New Jersey Undergraduate Mathematics Contest
Spring 2006
Individual Part

No calculators. Justify all answers. 60 minutes. Place your team number and individual letter in the upper-left corner of all pages. Clearly indicate the question number for each of your solutions.

1. Find the 2006th digit (after the decimal point) in the decimal representation of $\frac{1}{7}$.
2. How many of the integers between 1 and 1000, inclusive, can be expressed as the difference of the squares of two nonnegative integers?
3. A group of 5 people contains 2 women and 3 men. Three people are chosen at random from the group. What is the probability that both women were selected?
4. Prove that if p is a prime larger than 3 then $p^2 - 1$ is divisible by 24.
5. Sarah intended to multiply a two-digit number and a three-digit number, but she left out the multiplication sign and simply placed the two-digit number to the left of the three-digit number, thereby forming a five-digit number. This number is exactly nine times the product Sarah should have obtained. What is the sum of the two-digit number and the three-digit number?
6. Consider the sequence recursively defined by $L_0 = 2, L_1 = 7, L_{n+1} = L_{n-1} + L_n$. Prove that L_n and L_{n+2} are relatively prime for $n \geq 0$.
7. The currency of country A has coins worth 7 cents and 11 cents. Find the largest purchase price that cannot be paid exactly using these two coins.
8. Evaluate $\left(\frac{1}{3}\right)^{1/5}\left(\frac{1}{9}\right)^{1/25}\left(\frac{1}{27}\right)^{1/125}\left(\frac{1}{81}\right)^{1/625}\cdots$.
9. Let N be the product of all positive divisors of 10^5. Find $\log_{10}(N)$.
10. The function f defined by $f(x) = \frac{ax+b}{cx+d}$, where a, b, c and d are nonzero real numbers, has the properties $f(19) = 19$, $f(97) = 97$ and $f(f(x)) = x$ for all values of x except $-d/c$. Find the unique number that is not in the range of f.
11. Find the minimum values of a such that every solution (x, y) of $2x^2 + axy + 2y^2 = 1$ satisfies $x^2 + y^2 \leq 1$.
12. Calculate $\lim_{x\to\infty} \int_0^x xe^{t^2 - x^2}\, dt$.

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Team Part

No calculators. Justify all answers. 90 minutes. Each question should be answered on a separate page, with your team number in the upper-left corner and the question number in the upper-right corner.

1. Let $A_1, A_2, \ldots, A_n$ be the consecutive vertices of a regular n-sided polygon. Let B be a point in the exterior of the polygon such that A_1, A_2, B is an equilateral triangle. What is the largest value of n for which A_n, A_1, B are consecutive vertices of a regular polygon?

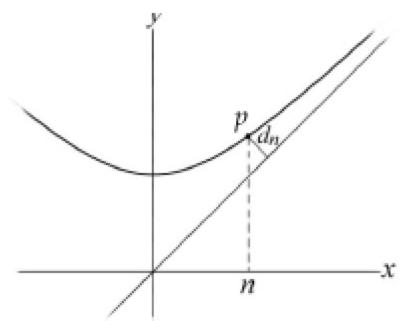
2. Let $S = \{1, 2, \ldots, 2006\}$ and A be a subset of S. Let $|A|$ denote the number of elements in A.

   a. Show that is $|A| > 1003$, then there are two distinct elements $x, y \in A$ such that x divides y.
   b. Show by example that if $|A| \leq 1003$ then there may not be two distinct elements $x, y \in A$ such that x divides y.

3. Find the value of $\int_0^1 \sqrt[7]{1-x^5} - \sqrt[5]{1-x^7}\,dx$. Justify your answer.

4. Suppose $a_{n+1} = 2(a_n + b_n), b_{n+1} = 2b_n$ for all $n \in \mathbb{N}$. If $a_{n+3} = xa_n + yb_n$ and $b_{n+3} = za_n + wb_n$. Find x, y, z, w.

5. For each positive integer n, consider the point p, with x-coordinate n, on the curve $y^2 - x^2 = 1$. If d_n respresents the shortest distance from point p to the line $y = x$, find $\lim_{n\to\infty}(n \cdot d_n)$.



6. Let $\alpha_n = (2+\sqrt{3})^n$. Show that $\lim_{n\to\infty}\left(\alpha_n - \lfloor \alpha_n \rfloor\right) = 1$.