transfer function whose denominator is derived from Structure A or B and whose numerator is different from those of the two

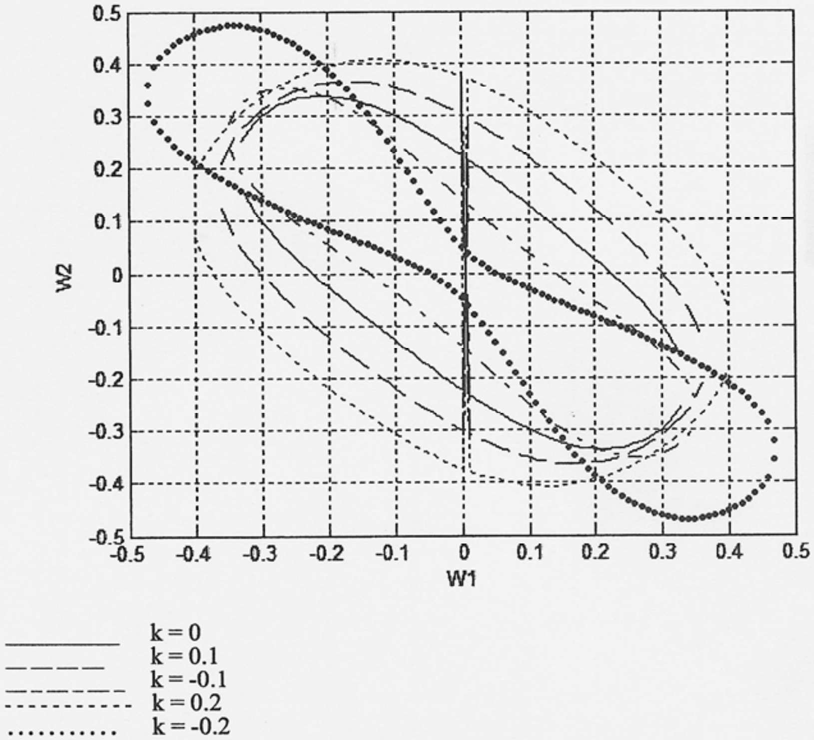


Fig. 13. The cutoff contours of Numerical Example 1 for different values of k.

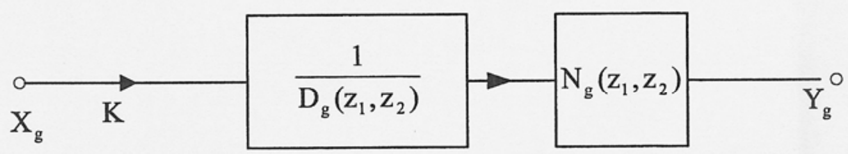


Fig. 14. Signal-flow graph for the realization of the transfer function $H_g(z_1, z_2)$ given in Eq. (34).

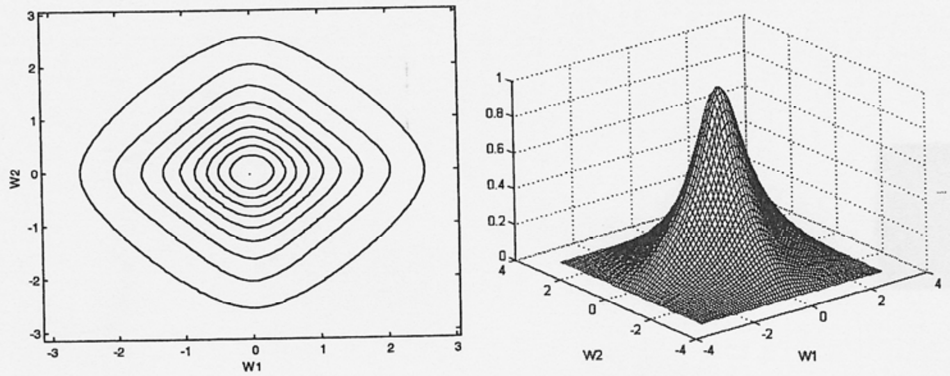
structures. A large number of possibilities exist. A general transfer function of this type can be written as:

$$H_{dg}(z_1, z_2) = \frac{Y_g(z_1, z_2)}{X_g(z_1, z_2)} = \frac{KN_g(z_1, z_2)}{D_g(z_1, z_2)}, \tag{34}$$

where $D_g(z_1, z_2)$ is obtained from the different types considered and can be written, in general, as:

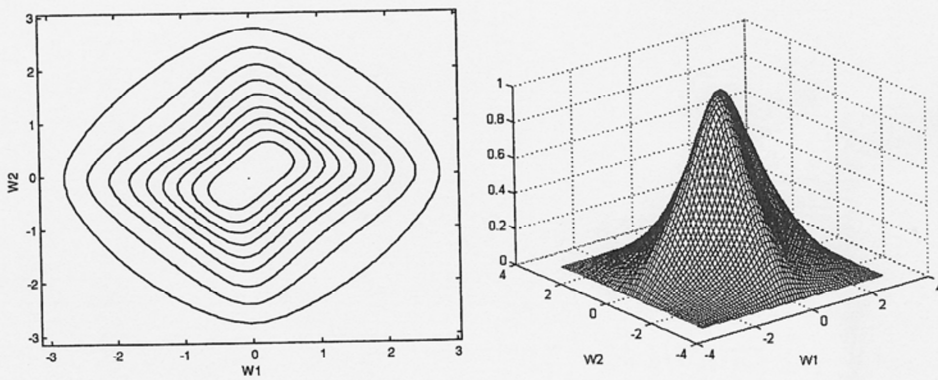
$$D_g(z_1, z_2) = z_1 z_2 + d_{10} z_1 + d_{01} z_2 + d_{00}, \tag{35}$$

where the coefficient d_{10}, d_{01} and d_{00} contain the variable k. Two possible general configurations are given in Figs. 14(a) and 14(b).



(a) (b)

Fig. 15. (a) and (b) The contour and the magnitude plots of the transfer function of Eq. (31), when $k = 0$.



(a) (b)

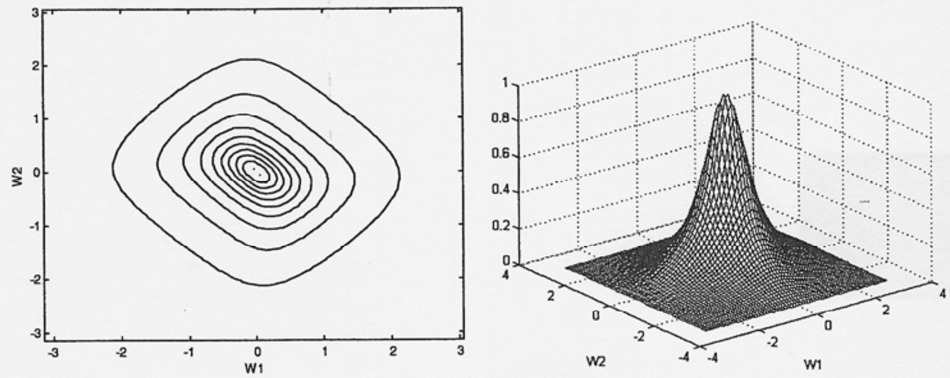
Fig. 16. (a) and (b) The contour and the magnitude plots of the transfer function of Eq. (31), when $k = 0.1$.

The box containing $N_g(z_1, z_2)$ can be realized using a suitable FIR filter and $D_g(z_1, z_2)$ is realized as discussed earlier.

Numerical Example 2. For illustration purposes, we consider a low-pass filter having a denominator given by Type I(a). The transfer function is:

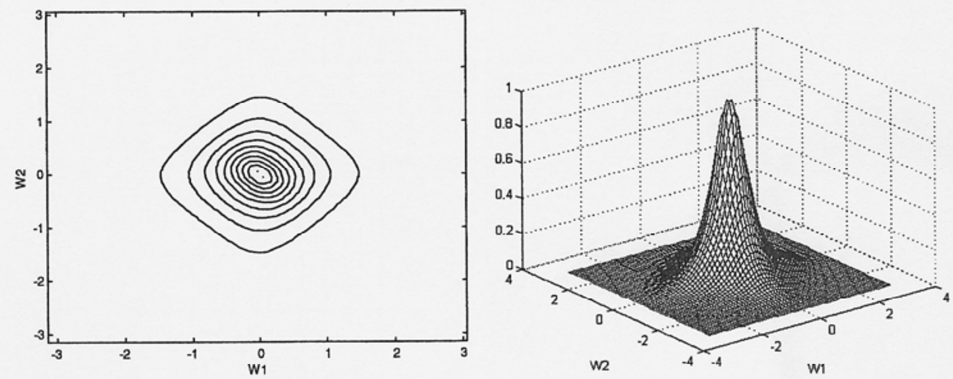
$$H_1(z_1, z_2) = \frac{(z_1 + 1)(z_2 + 1)}{(z_1 - 0.5)(z_2 - 0.5) + k} . \tag{36}$$

Figures 15(a) and 15(b) show, respectively, the magnitude and the contour plots of $H_1(z_1, z_2)$ when $k = 0$. When $k = 0.1$, the magnitude and the contour plots change and they are shown in Figs. 16(a) and 16(b), respectively. When k is made equal to -0.1, the magnitude and the contour plots are as represented in Figs. 17(a)



(a) (b)

Fig. 17. (a) and (b) The contour and the magnitude plots of the transfer function of Eq. (31), when $k = -0.1$.



(a) (b)

Fig. 18. (a) and (b) The contour and the magnitude plots of the transfer function of Eq. (32).

and 17(b), respectively. As has been stated earlier, two realizations can be cascaded. Equation (37) shows the overall transfer function when two realizations are cascaded,

$$H_2(z_1, z_2) = \frac{(z_1+1)^2(z_2+1)^2}{(z_1-0.5)^2(z_2-0.5)^2-(0.1)^2}\,. \tag{37}$$

The corresponding contour and the magnitude plots are shown in Figs. 18(a) and 18(b), respectively.

7. Summary and Conclusions

This paper considers the generation of 2D discrete-domain transfer functions having the following properties: (a) the degree in each variable is unity, and (b) by changing the value of the multiplier coefficient k, the magnitude and the contour characteristics can be altered. The denominator polynomial of the transfer function can be derived by one of the two basic structures considered. Seven different types are possible, depending on the term(s) in which k occurs. In all these types, the starting polynomial remains the same, corresponding to the value $k = 0$. The numerator polynomial of the resulting transfer function can be one of the following: (i) the term(s) in which k occurs, (ii) the starting polynomial and (iii) an arbitrary polynomial depending upon the type of filter desired. For each one of these types, the bounds on k to ensure stability are determined by testing the overall denominator at only four points, namely $z_1 = \pm 1$ and $z_2 = \pm 1$. This aspect of testing stability at only four extreme points proves to be a big advantage, because otherwise the order of numerical complexity increases as the degree of each variable increases. It is also shown that by varying the multiplier value k (within the bounds of stability determined earlier), the magnitude and the contour characteristics can be altered. By cascading several structures, different characteristics can be obtained. Numerical examples have been given to illustrate these concepts.

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