Journal of Algebra **235**, 805–814 (2001)
doi:10.1006/jabr.2000.8494, available online at http://www.idealibrary.com on IDEAL®

26 Implies the Bimonster

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Communicated by Walter Feit

Received February 9, 2000

1. INTRODUCTION

The bimonster, or wreathed square $\mathbb{M} \wr 2$ of the monster group $\mathbb{M}$, is presented by the Coxeter relations of the 26-node incidence graph of the projective plane of order 3, along with the additional relations that all free 12-gons of this diagram generate symmetric groups S_{12}. We prove this using purely geometric arguments as well as the celebrated Ivanov–Norton theorem [6, 7, 9]. This elementary presentation provides a natural connection from the projective plane of order 3 to the bimonster. Previously this connection was known in the other direction [2, 3]. We also consider other groups that can be obtained in a similar fashion.

A useful reference on Coxeter and reflection groups is [5]. We use the Atlas [1] notation for finite groups.

[1] John H. Conway was supported in part by an NSF grant.
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0021-8693/01 $35.00

2. RESULTS

We use $\mathbb{P}_n$ to denote the projective plane of order n. For example, $\mathbb{P}_3$ consists of 13 points and 13 lines with 4 points on each line and 4 lines passing through each point. The incidence graph of $\mathbb{P}_3$, $\mathrm{Inc}(\mathbb{P}_3)$, has 26 nodes (one for each of the 13 points and 13 lines). Two nodes are joined exactly when one is a point; the other is a line and the point lies on the line. The graph $\mathrm{Inc}(\mathbb{P}_3)$ therefore has valence 4. It is shown in Fig. 1. The indices i range over $\{1, 2, 3\}$ so that some nodes in the figure represent 3 nodes of $\mathrm{Inc}(\mathbb{P}_3)$. Single lines between the nodes of Fig. 1 indicate that the nodes of $\mathrm{Inc}(\mathbb{P}_3)$ are joined just if they have the same indices. Double lines indicate that the nodes of $\mathrm{Inc}(\mathbb{P}_3)$ are joined just if their indices differ.

For any graph Γ we can consider the associated Coxeter group H. H is the group generated by the nodes of Γ subject to the relations $a^2 = 1$ for any node a, $(ab)^2 = 1$ for any unjoined nodes a, b and $(ab)^3 = 1$ for any joined nodes a, b.

We introduce the following convention in greater generality than required for this paper. It is useful for further work.

DEFINITION 2.1. Let $\mathbb{Z}^m : G$ be a group (usually an affine Coxeter group). To deflate this group is to impose the relations that make the translations $\mathbb{Z}^m$ trivial. Similarly to biflate, triflate, of k-flate $\mathbb{Z}^m : G$ is to make the translations have order 2, 3, or k, respectively with the result that $\mathbb{Z}^m$ becomes 2^m, 3^m, or k^m.

Usually we view $\mathbb{Z}^m : G$ as a subgroup of a group H. The new relations are then imposed on H. In this paper we will deflate affine A_{n-1} Coxeter groups $\mathbb{Z}^{n-1} : S_n$. We refer to the group by listing its generators. Since the affine A_{n-1} diagram is merely a free n-gon, we say that we are deflating an n-gon.

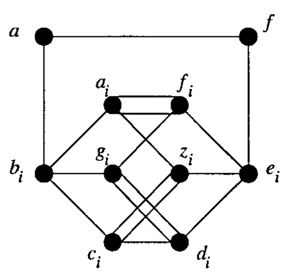

FIG. 1. $\mathrm{Inc}(\mathbb{P}_3)$.

We are now in a position to state one of the two main theorems of this paper. It provides a connection from the projective plane of order 3 to the bimonster by giving a remarkably simple presentation of the bimonster.

THEOREM 2.1. *If we deflate all free 12-gons of the* $\mathrm{Inc}(\mathbb{P}_3)$ *Coxeter group we obtain the bimonster* $\mathbb{M} \wr 2$.

This theorem follows from the proof of Theorem 2.2. The proof depends on the celebrated Ivanov–Norton theorem [6, 7, 9]. It remains a mystery why $\mathrm{Inc}(\mathbb{P}_3)$ should be so strongly connected to the bimonster. We note that 12 is the largest n for which there is a free n-gon in $\mathrm{Inc}(\mathbb{P}_3)$ and that all such 12-gons of $\mathrm{Inc}(\mathbb{P}_3)$ are isomorphic.

Some easier examples of analogous theorems are listed in Table I. In this table we deflate all the free n-gons (for the indicated n) in a Coxeter group determined by the graph to obtain some group. This "game" of starting with a Coxeter group of a graph and k-flating certain subgraphs is of considerable interest. One can ask which finite (or infinite) groups can be obtained in this fashion. What happens if we start with $\mathrm{Inc}(\mathbb{P}_4)$ or some other well chosen graph? However, in this paper we restrict ourselves to the specific case of the bimonster. The $O_8^-(2):2$ case is discussed in [11].

We now describe another method of obtaining groups from graphs. We start with a (small) graph Γ_0 and a group G satisfying its Coxeter relations. In our case we choose Γ_0 to be the 16-node $\mathbb{M}_{666}$ diagram (Fig. 2). We then adjoin A_{11} extending nodes whenever possible. The closure of Γ_0 under such extension is denoted Γ. G must still satisfy the Coxeter relations of Γ.

Consider an a_{11} subdiagram of Γ with nodes $\alpha_1, \dots, \alpha_{11}$. The Coxeter group of a_{11} is S_{12}. If we view these as the transpositions $(0,1), \dots, (10,11)$ of S_{12} then the extending node α_0 corresponds to the transposition $(11,0)$. We note that α_0 can be defined to be $\alpha_1^{\alpha_2 \cdots \alpha_{11}}$.

We can make this process more concrete by taking advantage of the equivalence of Coxeter groups and reflection groups. For each node of the a_{11} subdiagram of Γ we also have a root α_i (corresponding to the reflection $x \mapsto x - (x, \alpha_i)\alpha_i$). The roots are chosen such that $(\alpha_i, \alpha_i) = 2$

TABLE I
Deflating n-Gons in Graphs to Get Groups

n-Gons	# of Nodes	Graph	Group
12	26	$\mathrm{Inc}(\mathbb{P}_3)$	$\mathbb{M} \wr 2$
8	14	$\mathrm{Inc}(\mathbb{P}_2)$	$O_8^-(2):2$
6	10	Petersen	$O_6^-(2):2 \cong O_5(3):2$
6	8	cube = Inc(tetrahedron)	$O_5(3) \times 2 \cong O_6^-(2) \times 2$

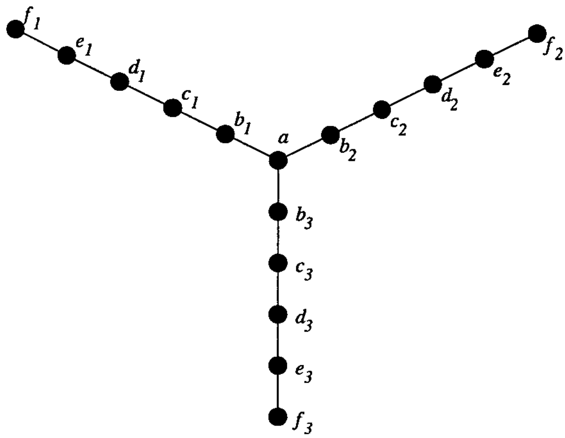

FIG. 2. $\mathbb{M}_{666}$ diagram.

for all i, $(\alpha_i, \alpha_j) = 0$ for α_i, α_j unjoined and $(\alpha_i, \alpha_j) = -1$ for α_i, α_j joined. The last equality requires careful choice of the signs of the roots (although these choices do not affect the underlying reflection group elements). The A_{11} extending node α_0 is then equal to $-(\alpha_1 + \cdots + \alpha_{11})$. Checking the inner products of the extending root α_0 with other roots α outside of the A_{11} subdiagram determines many additional joining relationships of Γ. If $(\alpha_0, \alpha) = 0$ then the nodes α_0, α are unjoined. If $(\alpha_0, \alpha) = \pm 1$ the nodes α_0, α are joined. Otherwise we do not know the relationship between α_0 and α. This extension process can now be repeated.

Two roots corresponding to the same element of G are said to be equivalent and they share the same node.

We prove that if Γ (for Γ_0 still the $\mathbb{M}_{66}$ diagram) has no more than 26 distinct nodes then G is the bimonster.

THEOREM 2.2 (26 implies the bimonster). *Let G be a group generated by and satisfying the Coxeter relations of the $\mathbb{M}_{666}$ diagram. If after closure of the generating $\mathbb{M}_{666}$ diagram by adjoining A_{11} extending nodes there are no more than* 26 *nodes, then G is the bimonster $\mathbb{M} \wr 2$.*

Insisting that there are no more than 26 nodes imposes some non-Coxeter relations, the equivalence of certain roots, on the $\mathbb{M}_{666}$ infinite Coxeter group with the result that we get a presentation for the bimonster. In fact Γ has exactly 26 nodes and is the graph $\mathrm{Inc}(\mathbb{P}_3)$. Upon checking the relations used in the proof we get Theorem 2.1 as a corollary.

TABLE II
Extending $\mathbb{M}_{kkk}$ Diagrams to Get Groups

k	n-Gons	# of Nodes	Graph	Group
6	12	26	$\mathrm{Inc}(\mathbb{P}_3)$	$\mathbb{M} \wr 2$
5	10	≥ 29	?	?
4	8	14	$\mathrm{Inc}(\mathbb{P}_2)$	$O_8^-(2):2$
3	6	10	Petersen	$O_6^-(2):2$
2	4	8	$K(4,4)$	$2^3:S_4$
1	2	1	1	S_2

Some easier examples of analogous theorems are listed in Table II. In this table we let Γ_0 be the Y-shaped $\mathbb{M}_{kkk}$ diagram and get G as the group presented by the Coxeter relations of Γ_0 along with the condition that the graph Γ obtained by A_n extension has (no more than) the specified number of distinct nodes. Note how $O_8^-(2):2$ arises almost exactly as the bimonster does. The technique in Table II leads to another "game" for getting groups. We do not know what happens for the $\mathbb{M}_{555}$ diagram. We also remark that the graph obtained from the $\mathbb{M}_{888}$ diagram is not an incidence graph.

3. PROOFS

In order to proceed with the proof of Theorem 2.2 we make use of the following coordinate system for the roots of the $\mathbb{M}_{666}$ diagram (System 1 of [3, 10]). Often we will use $+$ to denote 1 and $-$ to denote -1.

In System 1 we have a space of 19 coordinates

$$\begin{matrix} a & b & c & d & e & f & \\ g & h & i & j & k & l & t \\ m & n & o & p & q & r & \end{matrix} \quad \text{with quadratic form } a^2 + \cdots + r^2 - t^2. \tag{1}$$

In this system the fundamental monster roots are as indicated in Fig. 3.

All the vectors satisfy the following relations:

$$\begin{aligned} a + b + c + d + e + f &= t \\ g + h + i + j + k + l &= t \\ m + n + o + p + q + r &= t. \end{aligned} \tag{2}$$

Thus t is redundant, so we shall sometimes omit it. We call t the type.

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+-0000                 0+-000                 00+-00                 000+-0                 0000+-
0000000  —  0000000  —  0000000  —  0000000  —  0000000  \
000000                 000000                 000000                 000000                 000000      \
  f1                     e1                     d1                     c1                     b1          \
                                                                                                            \
000000                 000000                 000000                 000000                 000000          000001
+-00000  —  0+-0000  —  00+-000  —  000+-00  —  0000+-0  ———  0000011
000000                 000000                 000000                 000000                 000000          000001
  f2                     e2                     d2                     c2                     b2          /    a
                                                                                                          /
000000                 000000                 000000                 000000                 000000      /
0000000  —  0000000  —  0000000  —  0000000  —  0000000  /
+-0000                 0+-000                 00+-00                 000+-0                 0000+-
  f3                     e3                     d3                     c3                     b3
```

FIG. 3. The fundamental monster roots in System 1.

We now extend the $\mathbb{M}_{666}$ diagram by adjoining A_{11} extending roots to obtain Γ. We do this by finding free a_{11} subdiagrams and adding the A_{11} extending nodes. We stress that in all cases used the a_{11} subdiagrams satisfy the standard inner product conditions. The choice of signs of the a_{11} roots is important. If α is a root then we use $\underline{\alpha}$ to denote the negative of α. The signed sum of the a_{11} roots is the A_{11} extending root.

The $\mathbb{M}_{666}$ diagram has 16 nodes.

We use the subdiagram f_1—e_1—d_1—c_1—b_1—a—b_2—c_2—d_2—e_2—f_2 to get

$$a_3 = \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 0 & 0 & 0 & 0 & 0 & 1\,. \\ 0 & 0 & 0 & 0 & 0 & 1 & \end{matrix} \tag{3}$$

By the obvious S_3 symmetry of the $\mathbb{M}_{666}$ diagram we similarly obtain a_1 and a_2.

Now use $\underline{c_1}$—$\underline{d_1}$—$\underline{e_1}$—$\underline{f_1}$—a_2—$\underline{f_3}$—a_1—$\underline{f_2}$—$\underline{e_2}$—$\underline{d_2}$—$\underline{c_2}$ to get

$$z_3 = \begin{matrix} 0 & 0 & 0 & 0 & 1 & 1 & \\ 0 & 0 & 0 & 0 & 1 & 1 & 2\,. \\ 1 & 1 & 0 & 0 & 0 & 0 & \end{matrix} \tag{4}$$

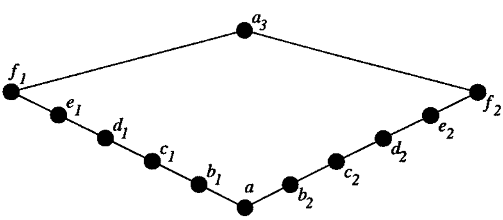

FIG. 4. a_3.

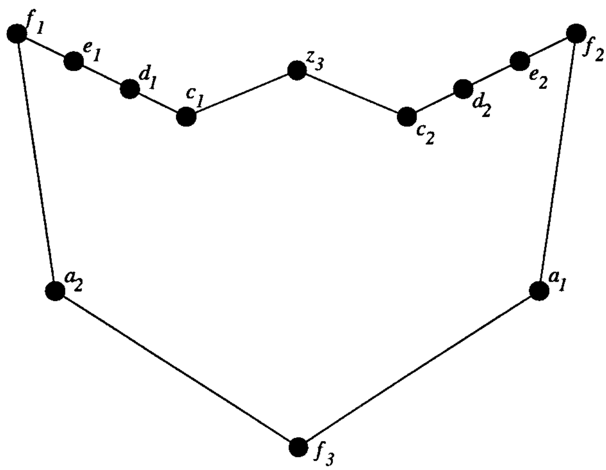

FIG. 5. z_3.

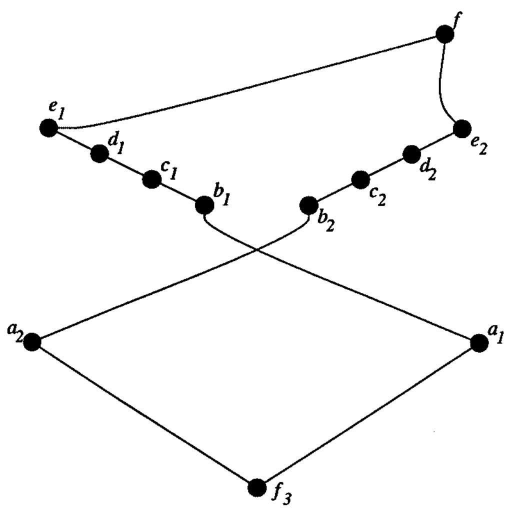

FIG. 6. f.

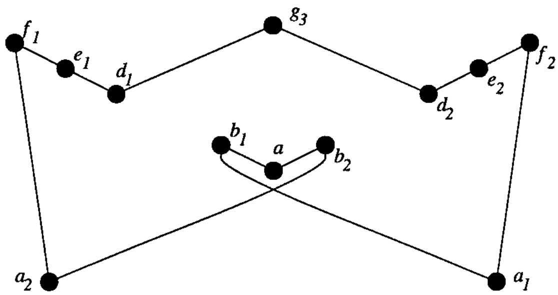

FIG. 7. g_3.

Similarly obtain z_1 and z_2.
Use $e_1—d_1—c_1—b_1—a_1—\underline{f_3}—a_2—b_2—c_2—d_2—e_2$ to get

$$f = \begin{matrix} 1 & 1 & 0 & 0 & 0 & 0 & \\ 1 & 1 & 0 & 0 & 0 & 0 & 2\,. \\ 1 & 1 & 0 & 0 & 0 & 0 & \end{matrix} \tag{5}$$

Use $\underline{d_1}—\underline{e_1}—\underline{f_1}—a_2—b_2—a—b_1—a_1—\underline{f_2}—\underline{e_2}—\underline{d_2}$ to get

$$g_3 = \begin{matrix} 0 & 0 & 0 & 1 & 1 & 1 & \\ 0 & 0 & 0 & 1 & 1 & 1 & 3\,. \\ 2 & 0 & 0 & 0 & 0 & 1 & \end{matrix} \tag{6}$$

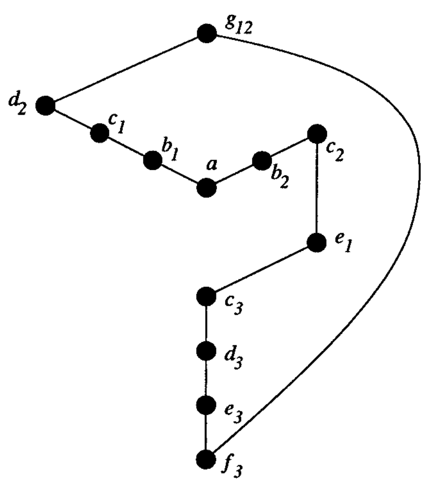

FIG. 8. g_{12}.

Similarly obtain g_1 and g_2.

We now have 26 roots. Checking their inner products with the $\mathbb{M}_{666}$ roots we find that they are distinct. We are assuming that G is not trivial (of order 1 or 2). This is justified since by its construction the very nontrivial bimonster group will satisfy the conditions. Under the conditions of Theorem 2.2 any additional nodes must correspond to one of these 26.

Consider d_1—c_1—b_1—a—b_2—c_2—z_1—c_3—d_3—e_3—f_3. The extending node is

$$g_{12} = \begin{matrix} 1 & 1 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 1 & 1 & 3\,. \\ 1 & 0 & 0 & 0 & 0 & 2 & \end{matrix} \tag{7}$$

Checking inner products with the $\mathbb{M}_{666}$ roots we find that the reflection in g_{12} can only be equivalent to the reflection in g_3. Therefore $g_{12} \equiv g_3$. Permuting by an $S_6 \times S_6 \times S_6$ symmetry this becomes

$$\begin{matrix} 0 & 0 & 0 & 1 & 1 & 1 & \\ 0 & 0 & 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 & \end{matrix} \equiv \begin{matrix} 1 & 1 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 1 & 1 & 3\,. \\ 0 & 0 & 0 & 0 & 2 & 1 & \end{matrix} \tag{8}$$

As we describe below, this relation along with the $\mathbb{M}_{666}$ Coxeter relations presents the bimonster and implies that Γ is in fact $\mathrm{Inc}(\mathbb{P}_3)$. By the root enumeration proof of [4, Theorem 3], the group generated by a, b_1, c_1, d_1, e_1, f_1, b_2, c_2, b_3, c_2 is $O_8^+(3):2$ and therefore the spider relation

$$(ab_1c_1ab_2c_2ab_3c_3)^{10} = 1 \tag{9}$$

holds. So by the Ivanov–Norton theorem [6, 7, 9], the group G is the bimonster $\mathbb{M} \wr 2$. We can use [3, Theorem 6] (the 26-node theorem) to see that Γ is $\mathrm{Inc}(\mathbb{P}_3)$. This proves Theorem 2.2. Theorem 2.1 quickly follows.

Remark 3.1. In [4, Theorem 7 and Corollary 8] the term "in 543" should be replaced by "in 643."

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