NJUMC – 2006 – Answers to Team 2, 3 and Individual 4, 8

2. a. Consider the function $f : A \to O = \{\text{positive odd integers} < 2006\}$ defined by $f(a) = b$, where $a = 2^s b$ and b is odd. There are 1003 odd integers in O, so by the pigeonhole principal there must exist a_1 and a_2 such that $f(a_1) = f(a_2)$, so a_1 divides a_2 or vice versa.

b. No element of $C = \{a : 1004 \le a \le 2006\}$ is a multiple of another.

3. The answer is 0. If $f(x) = \sqrt[7]{1-x^5}$ then $\sqrt[5]{1-x^7} = f^{-1}(x)$. Both graphs start at $(0,1)$ and decrease to end at $(0,1)$ and intersect at a point (A, A) (How would you find the numerical value of this point?) The region that is between the graphs is symmetric about the line $y = x$. Any portion of this region that is above $c < x < d < A$ and for which $f(x) > f^{-1}(x)$ for all $c < x < d$ will be reflected, via reflection in the line $y = x$, into a region of equal area for which $f(x) < f^{-1}(x)$, and vice versa. (Actually, $f(x) > f^{-1}(x)$ for $x < A$. How could your find this out without a calculator?)

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4. Well, $p^2 - 1 = (p+1)(p-1)$, so one of these factors is divisible by 3, since p is not. And $p = 4k-1$ or $4k-3$ so one of these factors is divisible by 4. The other is divisible by 2..

8. The limit exists because the sequence is decreasing and bounded below. Denoting the desired number by N, $\ln N = -\ln 3(\frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + \cdots) = -\ln 3\sum_{k=1}^{\infty} \frac{k}{5^k}.$ Let $f(x) = \sum_{k=1}^{\infty} (x/5)^k$.

Then $f'(x) = \sum_{k=1}^{\infty} \frac{kx^{k-1}}{5^k}$ .Both are convergent power series for $|x| < 5$, with $f(x)$ a geometric series converging to $(x/5)\frac{1}{1-x/5}$.

Then $f'(x) = \frac{1}{5}\frac{1}{(1-x/5)} + \frac{x}{5}(-1)\frac{1}{(1-x/5)^2}(-\frac{1}{5})$. Therefore

$\ln N = (-\ln 3)f'(1) = (-\ln 3)((1/5)(5/4) + \_(1/25)(25/16)) = -\ln 3 \cdot 5/16$

So N= $(1/3)^{5/16}$