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Fibonacci Imposters

CHRISTOPHER S. SIMONS and MARCUS WRIGHT
Mathematics Department, Rowan University
simons@rowan.edu wright@rowan.edu
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With Simson's 1753 paper as a starting point, we investigate Simson's identity (also known as Cassini's) for the Fibonacci sequence as a means to explore some fundamental ideas about recursion. Simple algebraic operations allow us to reduce the standard linear Fibonacci recursion to the nonlinear Simon's recursion that is equivalent to Simson's identity and then further to a nonlinear recursion dependent only on a single preceding term. This leads to a striking nonrecursive characterization of Fibonacci numbers that is much less well-known than it should be. We then discover that the Simson's recursion itself implies a family of linear recursions and characterizes a class of generalized Fibonacci sequences.

1 Introduction

The first recursive sequence many students encounter is the famous Fibonacci sequence defined by the two initial values $F_1 = F_2 = 1$ and the recursion $F_{n+1} = F_n + F_{n-1}$. For quick reference we list the first few values of this sequence.

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}
1	1	2	3	5	8	13	21	34	55

Among its many celebrated properties is Simson's identity (independently discovered by Cassini)

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n. \tag{1}$$

For example, if $n = 6$, this becomes $13 \times 5 - 8^2 = (-1)^6$ and we see that the square of a Fibonacci number differs by one from the product of the Fibonacci numbers adjacent to it. Simson gave a simple inductive proof of the identity in an interesting 1753 paper [5]. Our goal is to extend Simson's ideas to explore other sequences that behave like the Fibonacci sequence. In doing so we will rediscover some surprising results. Except for the notation, Simson's proof of the identity is essentially the same as those taught today in discrete math, number theory, and history of math courses.

Proof The base case $n = 2$ holds as $2 \times 1 - 1^2 = (-1)^2$. So we assume the identity holds for a given n and we proceed to prove that

$$F_{n+2}F_n - F_{n+1}^2 = (-1)^{n+1}.$$

The left hand side is equivalent to

$$\begin{aligned}(F_{n+1} + F_n)F_n - F_{n+1}^2 &= F_{n+1}F_n + F_n^2 - F_{n+1}^2 \\ &= F_n^2 + F_{n+1}(F_n - F_{n+1}) \\ &= F_n^2 - F_{n+1}F_{n-1} \\ &= -(-1)^n \\ &= (-1)^{n+1}.\end{aligned}$$

□

After proving the identity, Simson proposes using it to reduce the number of preceding terms used in the recursion for the Fibonacci sequence. Starting with Simson's identity and solving for F_{n+1} we get Simson's recursion

$$F_{n+1} = \frac{F_n^2 + (-1)^n}{F_{n-1}}. \tag{2}$$

Using this recursion along with the standard Fibonacci recursion we find that

$$\frac{F_{n+1}^2 + (-1)^{n+1}}{F_n} = F_{n+2} = F_{n+1} + F_n.$$

So that

$$F_{n+1}^2 + (-1)^{n+1} = F_nF_{n+1} + F_n^2.$$

This is equivalent to

$$F_{n+1}^2 - F_nF_{n+1} + (-F_n^2 + (-1)^{n+1}) = 0.$$

Solving for the positive root F_{n+1} using the quadratic formula,

$$F_{n+1} = \frac{F_n + \sqrt{F_n^2 - 4(-F_n^2 + (-1)^{n+1})}}{2} = \frac{F_n + \sqrt{5F_n^2 \pm 4}}{2}, \tag{3}$$

where the $\pm$ is the sign of $(-1)^n$.

Equation (3) tells us (and told Simson) two important things. The first is that we now only need one initial value to start the Fibonacci sequence, since if we know that $F_1 = 1$ then it follows that $F_2 = \frac{1+\sqrt{5\times 1-4}}{2} = 1$ and the recursion begins. The second is that since we are dealing with an integer sequence, $\sqrt{5F_n^2 \pm 4}$ must be an integer. Therefore if a is a Fibonacci number then $5a^2 \pm 4$ must be a square (of an integer) for at least one choice of the $\pm$. A computer search using Mathematica [4] for numbers with this property leads to a natural conjecture.

Conjecture 1.1 *Let a be a positive integer. Then a is a Fibonacci number if and only if at least one of $5a^2 - 4$ or $5a^2 + 4$ is a square.*

For now we define a Fibonacci imposter to be a positive integer a such that one of $5a^2 - 4$ or $5a^2 + 4$ is a square but such that a is not a Fibonacci number. Therefore the conjecture holds exactly if there are no Fibonacci imposters.

2 No Fibonacci Imposters?

It turns out that Conjecture 1.1 is true [2], and its proof is instructive. We prove it by combining the following two lemmas concerning the hyperbolas

$$y^2 - xy - x^2 = \pm 1. \tag{4}$$

Lemma 2.1 *Let x be a positive integer. $5x^2 \pm 4$ is a square if and only if there is a positive integer y such that $y^2 - xy - x^2 = \pm 1$.*

Lemma 2.2 *The positive integer solutions to $y^2 - xy - x^2 = \pm 1$ are Fibonacci pairs $(x, y) = (F_n, F_{n+1})$.*

Lemma 2.2 is described in [6], and given as an exercise in [7](Chapter 3, Section 1, Exercise 32). We now prove the lemmas.

Proof [Lemma 2.1] If x and y are positive integers satisfying Equation (4) then by the quadratic formula (for positive root y)

$$y = \frac{x + \sqrt{x^2 + 4(x^2 \pm 1)}}{2} = \frac{x + \sqrt{5x^2 \pm 4}}{2}. \tag{5}$$

So $5x^4 \pm 4$ is a square. Furthermore if $5x^4 \pm 4$ is a square, then we can choose y to be $\frac{x+\sqrt{5x^2\pm 4}}{2}$ in order to satisfy Equation (4). □

Proof [Lemma 2.2]
Let $(x, y) = (a, b)$ be a positive integer solution to Equation (4). The use of the quadratic formula in Equation (5) shows us that $b \geq a$. If $b = a$ then we have $-a^2 = \pm 1$, so that $b = a = 1$, which is the initial Fibonacci pair. So we now assume that $b > a \geq 1$. Setting $(x, y) = (b - a, a)$ gives a smaller positive integer solution to Equation (4) since $a^2 - a(b-a) - (b-a)^2 = a^2 + ab - b^2 = -(b^2 - ab + a^2) = \mp 1$. So long as $x \neq y$, this procedure can be repeated to get successively smaller integer solutions. However since we are dealing with positive integers, this repetition must terminate with a solution where $y = x$, and we have already seen this can only be the initial Fibonacci pair. Since this reduction implements the standard Fibonacci recursion, we find that all positive integer solutions to Equation (4) must be Fibonacci pairs.

□

We therefore restate Conjecture 1.1 as a theorem.

THEOREM 2.3 *Let a be a positive integer. Then a is a Fibonacci number if and only if at least one of $5a^2 - 4$ or $5a^2 + 4$ is a square.*

Having proven this theorem, we find that under our current definition there are no Fibonacci imposters. In order to make our search nontrivial, we must replace our definition of Fibonacci imposters with one that still satisfies Simson's recursion while yielding more than just the Fibonacci sequence.

3 Fibonacci Imposter Sequences

We now consider the family of all positive integer sequences defined by two initial values $\mathcal{F}_1$, $\mathcal{F}_2$, that satisfy Simson's recursion

$$\mathcal{F}_{n+1} = \frac{\mathcal{F}_n^2 + (-1)^n}{\mathcal{F}_{n-1}}. \tag{6}$$

Clearly the Fibonacci sequence belongs to this family, as do all of its tail sequences obtained by deleting the first $2k$ terms of the Fibonacci sequence. We now define a Fibonacci imposter sequence to be any other member of this family.

Since any such Fibonacci imposter sequence consists only of positive integers a necessary, but not sufficient, condition for such a sequence is for

$$\mathcal{F}_3 = \frac{\mathcal{F}_2^2 + 1}{\mathcal{F}_1}$$

to be a positive integer. The easiest way to ensure this is to set $\mathcal{F}_1 = 1$.

If $\mathcal{F}_2 = 1$ as well, we get the Fibonacci sequence.

If $\mathcal{F}_2 = 2$, we a get a sequence starting with

1 2 5 12 29 70 169 408 985 2378.

In fact this is known as the Pell sequence, and it also appears to satisfy the much simpler linear recursion $\mathcal{F}_{n+1} = 2\mathcal{F}_n + \mathcal{F}_{n-1}$.

If instead $\mathcal{F}_2 = 3$, we get a sequence starting with

1 3 10 33 109 360 1189 3927 12970 42837.

This sequence also appears to satisfy a simple linear recursion $\mathcal{F}_{n+1} = 3\mathcal{F}_n + \mathcal{F}_{n-1}$.

In general if $\mathcal{F}_1 = 1$ and $\mathcal{F}_2 = a$ then Simson's recursion (Equation (6)) implies that the much simpler linear recursion

$$\mathcal{F}_{n+1} = a\mathcal{F}_n + \mathcal{F}_{n-1} \tag{7}$$

holds. Before we prove this claim, we note that something remarkable is occurring. Not only have we found some Fibonacci imposter sequences, but we see that the very nonlinear Simson's recursion (Equation (6)) behaves in a surprisingly linear way.

Proof We prove Equation (7) by a simple induction. It holds in the base case $n = 2$ since $\mathcal{F}_3 = \frac{\mathcal{F}_2^2+1}{\mathcal{F}_1} = a^2 + 1 = aa + 1 = a\mathcal{F}_2 + \mathcal{F}_1$. So we assume that Equation (7) holds for a given n and proceed to that find

$$\begin{aligned}
\mathcal{F}_{n+2} &= \frac{\mathcal{F}_{n+1}^2 + (-1)^{n+1}}{\mathcal{F}_n} \\
&= \frac{(a\mathcal{F}_n + \mathcal{F}_{n-1})\mathcal{F}_{n+1} - (-1)^n}{\mathcal{F}_n} \\
&= \frac{a\mathcal{F}_n\mathcal{F}_{n+1} + \mathcal{F}_{n-1}\mathcal{F}_{n+1} - (-1)^n}{\mathcal{F}_n} \\
&= \frac{a\mathcal{F}_n\mathcal{F}_{n+1} + \mathcal{F}_n^2}{\mathcal{F}_n}
\end{aligned} \tag{8}$$

$$= a\mathcal{F}_{n+1} + \mathcal{F}_n$$

as desired. □

We conclude by asking if we have now found all Fibonacci imposter sequences. A quick machine investigation using Mathematica [4] yields no small counterexamples so we make the following conjecture.

CONJECTURE 3.1 *The infinite positive integer sequences satisfying the Simson's recursion (Equation (6)) are precisely those linear recursive sequences defined by $\mathcal{F}_1 = 1$, $\mathcal{F}_2 = a$, $\mathcal{F}_{n+1} = a\mathcal{F}_n + \mathcal{F}_{n-1}$ or their tails obtained by obtained by deleting the first $2k$ terms.*

We have in fact almost proven this conjecture already.

Proof Since we want Equation (7), if we set $n = 2$ and solve for a we must have

$$a = \frac{\mathcal{F}_3 - \mathcal{F}_1}{\mathcal{F}_2} = \frac{(\mathcal{F}_2^2 + 1)/\mathcal{F}_1 - \mathcal{F}_1}{\mathcal{F}_2} = \frac{\mathcal{F}_1^2 + \mathcal{F}_2^2 - 1}{\mathcal{F}_1\mathcal{F}_2} \tag{9}$$

This guarantees that the base case of inductive proof for Equation (7) holds. The inductive step follows exactly as in Equation (8). We must now prove two things.

We must first verify that $a = \frac{p}{q}$, with p and q relatively prime, is a positive integer. It is positive by its definition in Equation (9). Since

$$\mathcal{F}_{n+1} = \frac{p}{q}\mathcal{F}_n + \mathcal{F}_{n-1}$$

and all the terms are integers, we find that $q \mid \mathcal{F}_n$ for all $n \geq 2$. (In fact if we repeat this process inductively we even find that $q^n \mid \mathcal{F}_{n+1}$ for all $n \geq 0$). However Simson's recursion (Equation (6)) implies that the the greatest common divisor of $\mathcal{F}_n$ and $\mathcal{F}_{n+1}$ is 1. Therefore $q = 1$ and a is a positive integer.

Finally we must verify that $\mathcal{F}_1 = 1$ or that if the sequence is extended backwards using the linear recursion (Equation (7)) the first positive term is 1. So assume the sequence has been extended backwards and that $\mathcal{F}_{k+1}$ is the first positive term (note that $k \leq 0$). We then find that $\mathcal{F}_k = \mathcal{F}_{k+2} - a\mathcal{F}_{k+1} \leq 0$ and that $\mathcal{F}_{k+2}\mathcal{F}_k - \mathcal{F}_{k+1}^2 = (-1)^{k+1}$ because Simson's identity continues to hold for this extension as can be verified by modifying the proof of Simson's identity. However since the integers $\mathcal{F}_{k+1}, \mathcal{F}_{k+2} > 0$ and $\mathcal{F}_k \leq 0$, we find that k is even and $\mathcal{F}_{k+1} = 1$ (while $\mathcal{F}_k = 0$ and so by the linear recursion $\mathcal{F}_{k+2} = a$). □

We therefore restate Conjecture 3.1 as a theorem.

THEOREM 3.2 *The infinite positive integer sequences satisfying the Simson's recursion (Equation (6)) are precisely those linear recursive sequences defined by* $\mathcal{F}_1 = 1$, $\mathcal{F}_2 = a$, $\mathcal{F}_{n+1} = a\mathcal{F}_n + \mathcal{F}_{n-1}$ *or their tails obtained by obtained by deleting the first* $2k$ *terms.*

4 Generalized Fibonacci Sequences

Given these linear recursion properties, it is clear that these so-called Fibonacci imposter sequences are not really imposters, but are in fact generalizations of the Fibonacci sequence. They are a specialization of the well-known generalized Fibonacci sequences [1,3] with linear recursion

$$\mathcal{F}_{n+1} = a\mathcal{F}_n + b\mathcal{F}_{n-1}$$

for positive integers a and b and initial terms $\mathcal{F}_1 = 1$ and $\mathcal{F}_2 = a$. These can alternately be defined by the generalized Simson's identity

$$\mathcal{F}_{n+1}\mathcal{F}_{n-1} - \mathcal{F}_n^2 = (-1)^n b^{n-1}.$$

and its corresponding quadratic recursion. Our hunt for impostors considers the case where $b = 1$, however by modifying our proofs as needed the reader may obtain corresponding results and continue the search for generalized Fibonacci imposters.

Reference

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