Monster Roots

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1. Introduction

It is now known **[Nor] [Iva]** that the *Bimonster*, or wreathed square $\mathbb{M} \wr 2$ of the Monster group $\mathbb{M}$, is presented by the single relation

$$(ab_1c_1ab_2c_2ab_3c_3)^{10} = 1 \qquad (1)$$

in addition to the Coxeter relations of the $\mathbb{M}_{666}$ diagram (Figure 1).

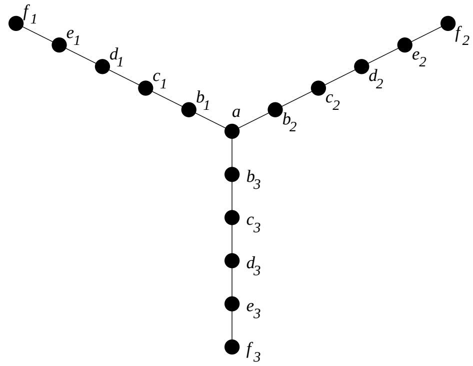

Figure 1: $\mathbb{M}_{666}$ diagram.

In the Atlas **[Co1]** and previous papers on this subject the group generated by a together with

the first p of b_1, c_1, d_1, e_1, f_1

the first q of b_2, c_2, d_2, e_2, f_2

the first r of b_3, c_3, d_3, e_3, f_3

is called Y_{pqr}. However, since this group appears to be more naturally associated with the parameters $p+1, q+1, r+1$, we introduce the alternate notation $\mathbb{M}_{p+1,q+1,r+1}$.

The Coxeter group $c\mathbb{M}_{666}$ defined by the $\mathbb{M}_{666}$ diagram is a hyperbolic reflection group (a group generated by reflections in hyperplanes containing the origin in a Lorentzian space), so the Norton-Ivanov theorem **[Nor] [Iva]** shows that the Bimonster is a homomorphic image of this very concretely geometrical group.

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Each element of the Bimonster has the form

$$(x, y) \text{ or } (x, y)\tau = \tau(y, x) \text{ with } \tau^2 = 1$$

where x, y range over $\mathbb{M}$ and τ is the wreathing involution. The conjugacy class of τ consists of all the elements $(x, x^{-1})\tau = \tau_x$ and so has size $|\mathbb{M}|$. We call the elements of this class the *reflection elements* of $\mathbb{M} \wr 2$, since they are the images of the reflections in the Coxeter group.

In the geometric representation, the generators of the $\mathbb{M}_{666}$ diagram correspond to reflections in certain vectors $r_1, \ldots, r_{16}$. Following **[Co2]** we call these the *fundamental Monster roots* and use the term *Monster roots* for their images under the Coxeter group, $c\mathbb{M}_{666}$ they determine. A Monster root serves as a name for the appropriate reflection element of $\mathbb{M}_{666}$. Conversely, each reflection element has infinitely many such names.

The aim of this paper is to explore this naming relationship: When do two Monster root vectors r, s name the same element of $\mathbb{M}_{666}$? A complete solution to this question, which may be within reach, would provide a very simple way to compute with the Monster. As a partial solution, tables of some of these relations appear in the appendices. These tables have proven to be very useful for further investigations of the Bimonster and the Monster.

2. Coordinate System and Roots

In order to enumerate the Monster roots and their equivalences we need coordinate systems. We will use two different coordinate systems, both taken from **[Co2]**. The setting of this section is $c\mathbb{M}_{666}$.

Often we will use $+$ to denote 1 and $-$ to denote -1.

In *System 1* we have a space of 19 coordinates

$$\begin{matrix} a & b & c & d & e & f \\ g & h & i & j & k & l \\ m & n & o & p & q & r \end{matrix} \quad t \text{ with quadratic form } a^2 + \cdots + r^2 - t^2. \tag{2}$$

In this system the fundamental Monster roots are as indicated in Figure 2. All the vectors satisfy the following relations.

$$a + b + c + d + e + f = t$$
$$g + h + i + j + k + l = t$$
$$m + n + o + p + q + r = t$$

Thus t is redundant, and we shall sometimes omit it. We call it the *type*. On occasion we use the notation

$$a \quad b \quad c \quad d \quad e \quad f \quad | \quad g \quad h \quad i \quad j \quad k \quad l \quad | \quad m \quad n \quad o \quad p \quad q \quad r\,.$$

For *System 2* we have a space of 18 coordinates

$$\begin{matrix} a & b & c & d & e & f \\ g & h & i & j & k & l \\ m & n & o & p & q & * \end{matrix} \quad s \text{ with quadratic form } a^2 + \cdots + q^2 - 5s^2. \tag{3}$$

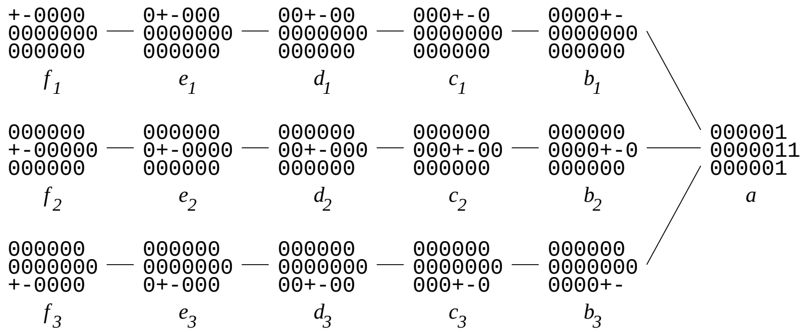

Figure 2: The fundamental Monster roots in System 1.

In this system the fundamental Monster roots are as indicated in Figure 3. All the vectors satisfy the following relations.

$$a+b+c+d+e+f+g+h+i+j+k+l=6s$$
$$m+n+o+p+q=s$$

Thus s is redundant, and we shall sometimes omit it. We call s the *size*. On occasion we use the notation

$$a \quad b \quad c \quad d \quad e \quad f \quad l \quad k \quad j \quad i \quad h \quad g \quad | \quad m \quad n \quad o \quad p \quad q\,.$$

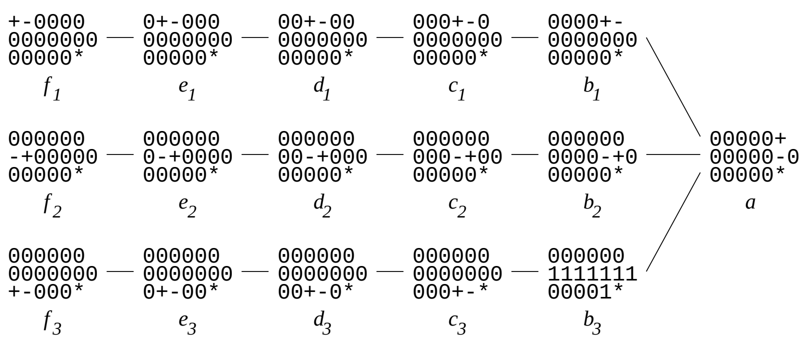

Figure 3: The fundamental Monster roots in System 2.

The two systems are linearly equivalent. Just let

$$\begin{matrix} a & b & c & d & e & f \\ g & h & i & j & k & l \\ m & n & o & p & q & r \end{matrix} \quad t \;=\; \begin{matrix} a & b & c & d & e & f \\ g' & h' & i' & j' & k' & l' \\ m & n & o & p & q & * \end{matrix} \quad s \tag{4}$$

where

$$g+g'=h+h'=i+i'=j+j'=k+k'=l+l'=s$$
$$m+n+o+p+q=s=t-r.$$

There really are 36 different System 2 coordinate systems determined by placement of the star and choice of row to be inverted in equation (4). We can indicate this choice of row to be inverted by replacing $*$ by $\uparrow$ or $\downarrow$ which (cyclically) points at the row to be inverted. As a convention we choose $*$ to have the same meaning as $\uparrow$. In $\mathbb{M}_{666}$ this choice has no effect. Looking at roots from the point of view of different coordinate systems is of great use for our computations.

We are now able to consider the (practical) enumeration of Monster roots. This theory is briefly discussed in **[Co2]**, but we shall provide a more detailed treatment of it.

We start with the fundamental Monster roots $r_1, \ldots, r_{16}$. These satisfy $(r_i, r_i) = 2$ for all i and $(r_i, r_j) = 0$ or -1 for all $i \neq j$. Since every Monster root r has $(r, r) = 2$ it follows that the formula for a reflection in r is $x \to x - (x, r)r = x^r$. This implies that every Monster root has integral coordinates in both systems.

By the theory of Coxeter groups **[Hum**, Section 5.13**]** we know that the reflecting hyperplanes of $c\mathbb{M}_{666}$ divide the hyperbolic space into copies of what is called the fundamental region, which we take to be the intersection of the halfspaces defined by $(r_i, x) < 0$. We now choose a point w inside the fundamental region, so that $(w, w) < 0$ and $(r_i, w) < 0$ for all i. In hyperbolic space **[Vin]** the distance d between w and $r^\perp$ can be defined such that

$$\sinh d = \frac{|(w, r)|}{\sqrt{|(w, w)||(r, r)|}}. \tag{5}$$

Therefore we can use (5) as an indicator of distance between $r^\perp$ and w. If a vector r is such that $(r, w) < 0$ while for some i $(r, r_i) > 0$, then $r^{r_i \perp}$ is necessarily closer to w than $r^\perp$ is. Therefore $|(w, r^{r_i})| < |(w, r)|$. We can use induction on $|(r, w)|$ (noting that w can be chosen to have rational coordinates) to obtain the following test for whether a given vector r is a Monster root.

Test 2.1. *First check if $(r, r) = 2$ and that the coordinates of r are integral. Then repeat the following steps:*

1. *If $r \in \{\pm r_1, \ldots, \pm r_{16}\}$ then r is a Monster root.*
2. *Otherwise replace r by r^{r_i} for i such that (r, r_i) and (r, w) have opposite signs. If no such i exists then r is not a Monster root.*

In System 1, reflections in the fundamental Monster roots other than a just interchange two coordinates and therefore generate the *coordinate permutation group* $S_6 \times S_6 \times S_6$ where each S_6 permutes the coordinates within a row. The 18 coordinates $a, \ldots, r$ are divided into 3 *blocks* of 6 coordinates each. Since Monster roots are integral and of norm 2, the Monster roots of type $t = 0$ or $t = 1$ are precisely those of form

$$0^4 + -|0^6|0^6 \text{ or } 0^5 1|0^5 1|0^5 1.$$

To test an integral norm 2 vector v of type $t > 1$ we first check that it satifies the relations that follow equation (2). Taking for example w close to $1^6|1^6|1^6$, we find that if r is a type 1 Monster root then $(w, r) < 0$. So the above procedure enables us to replace v by its reflection in any such r with $(v, r) > 0$. This inner product cannot be

$\geq t$, since supposing, for example, that

$$v = \begin{matrix} a & b & c & d & e & f \\ g & h & i & j & k & l \\ m & n & o & p & q & r \end{matrix} \quad t \text{ and } r = \begin{matrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{matrix} \quad 1$$

then by the following argument we have that the inner product $c + k + n - t < t$. We know that $c^2 + k^2 + n^2 - t^2 \leq 2$, and since all coordinates are integers $t \geq 2$. The equality $c^2 + k^2 + n^2 - t^2 = 2$ is inconsistent with our other assumptions, therefore we have $c^2 + k^2 + n^2 - t^2 \leq 1$. This implies $3c^2 + 3k^2 + 3n^2 - 3t^2 \leq 3$, so that $(c + k + n)^2 < 4t^2$. Taking square roots we get $c + k + n < 2t$ and this is equivalent to the desired inequality. Therefore replacing v by v^r reduces to a case of smaller positive type. Monster roots of negative type are precisely the negatives of those of positive type. As a bonus we can inductively show that in System 1 for nonzero type Monster roots, all coordinates have the same sign as their type.

Similar results hold in System 2. The reflections in the fundamental Monster roots other than b_3 just interchange two coordinates and therefore generate the *coordinate permutation group* $S_{12} \times S_5$ where the S_{12} acts on the coordinates in the first two rows and the S_5 acts on the coordinates of the starred third row. The 17 coordinates $a, \ldots, q$ are divided into 2 *blocks*: one of 12 coordinates and the other of 5 coordinates. The Monster roots of size $s = 0$ or $s = 1$ are precisely of form

$$0^{10} + -|0^5 \text{ or } 0^{12}|0^3 + - \text{ or } 0^6 1^6|0^4 1$$

(where the $|$ separates the two coordinate blocks). As in System 1, to test an integral norm 2 vector v of size $s > 1$ we first check that that the relations following equation (3) are satisfied. We then simply replace v by its reflection in any root r of size $s = 1$ with which it has positive inner product. It is easy to prove that this inner product is $< s$, so this test works and in System 2 for nonzero size Monster roots, all coordinates have the same sign as their type. Monster roots of negative size are precisely the negatives of those of positive size.

These facts allow us to enumerate the Monster roots by type in System 1 and by size in System 2. We define the *ancestor* of a Monster root r to be its reflection in a type or size 1 root vector v having maximal inner product (v, r). The ancestor of a root is unique up to the coordinate permutation group. Therefore, after sorting by the coordinate permutation group, the Monster roots acquire tree structures for System 1 and System 2. These structures allow us to eliminate duplication from our enumeration algorithm. Appendix A contains an initial segment of the (infinite) enumerations of the Monster roots in both System 1 and System 2. Naming conventions are also introduced.

3. Axiomatics

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In **[Co2]** $\mathbb{M}_{666}$ was defined to be the minimal group other than S_{17} that possesses an S_5-subgroup S whose centralizer is a subgroup S_{12} in which a 7-point stabilizer is conjugate to S. We call this the $S_{5,12}$ axiom. It has the appearance of being a very powerful and difficult axiom. We shall give a simpler, more geometric, axiom.

Let δ_i be the central involution of the Coxeter group $W(D_8)$ generated by a, b_1, b_2, b_3, c_i, d_i, e_i and f_i. On the space spanned by its D_8 roots, δ_i acts as negation.

Axiom 3.1 (D_8). $\mathbb{M}_{666}$ is the Coxeter group determined by the $\mathbb{M}_{666}$ diagram (Figure 1) with the added relations: $1 = \delta_1 = \delta_2 = \delta_3$.

In fact, by a result of Soicher **[Soi]** and some of the following results, it is known that the relation $1 = \delta_1$ implies the other two. We assume that $\mathbb{M}_{666}$ has order > 2.

Theorem 3.1. *The $S_{5,12}$ axiom implies the D_8 axiom.*

Proof. This proof is drawn in Figure 4. Assume the $S_{5,12}$ axiom. Conway and Pritchard **[Co2]** showed that the group obtained is necessarily a quotient of $c\mathbb{M}_{666}$. Without loss of generality take δ to be δ_1. Then δ centralizes the S_5 generated by f_1, e_1, d_1, c_1 so lies in the S_{12} generated by $f_2, e_2, d_2, c_2, b_2, a, b_3, c_3, d_3, e_3, f_3$. But δ centralizes the 3 S_4 subgroups $\langle d_2, e_2, f_2\rangle$, $\langle b_2, a, b_3\rangle$ and $\langle d_3, e_3, f_3\rangle$. It follows that δ is the identity. □

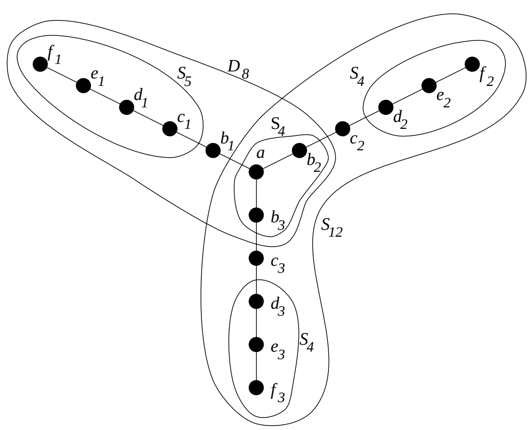

Figure 4: Proof of theorem 3.1.

We now deduce some further relations from the D_8 axiom.

Let δ be δ_1 and ε be the central involution of $W(E_8)$, where E_8 is as indicated in Figure 5. We note that both live in the Weyl group of an extended E_8, $\tilde{E}_8$ and that ε acts as negation on the space spanned by the E_8 roots.

We consider the element $\delta\varepsilon$.

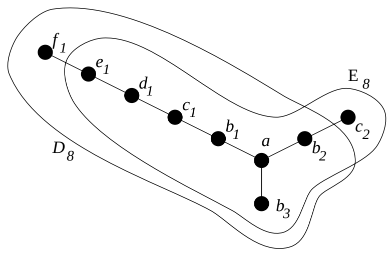

Figure 5: $\tilde{E}_8$**.**

Using the general theory of (affine) Coxeter groups **[Hum]** we introduce the following euclidean coordinates.

f_1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
e_1	$-$	$+$	0	0	0	0	0	0
d_1	0	$-$	$+$	0	0	0	0	0
c_1	0	0	$-$	$+$	0	0	0	0
b_1	0	0	0	$-$	$+$	0	0	0
a	0	0	0	0	$-$	$+$	0	0
b_2	0	0	0	0	0	$-$	$+$	0
c_2	0	0	0	0	0	0	$-$	$+$
b_3	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

The element $\delta\varepsilon$ clearly fixes the space spanned by the 7 roots in the generating sets of both D_8 and E_8. How does $\delta\varepsilon$ act on the remaining fundamental E_8 root c_2? Since the 7 other fundamental roots are fixed, c_2 must go to a vector of the form

$$a \quad a \quad a \quad a \quad a \quad a \quad (a-1) \quad (3a+1).$$

However the norm must be 2 and $a \,\epsilon\, \frac{1}{2}\mathbb{Z}$, so $a = 0$ and c_2 must be fixed. Recall that $W(\tilde{E}_8) \cong \mathbb{Z}^8 \colon W(E_8)$. Thus we have seen that the action of $\delta\varepsilon$ in $\mathbb{Z}^8 \colon W(E_8)/\mathbb{Z}^8$ is trivial. Therefore $\delta\varepsilon$ is a translation.

Looking at the fundamental chamber of the affine $\tilde{E}_8$ we see that this translation is a translation by twice r where r is such that $(r, f_1) = 1$ and all inner products with other fundamental D_8 roots are 0. It follows that r is $-\frac{1}{4}^{7} - \frac{3}{4}$. Therefore $\delta\varepsilon$ is a translation by a norm 4 vector. We have proven the following lemma.

Lemma 3.1. *We have* $\delta\varepsilon = t_{v_4}$ *where* t_{v_4} *is a translation by a norm* 4 *vector,* v_4*, of* E_8.

Theorem 3.2. *The* D_8 *axiom implies that the* E_8 *central involution and all translations in* E_8 *are trivial in* $\mathbb{M}_{666}$.

Proof. From the D_8 axiom we have $\delta = 1$ and so $\varepsilon = t_{v_4}$. But the norm 4 vectors of E_8 are all conjugate **[Co3]**. So for any norm 4 vector, v, of E_8 we have $\varepsilon = t_v$. Choose norm 4 E_8 vectors v, v', and v'' such that $v + v' + v'' = 0$. We have $\varepsilon = t_v$, $\varepsilon = t_{v'}$ and $\varepsilon = t_{v''}$, so $\varepsilon^3 = t_v t_{v'} t_{v''} = 1$. Hence $\varepsilon = 1$. This also implies that the translation by any norm 4 E_8 vector is trivial. The norm 4 vectors of E_8 span E_8 **[Co3]**. Therefore all translations by E_8 vectors are trivial. □

Obviously the above results hold for all the D_8 and E_8 subdiagrams of the $\mathbb{M}_{666}$ diagram.

Using the general theory of affine and hyperbolic groups and since

$$\omega = \begin{array}{ccccccc} 1 & 1 & 1 & 1 & 1 & 1 & \\ 0 & 0 & 0 & 2 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 & 3 & 3 & \end{array} \tag{6}$$

spans the radical, $\tilde{E}_8^{\perp}$, of its $\tilde{E}_8$ lattice, we have the following theorem.

Theorem 3.3. *If r is any root vector of the $\tilde{E}_8$ corresponding to ω then $r \equiv r + m\omega \equiv -r$ for all m in $\mathbb{Z}$. In particular, $r \equiv \omega - r$.*

As an example we have

$$\begin{array}{ccccccc} 1 & 1 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 & \end{array} \equiv \begin{array}{ccccccc} 0 & 0 & 0 & 1 & 1 & 1 & \\ 0 & 0 & 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 2 & 1 & \end{array}. \tag{7}$$

In fact this relation (together with the five other ones corresponding to different choices of $\tilde{E}_8$) suffices for the computations in **[Co2]**.

4. Conjugacy Classes

We briefly discuss conjugacy in $\mathbb{M} \wr 2$ and $\mathbb{M}_{666}$.

$\mathbb{M} \wr 2$**.** We reduce the problem of conjugacy in $\mathbb{M} \wr 2$ to that of conjugacy in $\mathbb{M}$. It is easy to see that

$$(x, y)\tau \sim (x', y')\tau \text{ just if } xy \sim x'y'$$
$$(x, y) \sim (x', y') \text{ just if } x \sim x', y \sim y' \text{ or } x \sim y', y \sim x'.$$

It follows that the conjugacy classes of $\mathbb{M} \wr 2$ are determined by

1. a single conjugacy class of $\mathbb{M}$ or
2. an unordered pair of conjugacy classes of $\mathbb{M}$.

In the first case (the *odd* case) we may take an element of the form $(c, 1)\tau$ as representative. In the second case (the *even* case) we may take an element of the form (c_1, c_2) as representative.

$\mathbb{M}_{666}$**.** We identify some conjugacy classes of $\mathbb{M}_{666}$ which we call $1A$, $2A$, $3A$ and $4A$.

We define $1A$ to be the class of the identity element. $2A$ is the class of the product of the two reflection elements

$$\begin{array}{ccccccc} + & - & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \end{array} \cdot \begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \end{array}. \tag{8}$$

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$3A$ is the class of the product of the two reflection elements

$$\begin{array}{cccccccccccccc} 0 & 0 & 0 & 0 & + & - & & 0 & 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \cdot & 0 & 0 & 0 & 0 & 0 & 1 & 1\,. \\ 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 1 & \end{array} \tag{9}$$

$4A$ is the class of the product of the two reflection elements

$$\begin{array}{cccccccccccccc} 0 & 0 & 0 & 0 & 1 & 1 & & 1 & 1 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 \cdot & 1 & 1 & 0 & 0 & 0 & 0 & 2\,. \\ 0 & 0 & 0 & 0 & 1 & 1 & & 1 & 1 & 0 & 0 & 0 & 0 & \end{array} \tag{10}$$

$5A$ is the class of the product of the two reflection elements

$$\begin{array}{cccccccccccccc} 0 & 0 & 0 & 1 & 1 & 2 & & 0 & 2 & 1 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 1 & 2 & 4 \cdot & 0 & 2 & 1 & 1 & 0 & 0 & 4\,. \\ 0 & 0 & 0 & 1 & 1 & 2 & & 0 & 0 & 1 & 0 & 2 & 1 & \end{array}$$

$6A$ is the class of the product of the two reflection elements

$$\begin{array}{cccccccccccccc} 0 & 0 & 0 & 1 & 1 & 2 & & 0 & 2 & 1 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 1 & 2 & 4 \cdot & 0 & 2 & 1 & 1 & 0 & 0 & 4\,. \\ 0 & 0 & 0 & 1 & 1 & 2 & & 0 & 0 & 0 & 2 & 1 & 1 & \end{array}$$

In fact only $1A$, $2A$, $3A$ and $4A$ are required for the purposes of this paper, but $5A$ and $6A$ are defined here for general reference.

We note immediately that any product of two orthogonal fundamental reflection elements is in $2A$ and any product of any two nonorthogonal fundamental reflection elements is in $3A$. This follows since in $c\mathbb{M}_{666}$ any two fundamental roots act as transpositions in some S_{12}. We extend this to all pairs of roots.

Theorem 4.1. *In $c\mathbb{M}_{666}$ any two roots with inner product ± 1 or 0 are roots of the same fundamental region.*

Corollary 4.1. *If two roots have inner product 0 then the product of their reflections is in $2A$. If two roots have inner product ± 1 then the product of their reflections is in $3A$.*

Proof of theorem. Let r_1 and r_2 be the two roots and w_1 and w_2 the corresponding walls. The two walls meet in a space of codimension 2 which is a facet of some fundamental region. The angle between w_1 and w_2 is filled with (say) m copies of the fundamental region. By our hypothesis this angle must be $\pi/3$ or $\pi/2$. But any two walls of the fundamental region meet at one of these two angles, so $m = 1$. □

This proves that the elements of $2A$ have order 2 and that the elements of $3A$ have order 3. It is possible to show, using the root and alias tables (Appendices A and B) with a few elementary calculations, that $4A$ is of order 4. We are using our assumption that $\mathbb{M}_{666}$ is of order > 2.

We intend to use the Atlas **[Co1]** notation, of the form nX, for the conjugacy classes of $\mathbb{M}$ and also for the corresponding classes (nX, nX^{-1}) of $\mathbb{M} \wr 2$. Using the fact that $\mathbb{M}_{666} = \mathbb{M} \wr 2$, we can verify that $2A$ and $3A$ are appropriately named. We do this by using the Monster power maps **[Co1]** to map conjugacy classes of $\frac{1}{2}(S_{12} \times S_5)$ to conjugacy classes of $\mathbb{M}$.

5. Alias Groups

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In equation (7) we saw two root vectors whose reflections are equivalent in $\mathbb{M}_{666}$,

$$\begin{matrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{matrix} \; 3 \equiv \begin{matrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 1 \end{matrix} \; 3\,.$$

We say that each of these roots is an *alias* for the other. This equivalence can be conjugated by elements of the coordinate permutation group $S_6 \times S_6 \times S_6$ to obtain further equivalences, for example we have

$$\begin{matrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 & 0 \end{matrix} \; 3 \equiv \begin{matrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 & 0 \end{matrix} \; 3\,.$$

This family of equivalences can be summarized as $0^3 1^3|0^3 1^3|0^4 12 \equiv 1^3 0^3|0^3 1^3|0^4 21$. We express this by saying that the digit permutation $(01)| \sim |(12)$ preserves the root vector.

For Appendix B one needs to compute similar equivalences for other roots, such as $0^2 12^2 4^2 6^2 78^2|0^3 4^2$. These are the interchange of 1 and 7, and the alternating group A_6 on 0, 2, 4, 6, 8, $\{1,7\}$. As an example, an equivalent root vector is $2^2 80^2 4^2 6^2 817|0^3 4^2$.

It is therefore natural to define a *digit* of a root to be the equivalence class of coordinate points in the same block with the property that any pair can be interchanged without changing the corresponding reflection element of $\mathbb{M}_{666}$. We define the *(small) alias group* of the root to be the group of block preserving digit permutations which preserve the root in $\mathbb{M}_{666}$. The *large* alias group is the small alias group extended by those permutations of the blocks which preserve the root in $\mathbb{M}_{666}$. In System 2 the large alias group coincides with the small alias group. We will often just use the term *alias group*.

As a final example of the terminology we state that the alias group of $0^{10} + -|0^5$ is of order 1 and that $+$ and $-$ are the same digit. The negation of a root is necessarily an alias of the root.

We shall quote **[Co2]** for the alias groups that appear there (they are for roots of $\mathbb{M}_{663}$ in System 2). However it was not there proved that the alias groups could not be larger that those stated. We need a method to establish such assertions.

We illustrate by computing the alias group of

$$r = \begin{matrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{matrix} \; 3\,.$$

The relations of the form of equation (7) show that the group is at least

$$\langle\, (01)| \sim |(12),\ \sim |(01)|(12)\,\rangle.$$

Can it be larger?

Might we not, for instance, have

$$r = \begin{matrix}0&0&0&1&1&1\\0&0&0&1&1&1\\0&0&0&0&1&2\end{matrix}\;\; 3 \equiv \begin{matrix}1&0&0&2&0&0\\1&1&0&0&1&0\\0&1&1&0&0&1\end{matrix}\;\; 3 = r'?$$

No, r commutes with f_3 but r' does not.

In general, if s is the difference of any 2 coordinate vectors in the same block, then s is a root and if $(r, s) = 0$ then $rs \,\epsilon\, 2A$ while if $(r, s) = \pm 1$ then $rs \,\epsilon\, 3A$ as in Section 4.

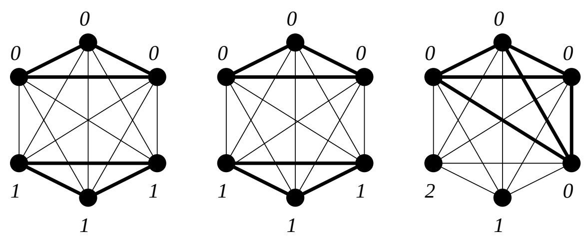

Figure 6: Graph for r. Thick lines for $2A$, thin lines for $3A$.

We use these facts to associate a coloured graph with r (Figure 6). The nodes of this graph are the 18 coordinate vectors. Two nodes from the same block are joined by an edge whose colour indicates the conjugacy class of rs, where $s = x - y$. Most of the edges are the determined by easy rules:

1. Two equal coordinates are joined by a $2A$ edge.
2. Two coordinates that differ by 1 are joined by a $3A$ edge.

There is no simple rule to find the colour of the edges between coordinates that differ by 2 or more. However, in our example, it is easy to see that the 4 edges between 0 and 2 are all $3A$, since after applying the alias element $(01)| \sim |(12)$ the corresponding differences are replaced by 1's.

Plainly all the elements of the alias group must preserve the graph. The automorphism group of this graph consists of

$$(S_3 \wr S_2) \times (S_3 \wr S_2) \times (S_4 \times S_2)$$

(preserving the blocks) together with the interchange of the first two blocks.

This gives us an upper bound for the alias group, namely

$$\underset{\{0,1\}}{S_2} \quad \times \quad \underset{\{0,1\}}{S_2} \quad \times \quad \underset{\{1,2\}}{S_2}$$

(together with the trivial interchange of the first two blocks). We call this the *possible group*. However, what we have seen (the *visible group*) is only of index 2 in this. Can there be more? No! Coset representatives for the visible group in the possible group are the identity element $\sim | \sim | \sim$ and the element $\sim | \sim |(12)$. But $\sim | \sim |(12)$

transforms r into its reflection under

$$s = \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0\,, \\ 0 & 0 & 0 & 0 & + & - & \end{matrix}$$

but we know that r and s do not commute. The alias group is therefore the group $\langle\,(01)| \sim |(12),\, \sim |(01)|(12)\,\rangle$ of order 4.

In practice we work with a collapsed graph obtained by identifying all nodes joined by $1A$ edges or $2A$ edges as in Figure 7. Since $3A$ edges are then the most common type, we usually omit them (or draw them in 'invisible ink'). This makes the graph for our example very simple indeed (Figure 8).

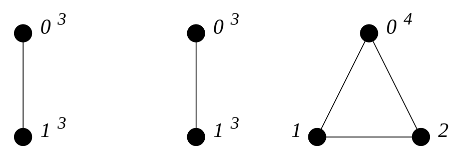

Figure 7: A collapsed graph.

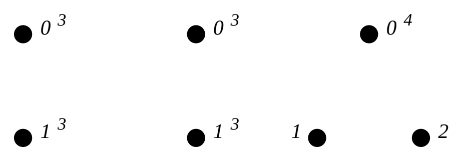

Figure 8: A very collapsed graph.

The above example is done in System 1. We can, of course, make similar alias computations in System 2.

We have identified the Monster roots of small type or size with their associated reflection elements. The alias groups of these reflection elements have also been computed. These are done simultaneously for System 1 and System 2 using an inductive process. To find the elements of the (visible) alias group of a root r, our main tools are conjugations by reflections in fundamental Monster roots r_i where r^{r_i} has known alias group and change of coordinate systems such that r then has known alias group. Similar techniques determine the reductions of Monster roots to reflection elements. The methods of this section are used to verify that the visible alias groups are the entire alias groups. The results of these computations are presented in Appendices A and B.

Appendix A: Monster Root Reduction Tables

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The first column of these tables names a family of roots equivalent under the coordinate permutation group. The second gives the standard root vector of this family. The third column gives the root vector's ancestor. When there is a simpler root vector that yields the same reflection element of $\mathbb{M}_{666}$, it is described in the fourth column. The root vectors of smallest type or size associated with a reflection element are called the *reduced root* *vectors*. The fifth column gives the *reduced family name*, which takes account of such equivalences.

Family names (first column) have the form n_{sm} where n is the type or size of the root vector and m of the reduced root vector, while s is a tag letter (lower case for System 1 and upper case for System 2). To make the equivalences more visible, we sometimes use letters $a,\ b,\ c,\ \ldots$ for the distinct digits of certain reduced root vectors. When $n = m$, we say that that n_{sm} is *irreducible*. To make the system explicit, we can prefix the reduced family names by $+$ for System 1 and $-$ for System 2. We define the *type* or *size* of a reflection element to be the type or size of one of its reduced root vectors.

System 1. We list all Monster roots of type ≤ 8. For type 9, only those roots reducing to reflection elements of type ≥ 5 are listed. For types $10,\ 11,\ 12$ and 13 we list the irreducible roots. This table is complete up to (and including) type 11, after which there is the (remote) possibility of omission.

0_a	000000|000000|0000+-			0
1_a	000001|000001|000001			1
2_a	000011|000011|000011	1_a		2
3_a	000111|000111|000012	2_a		3
4_a	000112|000112|000112	2_a		4
4_{a2}	001111|000112|000022	3_a	110000|000110|000011	2
4_{b2}	001111|001111|000013	3_a	110000|110000|000011	2
5_{a4}	001112|000113|000122	3_a	110002|000112|000211	4
5_{b4}	000122|000122|000122	4_a	000211|000211|000211	4
5_{a3}	001112|001112|000023	3_a	110001|001110|000012	3
5_{a1}	011111|000122|000023	4_{a2}	100000|000100|000010	1
5_{b1}	011111|011111|000014	4_{b2}	100000|100000|000010	1
6_a	001122|000123|000123	4_a		6
6_b	001122|001122|000114	4_a		$6'$
6_{a4}	001113|001113|000123	3_a	110002|110002|000121	4
6_{a3}	001113|000123|000222	4_{a2}	110001|000102|000111	3
6_{b3}	011112|000114|000222	4_{a2}	100002|000111|000111	3
6_{c3}	000222|000222|000123	5_{b4}	000111|000111|000210	3
6_{a2}	011112|001113|000033	4_{b2}	100001|110000|000011	2
6_{b2}	011112|001122|000024	4_{a2}	100001|001100|000011	2
6_{c2}	001122|001122|000033	5_{a3}	001100|001100|000011	2
6_{a0}	011112|000222|000033	5_{a1}	+0000-|000000|000000	0
6_{b0}	111111|000123|000033	5_{a1}	000000|000+0-|000000	0
6_{c0}	111111|000222|000024	5_{a1}	000000|000000|0000+-	0
6_{d0}	111111|111111|000015	5_{b1}	000000|000000|0000+-	0
7_a	001123|001123|000124	4_a		7
7_{a4}	001114|001123|000223	4_{a2}	110002|001120|000112	4
7_{b4}	001123|000133|000223	5_{a4}	001120|000211|000112	4
7_{c4}	011122|000133|000124	5_{a4}	200011|000211|000121	4
7_{d4}	001114|000133|001222	5_{a4}	110002|000211|112000	4
7_{e4}	001222|000223|000124	5_{b4}	112000|000112|000121	4

7_{f4}	011122|000115|001222	5_{a4}	200011|000112|112000	4
7_{g4}	001222|000133|000133	6_{a}	112000|000211|000211	4
7_{a3}	011113|001114|000133	4_{b2}	100002|110001|000111	3
7_{b3}	011122|001123|000034	5_{a3}	011100|001101|000012	3
7_{c3}	011122|011122|000025	5_{a3}	100011|011100|000012	3
7_{a2}	011113|000124|000223	4_{a2}	100001|000101|000110	2
7_{b2}	000223|000223|000223	5_{b4}	000110|000110|000110	2
7_{a1}	011113|011113|000034	4_{b2}	100000|100000|000010	1
7_{b1}	111112|000223|000034	5_{a1}	000001|000001|000001	1
7_{c1}	011113|001222|000034	5_{a1}	100000|001000|000001	1
7_{d1}	111112|001222|000025	5_{a1}	000001|001000|000010	1
7_{e1}	001222|001222|000034	6_{c2}	001000|001000|000010	1
8_{a}	001124|001124|001124	4_{a}		8
8_{a7}	001124|000134|001223	5_{a4}	110023|000124|002113	7
8_{b7}	001223|001223|000125	5_{b4}	002113|002113|000214	7
8_{a6}	011123|001133|000125	5_{a4}	100023|001122|000213	6
8_{b6}	001124|001133|000224	5_{a4}	110022|001122|000114	$6'$
8_{c6}	001115|001133|001223	5_{a4}	110004|001122|002112	$6'$
8_{d6}	001133|001133|000134	6_{a4}	110022|110022|000114	$6'$
8_{a4}	011123|000134|000224	5_{a4}	100021|000121|000112	4
8_{b4}	001124|001124|000233	5_{a3}	001102|001102|000211	4
8_{c4}	001223|000224|000224	5_{b4}	002110|000112|000112	4
8_{d4}	011123|001115|000233	5_{a3}	100012|110002|000211	4
8_{e4}	001223|000233|000134	6_{a}	002110|000211|000121	4
8_{f4}	001124|000233|000233	6_{a}	110020|000211|000211	4
8_{g4}	011222|000233|000125	6_{a}	211000|000211|000211	4
8_{h4}	011222|011222|000116	6_{b}	211000|211000|000112	4
8_{i4}	011222|000134|000134	6_{a}	211000|000121|000121	4
8_{a3}	011114|001124|000224	4_{a2}	100002|001101|000111	3
8_{a2}	011114|011114|000134	4_{b2}	100001|100001|000110	2
8_{b2}	011123|011123|000035	5_{a3}	100010|100010|000011	2
8_{c2}	001133|000233|000224	6_{a3}	001100|000011|000110	2
8_{d2}	011123|001223|000044	6_{b2}	100010|001001|000011	2
8_{e2}	001133|000125|002222	6_{a3}	001100|000101|110000	2
8_{f2}	001115|000233|002222	6_{a3}	110000|000011|110000	2
8_{g2}	111122|001133|000035	6_{a2}	000011|110000|000011	2
8_{h2}	011222|001223|000035	6_{c2}	011000|001001|000011	2
8_{i2}	111122|011222|000026	6_{b2}	000011|011000|000011	2
8_{j2}	111122|000116|002222	6_{b3}	000011|000110|110000	2
8_{k2}	011114|011222|000044	6_{b2}	100001|011000|000011	2
8_{l2}	111122|001124|000044	6_{b2}	000011|001100|000011	2
8_{m2}	002222|000224|000134	6_{c3}	110000|000110|000101	2
8_{n2}	011222|001133|000044	7_{b3}	011000|110000|000011	2
8_{o2}	111122|000233|000044	7_{b1}	000011|000011|000011	2
8_{a1}	011114|000224|000233	5_{a1}	100000|000001|000100	1

8_{b1}	$111113\vert 000125\vert 000233$	5_{a1}	$000001\vert 000100\vert 000100$	1
8_{c1}	$000233\vert 000233\vert 000233$	7_{b2}	$000100\vert 000100\vert 000100$	1
8_{a0}	$111113\vert 011114\vert 000044$	5_{b1}	$000000\vert {+}0000{-}\vert 000000$	0
8_{b0}	$111113\vert 001223\vert 000035$	5_{a1}	$000000\vert 00{+}00{-}\vert 000000$	0
8_{c0}	$011123\vert 002222\vert 000035$	6_{a0}	${+}000{-}0\vert 000000\vert 000000$	0
8_{d0}	$002222\vert 001223\vert 000044$	7_{e1}	$000000\vert 00{+}00{-}\vert 000000$	0
8_{e0}	$002222\vert 002222\vert 000035$	7_{e1}	$000000\vert 000000\vert 0000{+}{-}$	0
9_a	$001224\vert 001224\vert 000144$	6_b	$aabccd\vert aabccd\vert 000144$	9
9_b	$011223\vert 001125\vert 000144$	6_b	$baaccd\vert aaccbd\vert 000144$	9
9_c	$001233\vert 001134\vert 000144$	7_a	$aabdcc\vert ccaabd\vert 000144$	9
9_{a8}	$001125\vert 001134\vert 001224$	5_{a4}	$110024\vert 001124\vert 002114$	8
9_{a7}	$011124\vert 001134\vert 001224$	5_{a4}	$100024\vert 001123\vert 002113$	7
9_{b7}	$001224\vert 001233\vert 000135$	6_a	$112003\vert 003211\vert 000124$	7
9_{c7}	$001125\vert 001233\vert 000234$	6_a	$001123\vert 003211\vert 000124$	7
9_{d7}	$011223\vert 001233\vert 000126$	6_a	$200113\vert 002311\vert 000124$	7
9_{a6}	$011124\vert 000135\vert 001224$	5_{a4}	$100023\vert 000123\vert 002112$	6
9_{b6}	$001224\vert 000234\vert 000234$	6_a	$002112\vert 000132\vert 000132$	6
9_{c6}	$011223\vert 000234\vert 000135$	6_a	$200112\vert 000312\vert 000132$	6
9_{d6}	$001224\vert 001224\vert 000225$	5_{b4}	$002112\vert 002112\vert 000114$	$6'$
10_a	$001225\vert 001225\vert 001144$	6_b	$aabccd\vert aabccd\vert 001144$	10
10_b	$011224\vert 001126\vert 001144$	6_b	$baaccd\vert aaccbd\vert 001144$	10
10_c	$001126\vert 001234\vert 001234$	6_a		$10'$
10_d	$001234\vert 001234\vert 000145$	7_a		$10''$
10_e	$001144\vert 001144\vert 001144$	8_a		$10'''$
11_a	$001235\vert 001136\vert 001244$	7_a	$aab23c\vert 0011de\vert 00fg44$	11
11_b	$011234\vert 001127\vert 001244$	7_a	$baa23c\vert 0011de\vert 00gf44$	11
11_c	$001244\vert 001145\vert 001145$	8_a		$11'$
12_a	$002244\vert 002217\vert 000444$	9_a	$aabbcc\vert ddeeff\vert 000444$	12
12_b	$001155\vert 112206\vert 000444$	9_b	$aabbcc\vert ddeeff\vert 000444$	12
12_c	$112233\vert 001128\vert 000444$	9_b	$aabbcc\vert ddeeff\vert 000444$	12
12_d	$110046\vert 003315\vert 000444$	9_c	$aabbcc\vert ddeeff\vert 000444$	12
12_e	$003324\vert 110037\vert 000444$	9_c	$aabbcc\vert ddeeff\vert 000444$	12
12_f	$001155\vert 002415\vert 000444$		$aabbcc\vert ddeeff\vert 000444$	12
12_g	$001236\vert 001236\vert 001245$	7_a		$12'$
12_h	$001245\vert 001245\vert 001146$	8_a		$12''$
13_a	$001246\vert 001246\vert 000445$	9_a	$00abcd\vert 00abcd\vert 000445$	13
13_b	$001237\vert 001345\vert 000445$	9_c	$00cabd\vert 00bcad\vert 000445$	13
13_c	$001246\vert 001246\vert 001246$	8_a		$13'$

System 2. We list all Monster roots of size ≤ 4. For size 5, only those roots reducing to reflection elements of size ≥ 3 are listed. For sizes 6, 7, 8 and 9 we list the irreducible roots. This table is complete up to (and including) size 6, after which there is the possibility of omission.

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0_A	$0000000000{+}{-}\vert 00000$	0
0_B	$000000000000\vert 000{+}{-}$	$0'$

1_A	$000000111111\vert 00001$			1
2_A	$000011112222\vert 00011$	1_A		2
2_{A1}	$000111111222\vert 00002$	1_A	$000111111000\vert 00001$	1
3_A	$000111222333\vert 00012$	1_A		3
3_{A2}	$000111123333\vert 00111$	2_A	$000111102222\vert 11000$	2
3_{B2}	$000012222333\vert 00111$	2_A	$000021111222\vert 11000$	2
3_{A1}	$001111222233\vert 00003$	1_A	$001111000011\vert 00001$	1
3_{B1}	$001111113333\vert 00012$	2_A	$001111110000\vert 00010$	1
3_{C1}	$000022222233\vert 00012$	2_A	$000011111100\vert 00010$	1
3_{A0}	$011111112333\vert 00003$	2_{A1}	$+0000000{-}000\vert 00000$	0
3_{B0}	$000122222223\vert 00003$	2_{A1}	$000{+}0000000{-}\vert 00000$	0
4_A	$000112233444\vert 00112$	2_A		4
4_{A3}	$001112223444\vert 00013$	2_A	$110002221333\vert 00012$	3
4_{B3}	$000122233344\vert 00013$	2_A	$000211133322\vert 00012$	3
4_{A2}	$001111333344\vert 00013$	1_A	$112222000011\vert 00011$	2
4_{B2}	$001111224444\vert 00112$	2_A	$110000112222\vert 00110$	2
4_{C2}	$000022333344\vert 00112$	2_A	$000011222200\vert 00110$	2
4_{D2}	$001111233444\vert 00022$	2_{A1}	$110000211222\vert 00011$	2
4_{E2}	$000112333344\vert 00022$	2_{A1}	$000110222211\vert 00011$	2
4_{F2}	$000122223444\vert 00022$	3_A	$000211110222\vert 00011$	2
4_{A1}	$111111222444\vert 00004$	2_{A1}	$000000111111\vert 00001$	1
4_{B1}	$011112223344\vert 00004$	2_{A1}	$011110001100\vert 00001$	1
4_{C1}	$001122233334\vert 00004$	2_{A1}	$001100011110\vert 00001$	1
4_{D1}	$000222222444\vert 00013$	3_A	$000111111000\vert 00010$	1
4_{E1}	$000112224444\vert 01111$	3_{A2}	$000110001111\vert 10000$	1
4_{F1}	$000022233444\vert 01111$	3_{B2}	$000011100111\vert 10000$	1
4_{G1}	$000013333344\vert 01111$	3_{B2}	$000011111100\vert 10000$	1
4_{A0}	$011111333334\vert 00004$	1_A	$+0000000000{-}\vert 00000$	0
4_{B0}	$001222222344\vert 00004$	3_{A1}	$00{+}000000{-}00\vert 00000$	0
4_{C0}	$011111124444\vert 00022$	3_{B1}	$+000000{-}0000\vert 00000$	0
4_{D0}	$000023333334\vert 00022$	3_{C1}	$0000{+}000000{-}\vert 00000$	0
4_{E0}	$111111114444\vert 00013$	3_{B1}	$000000000000\vert 000{+}{-}$	$0'$
4_{F0}	$000033333333\vert 00013$	3_{C1}	$000000000000\vert 000{+}{-}$	$0'$
5_A	$001122334455\vert 00014$	2_A		5
5_{A4}	$001112334555\vert 00113$	2_A	$110002332444\vert 00112$	4
5_{B4}	$000122344455\vert 00113$	2_A	$000211244433\vert 00112$	4
5_{C4}	$001112244555\vert 00122$	3_A	$110002233444\vert 00211$	4
5_{D4}	$000122334555\vert 00122$	3_A	$000211332444\vert 00211$	4
5_{E4}	$000113344455\vert 00122$	3_A	$000112244433\vert 00211$	4
5_{A3}	$001112344455\vert 00023$	2_{A1}	$001110233322\vert 00021$	3
5_{B3}	$001122234555\vert 00023$	3_A	$001122201333\vert 00012$	3
5_{C3}	$000123334455\vert 00023$	3_A	$000123331122\vert 00012$	3
5_{D3}	$000113334555\vert 01112$	3_{A2}	$000112221333\vert 10002$	3
5_{E3}	$000122244555\vert 01112$	3_{B2}	$000211122333\vert 10002$	3
6_A	$001122345666\vert 00123$	3_A	$aabbccdefggg\vert 00123$	6

6_B	$000123445566\|00123$	3_A	$gggfedccbbaa\|00123$	6
6_C	$001122445566\|00114$	2_A		$6'$
7_A	$001123456677\|00124$	3_A		7
8_A	$001224466788\|00044$		$aabccddeebff\|00044$	8
8_B	$001125566778\|00044$		$aaccbddeeffb\|00044$	8
8_C	$011223367788\|00044$		$bffeeddbccaa\|00044$	8
8_D	$001134556788\|00044$		$ccaabdeedbff\|00044$	8
8_E	$001233457788\|00044$		$ffbdeedbaacc\|00044$	8
8_F	$001124467788\|01124$		$aabbc44deeff\|ghh2i$	$8'$
8_G	$001224457788\|00125$		$aacbb44deeff\|hhg2i$	$8'$
8_H	$001134466788\|00125$		$ffeed44bbcaa\|hhg2i$	$8'$
8_I	$001234456788\|00116$			$8''$
9_A	$001224577899\|00144$		$aabccdeffghh\|00144$	9
9_B	$011223478899\|00144$		$baaccdegff99\|00144$	9
9_C	$001125677889\|00144$		$aaccbdeffhhg\|00144$	9
9_D	$001233568899\|00144$		$aabdcceghhff\|00144$	9
9_E	$001134667899\|00144$		$ccaabdffeghh\|00144$	9
9_F	$001144558899\|00144$			$9'$
9_G	$001234567899\|00045$			$9''$
9_H	$001234567899\|00126$			$9'''$

Appendix B: Alias Tables

In the following alias tables we describe the alias groups of the Monster roots appearing in appendix A. The first column is the reduced family name. The second column gives the reduced root vector. The third column describes the associated (small) alias group, often by listing the group generators. The fourth column is the order of the small alias group. The large alias group is obvious, except that for 12 of System 1 we must adjoin $(ad)(be)(cf)|(04)$.

When describing the groups we write $\{ab\dots c\}$ to mean the symmetric group on the digits $a,\ b,\ \dots,\ c$. We use $\frac{1}{2}(G)$ to designate the even part of G. The dihedral group of order $2n$ on the letters $a_0,\ a_1,\ \dots,\ a_{n-1}$ in this cyclic order is denoted by $D_{2*n}(a_0a_1\dots a_{n-1})$, and $\mathrm{PGL}_2(n)(a_0a_1\dots a_n)$ is the 2-dimensional projective linear group over $\mathbb{F}_n$ where $a_0,\ a_1,\ \dots\ ,\ a_n$ are acted upon as $0,\ 1,\ \dots\ ,\ n-1,\ \infty$.

System 1. We list alias groups for System 1.

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0	$000000\|000000\|0000{+}{-}$	$\sim\|\sim\|\sim$	1
1	$000001\|000001\|000001$	$\sim\|\sim\|\sim$	1
2	$000011\|000011\|000011$	$\sim\|\sim\|\sim$	1
3	$000111\|000111\|000012$	$01\|\sim\|12,\ \sim\|01\|12$	4
4	$000112\|000112\|000112$	$\sim\|\sim\|\sim$	1
6	$001122\|000123\|000123$	$210\|123\|\sim,\ 12\|12\|12,$ $210\|\sim\|123$	18
$6'$	$001122\|001122\|000114$	$012\|210\|\sim,\ 02\|02\|\sim$	6

7	$001123\|001123\|000124$	$01\|23\|12,\ 23\|01\|12$	4
8	$001124\|001124\|001124$	$\sim\|\sim\|\sim$	1
9	$aabccd\|aabccd\|000144$	$bd\|bd.ac\|\sim,\ ac\|ac\|\sim$	4
10	$aabccd\|aabccd\|001144$	$ac\|ac\|\sim,\ bd\|bd\|\sim,\ bd\|ac\|04$	8
$10'$	$001126\|001234\|001234$	$01\|23\|12,\ 01\|12\|23$	6
$10''$	$001234\|001234\|000145$	$1234\|4321\|\sim,\ 12.34\|13\|45$	8
$10'''$	$001144\|001144\|001144$	$\{04\}\|\{04\}\|\{04\}$	8
11	$aab23c\|0011de\|00fg44$	$bc\|01\|04,\ 23\|01\|fg$	4
$11'$	$001244\|001145\|001145$	$12\|01.45\|\sim,\ 12\|\sim\|01.45,\ 04\|\sim\|\sim$	8
12	$aabbcc\|ddeeff\|000444$	$\frac{1}{2}(\{abc\}\|\{def\}\|\{04\})$	36
$12'$	$001236\|001236\|001245$	$12\|12\|12.45$	2
$12''$	$001245\|001245\|001146$	$15\|15\|\sim$	2
13	$00abcd\|00abcd\|000445$	$ad.bc\|ab.cd\|\sim,\ ab.cd\|ad.bc\|\sim,\ bcd\|dcb\|\sim$	12
$13'$	$001246\|001246\|001246$	$\sim\|\sim\|\sim$	1

System 2. We list alias groups for system 2.

0	$0000000000_{+-}\|00000$	$\sim\|\sim\|\sim$	1
$0'$	$000000000000\|000_{+-}$	$\sim\|\sim\|\sim$	1
1	$000000111111\|00001$	$01\|\sim$	2
2	$000011112222\|00011$	$\{012\}\|\sim$	6
3	$000111222333\|00012$	$\frac{1}{2}(\{0123\}\|\{12\})$	24
4	$000112233444\|00112$	$\frac{1}{2}(\{04\}\{123\}\|\{01\})$	12
5	$001122334455\|00014$	$\mathrm{PGL}_2(5)(023451) \equiv 01.34\|\sim,\ 12.45\|\sim,\ 01.23.45\|\sim$	120
6	$aabbccdefggg\|00123$	$bc.ef\|12,\ fed\|123,\ cba\|123$	18
$6'$	$001122445566\|00114$	$012.456\|\sim,\ 01.45\|\sim, 04\|\sim$	48
7	$001123456677\|00124$	$01.45\|12, 07.16.25.34\|\sim$	8
8	$aabbccddeeff\|00044$	$\frac{1}{2}\{abcdef\}$	360
$8'$	$aabbc44deeff\|ghh2i$	$\frac{1}{2}(D_{2*5}(ab4ef)\{cd\}\|\{04\})$	20
$8''$	$001234456788\|00116$	$12.56\|01,\ 15.48\|01,\ 08.17.26.35\|\sim$	144
9	$aabccdeffghh\|00144$	$bd.eg.fh\|\sim,\ ah.bg.cf.de\|\sim,\ be.fh\|04$	32
$9'$	$001144558899\|00144$	$\{048\}\{159\}\|\{04\},\ 09.18.45\|\sim$	144
$9''$	$001234567899\|00045$	$\frac{1}{2}(\mathrm{PGL}_2(7)(12634578)\|\{45\}),\ 09\|\sim$	672
$9'''$	$001234567899\|00126$	$12.45.78\|12,\ 123.678\|\sim,\ 09.18.27.36.45\|\sim$	12

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