Relations in $\mathbb{M}_{666}$

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Abstract

We sketch geometrical proofs of the equivalence of various relations that hold in the Bimonster.

1 Introduction

The Ivanov–Norton theorem [4, 5] asserts that the "Bimonster", or wreathed square $(\mathbb{M} \times \mathbb{M}){:}2$ of the Monster group $\mathbb{M}$, is the abstract group defined by the Coxeter relations of the $\mathbb{M}_{666}$ diagram (see Figure 1) together with a single additional relation, initially taken as

$$(ab_1c_1ab_2c_2ab_3c_3)^{10} = 1 \qquad \text{(S)}$$

The purpose of this paper is to provide purely geometrical proofs of the equivalence between various alternatives to this so-called "spider" relation (S). The original proofs of these equivalences, mostly due to L. Soicher [2], involved machine coset enumerations. We found those given here during the preparation of Simons's Ph.D. dissertation [6].

There is considerable point in providing machine-free proofs, because the coset enumerations that would be required to complete this work into a new proof of the Ivanov–Norton theorem are much too large ever to be done by machine. However there is no similar bar to extending our geometric arguments, and a purely geometrical proof of the Ivanov–Norton theorem would almost certainly yield a new and simple way to compute inside the Monster.

We shall use $c\mathbb{M}_{666}$ for the Coxeter group defined by the graph in Figure 1, and $c\mathbb{M}_{pqr}$ for the subgroup of this generated by

$$\begin{array}{rl} & \text{the first } p \text{ of } a, b_1, c_1, d_1, e_1, f_1, \\ & \text{the first } q \text{ of } a, b_2, c_2, d_2, e_2, f_2, \\ \text{and} & \text{the first } r \text{ of } a, b_3, c_3, d_3, e_3, f_3. \end{array}$$

The "c" in $c\mathbb{M}_{pqr}$ stands for "Coxeter". To indicate that other relations are to be added we change "c" to some other letter, or omit it to indicate

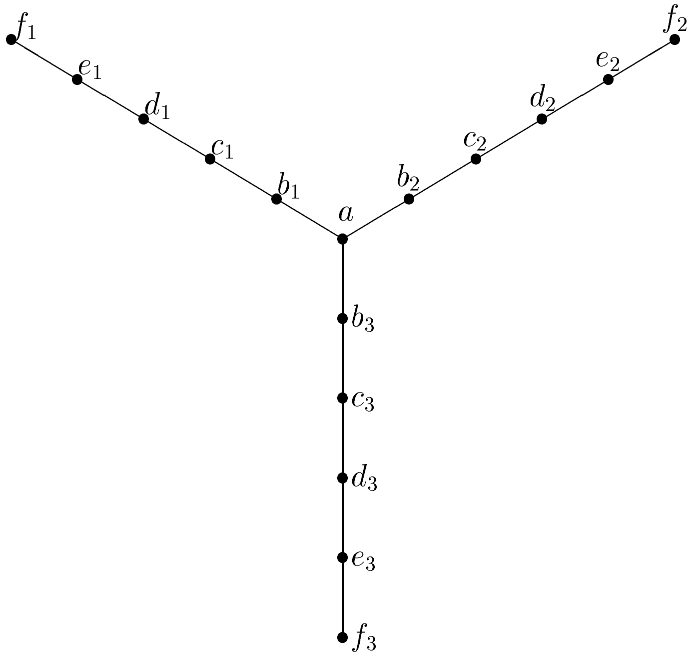

Figure 1: The $\mathbb{M}_{666}$ diagram

the corresponding subgroups $\mathbb{M}_{pqr}$ of the Bimonster. These subgroups are described in Table 1. In [1] these groups are called $Y_{p-1,q-1,r-1}$, but our reparametrization has many advantages.

The phrase "pqr corrects PQR (in $\mathcal{PQR}$)" means that if we subject the Coxeter group $c\mathbb{M}_{\mathcal{PQR}}$ to the relations that convert $c\mathbb{M}_{pqr}$ to $\mathbb{M}_{pqr}$, then the subgroup $c\mathbb{M}_{PQR}$ is converted to $\mathbb{M}_{PQR}$. If $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ are unspecified, we are to understand their minimal values, namely

$$\begin{aligned} \mathcal{P} &= \max(p, P) \\ \mathcal{Q} &= \max(q, Q) \\ \mathcal{R} &= \max(r, R) \end{aligned}$$

We discuss the first few cases in which $c\mathbb{M}_{pqr}$ differs from $\mathbb{M}_{pqr}$, and indicate the appropriate relations. As is convenient in this subject, we use lower case letters for spherical Dynkin diagrams and upper case letters for Euclidean Dynkin diagrams.

Thus $c\mathbb{M}_{622}$ is the Weyl group of d_8, namely $2^7{:}S_8$, but $\mathbb{M}_{622}$ is its central quotient obtained by adjoining the relation

$$\delta_8 = (ab_1b_2b_3c_1d_1e_1f_1)^7 = 1,$$

where δ_8 is the central inversion of d_8.

In general, if a Coxeter group has a central inversion, it is the product of the central inversions of its connected components (which together generate its center). The central inversion of a connected Coxeter group, when it has

pqr	$\mathbb{M}_{pqr}$	# of inequivalent roots
$pq1$	S_{p+q}	$(p+q)(p+q-1)/2$
222	$2^3{:}S_4$	12
322	$2^4{:}S_5$	20
422	$2^5{:}S_6$	30
522	$2^6{:}S_7$	42
622	$2^6{:}S_8$	56
332	$O_6^-(2){:}2 \cong O_5(3){:}2$	36
432	$O_7(2) \times 2$	63
632	$O_8^+(2){:}2$	120
442	$2^7{:}\,(2 \times O_7(2))$	126
642	$O_9(2) \times 2$	255
662	$O_{10}^-(2){:}2$	528
333	$3^5{:}O_5(3){:}2 \cong 3^5{:}O_6^-(2){:}2$	108
433	$O_7(3) \times 2$	351
633	$O_8^+(3){:}2$	1080
443	$2^2.Fi_{22}$	3510
643	$2 \times Fi_{23}$	31671
663	$3.Fi_{24}$	920808
444	$2^3.{}^2E_6(2)$	3968055
644	$2^2.B$	13571955000
664	$2 \times \mathbb{M}$	$\frac{\lvert\mathbb{M}\rvert}{2\lvert B\rvert} \approx 10^{20}$
666	$(\mathbb{M} \times \mathbb{M}){:}2$	$\lvert\mathbb{M}\rvert \approx 10^{54}$

Table 1: $\mathbb{M}_{pqr}$ subgroups of $\mathbb{M}_{666}$.

one, is the product of its generators raised to the power $\frac{1}{2}h$, where h is the order of this product (the "Coxeter number").

In a similar way, $c\mathbb{M}_{532}$ is the Weyl group of e_8, namely $2G{:}2$, where G is the simple group $O_8^+(2)$, while $\mathbb{M}_{532}$ is its central quotient $G{:}2$, obtained by adjoining $\varepsilon_8 = 1$, where

$$\varepsilon_8 = (ab_1b_2b_3c_1c_2d_1e_1)^{15}$$

is the central inversion of e_8.

We see a new type of relation in the case 632. The Coxeter group $c\mathbb{M}_{632}$ (being associated to the Euclidean diagram E_8) has structure $\mathbb{Z}^8{:}2G{:}2$, while $\mathbb{M}_{632}$ collapses this to the group $G{:}2$ we have just seen. It may be obtained by killing not only ε_8 but also all the *translations* t_v by vectors v of the E_8 root lattice.

Similarly, $c\mathbb{M}_{442}$ is the Weyl group of the Euclidean diagram E_7, namely $\mathbb{Z}^7{:}(2 \times H)$, where $2 \times H$ is the Weyl group of the corresponding spherical diagram e_7. This collapses to

$$2^7{:}(2 \times H)$$

in $\mathbb{M}_{442}$, obtained by killing the translations for which v belongs to *twice* the root lattice of E_7.

Finally, the Weyl group $c\mathbb{M}_{333}$ of the Euclidean diagram E_6 has structure $\mathbb{Z}^6{:}K{:}2$, where $K{:}2$ is the Weyl group of the spherical diagram e_6, and this becomes

$$3^5{:}K{:}2$$

in $\mathbb{M}_{333}$, which we can obtain by killing the translations for which v belongs to *three times* the *weight* lattice of E_6. (The spider relation has precisely this effect.)

2 Some results

Theorem 1 622 *corrects* 532 *(in* 632*).*

Proof: We can obtain various spherical Coxeter groups by removing single nodes from a Euclidean Dynkin diagram. If two of these both have central inversions, then the product of these is a translation of the Euclidean group. This is illustrated by the figure for the Euclidean group G_2 (see Figure 2), defined as

$$G_2 = \langle a, b, c \mid 1 = a^2 = b^2 = c^2 = (ab)^3 = (bc)^6 = (ca)^2 \rangle.$$

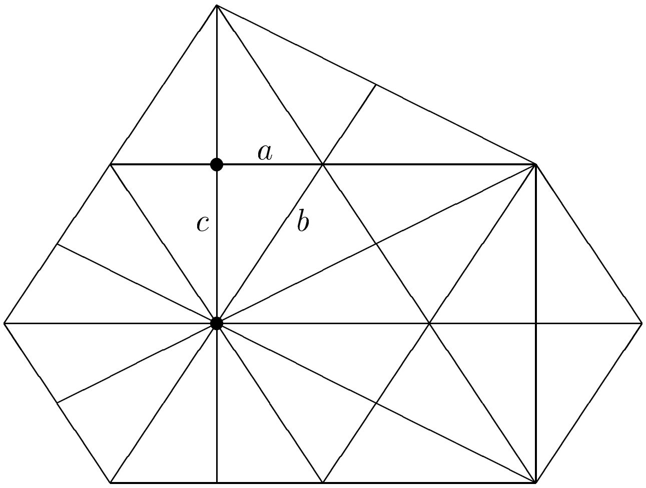

Figure 2: The Euclidean group G_2

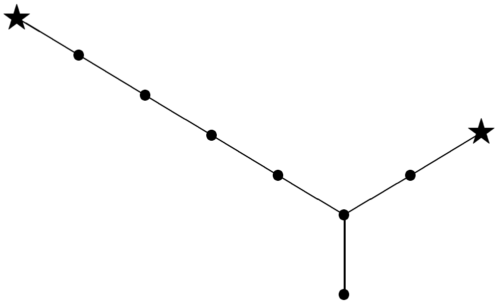

Figure 3: The 632 diagram

The subgroups $\langle a, c\rangle$ and $\langle b, c\rangle$ have central inversions $\alpha = ac$ and $\beta = (bc)^3$ which are the half-turns about the two marked vertices, and $\alpha\beta$ is the vertical translation through twice their distance.

In a similar way, by dropping either of the starred nodes in the 632 diagram (see Figure 3) we obtain the d_8 or e_8 diagrams, having the respective central inversions δ_8 and ε_8. So we have

$$\delta_8\varepsilon_8 = t_v$$

for some vector v in the E_8 root lattice. Therefore to impose the relation $\delta_8 = 1$ is to equate ε_8 with t_v.

Now v happens to be a vector of norm 4 (see [6, 7]), and all such vectors are equivalent under the Weyl group of e_8, and so we also have

$$\varepsilon_8 = t_w$$

for any other norm 4 vector w. We can even choose w so that $v + w$ also has norm 4, so that

$$\varepsilon_8 = t_{v+w} = t_v t_w = (\varepsilon_8)^2,$$

from which we see that $\varepsilon_8 = 1$ as desired. $\square$

Theorem 2 622 *corrects* 632.

Proof: In the above, since $\varepsilon_8 = t_v$ and $\varepsilon_8 = 1$, we have $t_v = 1$ for any norm 4 vector v. But the norm 4 vectors generate the root lattice. $\square$

Theorem 3 622 *corrects* 633.

Proof: Since 633 is neither spherical nor Euclidean, we must use a different method, that of explicit enumeration of roots. Since this has been described in detail in [6], we only briefly sketch it here. We observe that (in the so-called "system 1" coordinates of [3]) we have the relation

$$\boxed{\begin{array}{cccccc} . & . & . & 1 & 1 & 1 \\ . & . & . & 1 & 1 & 1 \\ . & . & . & . & 1 & 2 \end{array}} = \boxed{\begin{array}{cccccc} 1 & 1 & 1 & . & . & . \\ . & . & . & 1 & 1 & 1 \\ . & . & . & . & 2 & 1 \end{array}}$$

since the elements represented by the two sides differ by a translation. Using this, we find that the 1080 root elements represented by

$$\begin{array}{cccc} \boxed{\begin{array}{cccccc} . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & + & - \end{array}} & \boxed{\begin{array}{cccccc} . & . & . & . & + & - \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{array}} & \boxed{\begin{array}{cccccc} . & . & . & . & . & 1 \\ . & . & . & . & . & 1 \\ . & . & . & . & . & 1 \end{array}} & \boxed{\begin{array}{cccccc} . & . & . & . & 1 & 1 \\ . & . & . & . & 1 & 1 \\ . & . & . & . & 1 & 1 \end{array}} \\ (6) & (15) & (54) & (135) \\ \\ \boxed{\begin{array}{cccccc} . & . & . & . & 1 & 2 \\ . & . & . & 1 & 1 & 1 \\ . & . & . & 1 & 1 & 1 \end{array}} & \boxed{\begin{array}{cccccc} . & . & . & 1 & 1 & 1 \\ . & . & . & 1 & 1 & 1 \\ . & . & . & 0 & 1 & 2 \end{array}} & \boxed{\begin{array}{cccccc} . & . & . & 1 & 1 & 2 \\ . & . & . & 1 & 1 & 2 \\ . & . & . & 1 & 1 & 2 \end{array}} & \boxed{\begin{array}{cccccc} . & . & 1 & 1 & 2 & 2 \\ . & . & . & 1 & 2 & 3 \\ . & . & . & 1 & 2 & 3 \end{array}} \\ (30) & (120) & (540) & (180) \end{array}$$

form a conjugacy class. [The elements are obtained from those shown by permuting all six coordinates of the top row, the last three coordinates of each of the other two rows, or interchanging those two rows. Parentheses show the numbers of elements so obtained.]

Moreover, these roots are transformed the way $O_8^+(3)$:2 permutes its root elements, and so the group generated is a central extension of the latter group. But the multiplier of the simple subgroup $O_8^+(3)$ is known—it is a 4-group whose elements are annihilated by our relations, which therefore correct 633 to $O_8^+(3)$:2. $\square$

3 From 622 to the spider

We check inside $O_8^+(3)$ that the three products

$$\boxed{\begin{array}{cccccc} . & . & . & + & - & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{array}} \cdot \boxed{\begin{array}{cccccc} . & . & . & 0 & 2 & 1 \\ . & . & . & 1 & 1 & 1 \\ . & . & . & 1 & 1 & 1 \end{array}} \cdot \boxed{\begin{array}{cccccc} . & . & . & + & 0 & - \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{array}} \cdot \boxed{\begin{array}{cccccc} . & . & . & 0 & 1 & 2 \\ . & . & . & 1 & 1 & 1 \\ . & . & . & 1 & 1 & 1 \end{array}},$$

$$\boxed{\begin{array}{cccccc} . & . & . & . & . & . \\ . & . & . & + & - & 0 \\ . & . & . & . & . & . \end{array}} \cdot \boxed{\begin{array}{cccccc} . & . & . & 1 & 1 & 1 \\ . & . & . & 0 & 2 & 1 \\ . & . & . & 1 & 1 & 1 \end{array}} \cdot \boxed{\begin{array}{cccccc} . & . & . & . & . & . \\ . & . & . & + & 0 & - \\ . & . & . & . & . & . \end{array}} \cdot \boxed{\begin{array}{cccccc} . & . & . & 1 & 1 & 1 \\ . & . & . & 0 & 1 & 2 \\ . & . & . & 1 & 1 & 1 \end{array}},$$

$$\boxed{\begin{array}{cccccc} . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & + & - & 0 \end{array}} \cdot \boxed{\begin{array}{cccccc} . & . & . & 1 & 1 & 1 \\ . & . & . & 1 & 1 & 1 \\ . & . & . & 0 & 2 & 1 \end{array}} \cdot \boxed{\begin{array}{cccccc} . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & + & 0 & - \end{array}} \cdot \boxed{\begin{array}{cccccc} . & . & . & 1 & 1 & 1 \\ . & . & . & 1 & 1 & 1 \\ . & . & . & 0 & 1 & 2 \end{array}}$$

(of four reflections each) are equal. Now the top line is the product of the $c\mathbb{M}_{333}$ translations in

$$\boxed{\begin{array}{cccccc} . & . & . & + & - & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{array}} \quad \text{and} \quad \boxed{\begin{array}{cccccc} . & . & . & + & 0 & - \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{array}},$$

and so is the translation in

$$\boxed{\begin{array}{cccccc} . & . & . & 2 & - & - \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{array}},$$

so the quotient of the first two rows is the translation in

$$\boxed{\begin{array}{cccccc} . & . & . & 2 & - & - \\ . & . & . & -2 & + & + \\ . & . & . & . & . & . \end{array}}.$$

But this vector is orthogonal modulo 3 to all the fundamental roots of $\mathbb{M}_{333}$, and so it is easy to see that it and its images generate 3 times the weight lattice of E_6. Hence the equality of any two of the above products is a relation equivalent to our spider relation (S).

Since 622 corrects 333, it entails these relations.

4 From the spider to 622

We now work in the other direction, from 333 towards 622.

Theorem 4 333 *half-corrects* 533.

What we mean is that the relations just discussed (which are a way of correcting 333) convert $c\mathbb{M}_{533}$ to the group $2O_8^+(3){:}2$, whereas $\mathbb{M}_{533} \cong O_8^+(3){:}2$ is only half as large.

Proof: This is again proved by root enumeration, and again we only sketch the proof. The relations are used to show that the 1080 elements

$$\boxed{\begin{array}{cccccc} . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & + & - \end{array}}$$

(6)

$$\boxed{\begin{array}{cccccc} . & . & . & . & + & - \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{array}}$$

(10)

$$\boxed{\begin{array}{cccccc} . & . & . & . & . & 1 \\ . & . & . & . & . & 1 \\ . & . & . & . & . & 1 \end{array}}$$

(45)

$$\boxed{\begin{array}{cccccc} . & . & . & . & 1 & 1 \\ . & . & . & . & 1 & 1 \\ . & . & . & . & 1 & 1 \end{array}}$$

(90)

$$\boxed{\begin{array}{cccccc} . & . & . & . & 1 & 2 \\ . & . & . & 1 & 1 & 1 \\ . & . & . & 1 & 1 & 1 \end{array}}$$

(20)

$$\boxed{\begin{array}{cccccc} . & . & . & 1 & 1 & 1 \\ . & . & . & 1 & 1 & 1 \\ . & . & . & 0 & 1 & 2 \end{array}}$$

(120)

$$\boxed{\begin{array}{cccccc} . & . & . & 1 & 1 & 2 \\ . & . & . & 1 & 1 & 2 \\ . & . & . & 1 & 1 & 2 \end{array}}$$

(270)

$$\boxed{\begin{array}{cccccc} . & . & 1 & 1 & 1 & 1 \\ . & . & . & 1 & 1 & 2 \\ . & . & . & 0 & 2 & 2 \end{array}}$$

(45)

$$\boxed{\begin{array}{cccccc} . & . & 1 & 1 & 1 & 2 \\ . & . & . & 1 & 1 & 3 \\ . & . & . & 1 & 2 & 2 \end{array}}$$

(180)

$$\boxed{\begin{array}{cccccc} . & 1 & 1 & 1 & 1 & 1 \\ . & . & . & 1 & 2 & 2 \\ . & . & . & . & 2 & 3 \end{array}}$$

(9)

$$\boxed{\begin{array}{cccccc} . & . & 1 & 1 & 2 & 2 \\ . & . & . & 1 & 2 & 3 \\ . & . & . & 1 & 2 & 3 \end{array}}$$

(180)

$$\boxed{\begin{array}{cccccc} . & 1 & 1 & 1 & 1 & 2 \\ . & . & . & 2 & 2 & 2 \\ . & . & . & . & 3 & 3 \end{array}}$$

(5)

$$\boxed{\begin{array}{cccccc} . & 1 & 1 & 1 & 1 & 2 \\ . & . & . & 2 & 2 & 2 \\ . & . & . & 1 & 1 & 4 \end{array}}$$

(5)

$$\boxed{\begin{array}{cccccc} . & 1 & 1 & 1 & 2 & 2 \\ . & . & . & 1 & 3 & 3 \\ . & . & . & 1 & 2 & 4 \end{array}}$$

(90)

$$\boxed{\begin{array}{cccccc} . & 1 & 1 & 1 & 2 & 4 \\ . & . & . & 3 & 3 & 3 \\ . & . & . & 1 & 4 & 4 \end{array}}$$

(5)

form a conjugacy class in the group they generate, and so as before this must be a central extension of $O_8^+(3){:}2$. This time the extension that happens is non-trivial, so that $c\mathbb{M}_{533}$ subjected to the spider relation (S) is twice as large as $\mathbb{M}_{533}$. □

5 Translations

One of the things we must prove is that the E_8 translations of the various subdiagrams 632, 623, ..., 236 are all trivial. One such translation is

$$\overline{\underline{\boxed{\boxed{\begin{array}{cccccc} + & - & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{array}}}}} = \boxed{\begin{array}{cccccc} + & - & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{array}} \cdot \boxed{\begin{array}{cccccc} 0 & 2 & 1 & 1 & 1 & 1 \\ . & . & . & 2 & 2 & 2 \\ . & . & . & . & 3 & 3 \end{array}},$$

and we can obtain another for any root vector r of 632, namely

$$\overline{\underline{\boxed{\boxed{r}}}} = \boxed{r} \cdot \boxed{\varepsilon_1 - r},$$

where

$$\varepsilon_1 = \boxed{\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ . & . & . & 2 & 2 & 2 \\ . & . & . & . & 3 & 3 \end{array}}$$

is the "null vector" of 632. But some root vectors belong to several E_8 subdiagrams—for instance the above example belongs also to 623, whose null vector is

$$\varepsilon_2 = \boxed{\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ . & . & . & . & 3 & 3 \\ . & . & . & 2 & 2 & 2 \end{array}}.$$

Fortunately this does not matter:

Theorem 5 *For the above r, we have*

$$\boxed{\varepsilon_1 - r} = \boxed{\varepsilon_2 - r},$$

namely

$$\boxed{\begin{array}{cccccc} 0 & 2 & 1 & 1 & 1 & 1 \\ . & . & . & 2 & 2 & 2 \\ . & . & . & . & 3 & 3 \end{array}} = \boxed{\begin{array}{cccccc} 0 & 2 & 1 & 1 & 1 & 1 \\ . & . & . & . & 3 & 3 \\ . & . & . & 2 & 2 & 2 \end{array}}.$$

Proof: Conjugation quickly reduces this to

$$\boxed{\begin{array}{cccccc} . & . & 1 & 1 & 1 & 1 \\ . & . & . & . & 2 & 2 \\ . & . & . & 2 & 1 & 1 \end{array}} = \boxed{\begin{array}{cccccc} . & . & 1 & 1 & 1 & 1 \\ . & . & . & 2 & 1 & 1 \\ . & . & . & . & 2 & 2 \end{array}},$$

which can be established by transforming

$$\boxed{\begin{array}{cccccc} . & . & 1 & 0 & 0 & 0 \\ . & . & . & 1 & 0 & 0 \\ . & . & . & 1 & 0 & 0 \end{array}}$$

by the second and third of the three equivalent products in Section 3. □

In a precisely similar way, we can show that if a fundamental root r belongs to two or more of the E_8 diagrams

$$632, 623, 263, 362, 236, 326$$

then the two associated translations are equal. (Up to conjugation there are very few cases.) We can also show that any two of these 16 translations commute, since they can be conjugated into the same copy of E_8.

So the translations

$$\boxed{\boxed{r}}$$

generate some quotient of the free Abelian group of rank 16. Since also the corresponding vectors r additively generate a free Abelian group of rank 16, this justifies the notation

$$\boxed{\boxed{r_1 + r_2 + r_3 + \ldots}} = \boxed{\boxed{r_1}} \cdot \boxed{\boxed{r_2}} \cdot \boxed{\boxed{r_2}} \cdot \; \ldots$$

Theorem 6 *The translations are trivial.*

Proof: If we conjugate the translation

$$\boxed{\boxed{\begin{array}{cccccc} . & . & + & - & 0 & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{array}}}$$

by the last two of our three equivalent expressions, we obtain

$$\boxed{\boxed{\begin{array}{cccccc} . & . & 1 & 2 & 3 & 3 \\ . & . & . & 1 & 4 & 4 \\ . & . & . & 3 & 3 & 3 \end{array}}} = \boxed{\boxed{\begin{array}{cccccc} . & . & 1 & 2 & 3 & 3 \\ . & . & . & 3 & 3 & 3 \\ . & . & . & 1 & 4 & 4 \end{array}}},$$

so that their difference

$$\boxed{\boxed{\begin{array}{cccccc} . & . & . & 0 & 0 & 0 \\ . & . & . & 2 & -1 & -1 \\ . & . & . & -2 & 1 & 1 \end{array}}} = 0.$$

But a further conjugation connects this to

$$\boxed{\boxed{\begin{array}{cccccc} . & . & . & . & . & . \\ . & . & -1 & 2 & -1 & 0 \\ . & . & . & -2 & 1 & 1 \end{array}}} = 0,$$

and so *their* difference, namely

$$\boxed{\boxed{\begin{array}{cccccc} . & . & . & . & . & . \\ . & . & + & 0 & 0 & - \\ . & . & . & . & . & . \end{array}}},$$

is also 0. But this is one of the 16 fundamental translations. □

Theorem 7 333 *corrects* 542 *(in* 643*).*

Proof: Once we know that the translations are trivial it is very easy to deduce the correctness of 542 by root-enumeration. [No proper central extension can arise, because the index 2 subgroup $O_9(2)$ has trivial multiplier.] □

Corollary 8 333 *corrects* 533 *in* 643*.*

(Recall that 333 only half-corrected 533 in 533 itself.)

We now work in 666. In view of the last corollary we have in 666

333 corrects 532

and so since the translations are trivial

333 corrects 632

and so

333 corrects 622.

Now by symmetry, in 666 we have that

333 corrects 262 and 226,

and so, since we proved in 666 that 622 implies 333, we obtain our final result:

Theorem 9 *Inside* 666 *the correctness of any one of* 622*,* 262*,* 226 *implies that of all three.*

References

[1] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, ATLAS *of finite groups*, Clarendon Press, Oxford, 1985.

[2] J. H. Conway, S. P. Norton and L. H. Soicher, The Bimonster, the group Y_{555} and the projective plane of order 3, in *Computers in algebra (ed. M. C. Tangora)*, pp. 27–50. Marcel Dekker, 1988.

[3] J. H. Conway and A. D. Pritchard, Hyperbolic reflections for the Bimonster and $3Fi_{24}$, in *Groups, combinatorics and geometry*, (eds. M. W. Liebeck and J. Saxl), pp. 24–45. LMS Lecture Note Series **165**, Cambridge University Press, 1992.

[4] A. A. Ivanov, A geometric characterization of the Monster, in *Groups, combinatorics and geometry*, (eds. M. W. Liebeck and J. Saxl), pp. 46–62. LMS Lecture Note Series **165**, Cambridge University Press, 1992.

[5] S. P. Norton, Constructing the Monster, in *Groups, combinatorics and geometry*, (eds. M. W. Liebeck and J. Saxl), pp. 63–76. LMS Lecture Note Series **165**, Cambridge University Press, 1992.

[6] C. S. Simons, *Hyperbolic reflection groups, completely replicable functions, the Monster and the Bimonster*, Ph.D. thesis, Princeton University, November 1997.

[7] C. S. Simons, Monster roots, in *Monster II and Lie algebras, (eds. J. Ferrar and K. Harada)*, Walter de Gruyter, to appear.