What is the least number of moves to transfer four disks from one tower to another if only one disk can be moved at a time and a disk cannot be placed on top of a smaller disk? In this applet, you will solve an ancient problem by finding patterns to determine the minimum number of moves for transferring an arbitrary number of disks.

Exercises and Problems 1.1

Problems 1 through 20 involve strategies that were presented in this section. Some of these problems are analyzed by Polya’s four-step process. See if you can solve these problems before answering parts a, b, c, and d. Other strategies may occur to you, and you are encouraged to use the ones you wish. Often a good problem requires several strategies.

Making a Drawing (1–4)

1. A well is 20 feet deep. A snail at the bottom climbs up 4 feet each day and slips back 2 feet each night. How many days will it take the snail to reach the top of the well?
   a. Understanding the Problem. What is the greatest height the snail reaches during the first 24 hours? How far up the well will the snail be at the end of the first 24 hours?
   b. Devising a Plan. One plan that is commonly chosen is to compute 20/2, since it appears that the snail gains 2 feet each day. However, 10 days is not the correct answer. A second plan is to make a drawing and plot the snail’s daily progress. What is the snail’s greatest height during the second day?
   c. Carrying Out the Plan. Trace out the snail’s daily progress, and mark its position at the end of each day. On which day does the snail get out of the well?
1. **Looking Back.** There is a “surprise ending” at the top of the well because the snail does not slip back on the ninth day. Make up a new snail problem by changing the numbers so that there will be a similar surprise ending at the top of the well.

2. Five people enter a racquetball tournament in which each person must play every other person exactly once. Determine the total number of games that will be played.

3. When two pieces of rope are placed end to end, their combined length is 130 feet. When the two pieces are placed side by side, one is 26 feet longer than the other. What are the lengths of the two pieces?

4. There are 560 third- and fourth-grade students in King Elementary School. If there are 80 more third-graders than fourth-graders, how many third-graders are there in the school?

**Making a Table (5–8)**

5. A bank that has been charging a monthly service fee of $2 for checking accounts plus 15 cents for each check announces that it will change its monthly fee to $3 and that each check will cost 8 cents. The bank claims the new plan will save the customer money. How many checks must a customer write per month before the new plan is cheaper than the old plan?

   a. **Understanding the Problem.** Try some numbers to get a feel for the problem. Compute the cost of 10 checks under the old plan and under the new plan. Which plan is cheaper for a customer who writes 10 checks per month?

   b. **Devising a Plan.** One method of solving this problem is to make a table showing the cost of 1 check, 2 checks, etc., such as that shown here. How much more does the new plan cost than the old plan for 6 checks?

<table>
<thead>
<tr>
<th>Checks</th>
<th>Cost for Old Plan, $</th>
<th>Cost for New Plan, $</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.15</td>
<td>3.08</td>
</tr>
<tr>
<td>2</td>
<td>2.30</td>
<td>3.16</td>
</tr>
<tr>
<td>3</td>
<td>2.45</td>
<td>3.24</td>
</tr>
<tr>
<td>4</td>
<td>2.60</td>
<td>3.32</td>
</tr>
<tr>
<td>5</td>
<td>2.75</td>
<td>3.40</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

   c. **Carrying Out the Plan.** Extend the table until you reach a point at which the new plan is cheaper than the old plan. How many checks must be written per month for the new plan to be cheaper?

   d. **Looking Back.** For customers who write 1 check per month, the difference in cost between the old plan and the new plan is 93 cents. What happens to the difference as the number of checks increases? How many checks must a customer write per month before the new plan is 33 cents cheaper?

6. Sasha and Francisco were selling lemonade for 25 cents per half cup and 50 cents per full cup. At the end of the day they had collected $15 and had used 37 cups. How many full cups and how many half cups did they sell?

7. Harold wrote to 15 people, and the cost of postage was $5.40. If it cost 27 cents to mail a postcard and 42 cents to mail a letter, how many letters did he write?

8. I had some pennies, nickels, dimes, and quarters in my pocket. When I reached in and pulled out some change, I had less than 10 coins whose value was 42 cents. What are all the possibilities for the coins I had in my hand?

**Guessing and Checking (9–12)**

9. There are two 2-digit numbers that satisfy the following conditions: (1) Each number has the same digits, (2) the sum of the digits in each number is 10, and (3) the difference between the 2 numbers is 54. What are the two numbers?

   a. **Understanding the Problem.** The numbers 58 and 85 are 2-digit numbers that have the same digits, and the sum of the digits in each number is 13. Find two 2-digit numbers such that the sum of the digits is 10 and both numbers have the same digits.

   b. **Devising a Plan.** Since there are only nine 2-digit numbers whose digits have a sum of 10, the problem can be easily solved by guessing. What is the difference of your two 2-digit numbers from part a? If this difference is not 54, it can provide information about your next guess.

   c. **Carrying Out the Plan.** Continue to guess and check. Which pair of numbers has a difference of 54?

   d. **Looking Back.** This problem can be extended by changing the requirement that the sum of the two digits equals 10. Solve the problem for the case in which the digits have a sum of 12.

10. When two numbers are multiplied, their product is 759; but when one is subtracted from the other, their difference is 10. What are these two numbers?
11. When asked how a person can measure out 1 gallon of water with only a 4-gallon container and a 9-gallon container, a student used this “picture.”

a. Briefly describe what the student could have shown by this sketch.

b. Use a similar sketch to show how 6 gallons can be measured out by using these same containers.

12. Carmela opened her piggy bank and found she had $15.30. If she had only nickels, dimes, quarters, and half-dollars and an equal number of coins of each kind, how many coins in all did she have?


d. Looking Back. Suppose the problem had asked for the smallest number of colors to form a square of nine tiles so that no tile touches another tile of the same color along an entire edge. Can it be done in fewer colors; if so, how many?

14. What is the smallest number of different colors of tile needed to form a 4 × 4 square so that no tile touches another of the same color along an entire edge?

15. The following patterns can be used to form a cube. A cube has six faces: the top and bottom faces, the left and right faces, and the front and back faces. Two faces have been labeled on each of the following patterns. Label the remaining four faces on each pattern so that when the cube is assembled with the labels on the outside, each face will be in the correct place.

16. At the left in the following figure is a domino doughnut with 11 dots on each side. Arrange the four single dominoes on the right into a domino doughnut so that all four sides have 12 dots.

Using a Model (13–16)

13. Suppose that you have a supply of red, blue, green, and yellow square tiles. What is the fewest number of different colors needed to form a 3 × 3 square of tiles so that no tile touches another tile of the same color at any point?

a. Understanding the Problem. Why is the square arrangement of tiles shown here not a correct solution?

b. Devising a Plan. One plan is to choose a tile for the center of the grid and then place others around it so that no two of the same color touch. Why must the center tile be a different color than the other eight tiles?

c. Carrying Out the Plan. Suppose that you put a blue tile in the center and a red tile in each corner, as shown here. Why will it require two more colors for the remaining openings?

Working Backward (17–20)

17. Three girls play three rounds of a game. On each round there are two winners and one loser. The girl who loses on a round has to double the number of chips that each of the other girls has by giving up some of her own chips. Each girl loses one round. At the end of three rounds, each girl has 40 chips. How many chips did each girl have at the beginning of the game?

a. Understanding the Problem. Let’s select some numbers to get a feel for this game. Suppose girl A, girl B, and girl C have 70, 30, and 20 chips, respectively.
respectively, and girl A loses the first round. Girl B and girl C will receive chips from girl A, and thus their supply of chips will be doubled. How many chips will each girl have after this round?

b. Devising a Plan. Since we know the end result (each girl finished with 40 chips), a natural strategy is to work backward through the three rounds to the beginning. Assume that girl C loses the third round. How many chips did each girl have at the end of the second round?

c. Carrying Out the Plan. Assume that girl B loses the second round and girl A loses the first round. Continue working back through the three rounds to determine the number of chips each of the girls had at the beginning of the game.

d. Looking Back. Check your answer by working forward from the beginning. The girl with the most chips at the beginning of this game lost the first round. Could the girl with the fewest chips at the beginning of the game have lost the first round? Try it.

18. Sue Ellen and Angela have both saved $51 for their family trip to the coast. They each put money in their piggy banks on the same day but Sue Ellen started with $7 more than Angela. From then on Sue Ellen added $1 to her piggy bank each week and Angela put $2 in her piggy bank each week. How much money did Sue Ellen put in her piggy bank when they started?

19. Ramon took a collection of color tiles from a box. Amelia took 13 tiles from his collection, and Keiko took half of those remaining. Ramon had 11 left. How many did he start with?

20. Keiko had 6 more red tiles than yellow tiles. She gave half of her red tiles to Amelia and half of her yellow tiles to Ramon. If Ramon has 7 yellow tiles, how many tiles does Keiko have now?

Each of problems 21 to 24 is accompanied by a sketch or diagram that was used by a student to solve it. Describe how you think the student used the diagram, and use this method to solve the problem.

21. There are three numbers. The first number is twice the second number. The third is twice the first number. Their sum is 112. What are the numbers?

22. Mike has 3 times as many nickels as Larry has dimes. Mike has 45 cents more than Larry. How much money does Mike have?

23. At Joe’s Cafe 1 cup of coffee and 3 doughnuts cost $0.90, and 2 cups of coffee and 2 doughnuts cost $1.00. What is the cost of 1 cup of coffee? 1 doughnut?

24. One painter can letter a billboard in 4 hours and another requires 6 hours. How long will it take them together to letter the billboard?
Section 1.1 Introduction to Problem Solving

Problems 25 through 34 can be solved by using strategies presented in this section. While you are problem-solving, try to record the strategies you are using. If you are using a strategy different from those of this section, try to identify and record it.

25. There were ships with 3 masts and ships with 4 masts at the Tall Ships Exhibition. Millie counted a total of 30 masts on the 8 ships she saw. How many of these ships had 4 masts?

26. When a teacher counted her students in groups of 4, there were 2 students left over. When she counted them in groups of 5, she had 1 student left over. If 15 of her students were girls and she had more girls than boys, how many students did she have?

27. The video club to which Lin belongs allows her to receive a free movie video for every three videos she rents. If she pays $3 for each movie video and paid $132 over a 4-month period, how many free movie videos did she obtain?

28. Linda picked a basket of apples. She gave half of the apples to a neighbor, then 8 apples to her mother, then half of the remaining apples to her best friend, and she kept the 3 remaining apples for herself. How many apples did she start with in the basket?

29. Four people want to cross the river. There is only one boat available, and it can carry a maximum of 200 pounds. The weights of the four people are 190, 170, 110, and 90 pounds. How can they all manage to get across the river, and what is the minimum number of crossings required for the boat?

30. A farmer has to get a fox, a goose, and a bag of corn across a river in a boat that is only large enough for her and one of these three items. She does not want to leave the fox alone with the goose nor the goose alone with the corn. How can she get all these items across the river?

31. Three circular cardboard disks have numbers written on the front and back sides. The front sides have the numbers shown here.

   ![Disks](image)

   By tossing all three disks and adding the numbers that show face up, we can obtain these totals: 15, 16, 17, 18, 19, 20, 21, and 22. What numbers are written on the back sides of these disks?

32. By moving adjacent disks two at a time, you can change the arrangement of large and small disks shown below to an arrangement in which 3 big disks are side by side followed by the 3 little disks. Describe the steps.

   ![Disks](image)

33. How can a chef use an 11-minute hourglass and a 7-minute hourglass to time vegetables that must steam for 15 minutes?

34. The curator of an art exhibit wants to place security guards along the four walls of a large auditorium so that each wall has the same number of guards. Any guard who is placed in a corner can watch the two adjacent walls, but each of the other guards can watch only the wall by which she or he is placed.
   a. Draw a sketch to show how this can be done with 6 security guards.
   b. Show how this can be done for each of the following numbers of security guards: 7, 8, 9, 10, 11, and 12.
   c. List all the numbers less than 100 that are solutions to this problem.

35. Trick questions like the following are fun, and they can help improve problem-solving ability because they require that a person listen and think carefully about the information and the question.
   a. Take 2 apples from 3 apples, and what do you have?
   b. A farmer had 17 sheep, and all but 9 died. How many sheep did he have left?
   c. I have two U.S. coins that total 30 cents. One is not a nickel. What are the two coins?
   d. A bottle of cider costs 86 cents. The cider costs 60 cents more than the bottle. How much does the bottle cost?
   e. How much dirt is in a hole 3 feet long, 2 feet wide, and 2 feet deep?
   f. A hen weighs 3 pounds plus half its weight. How much does it weigh?
   g. There are nine brothers in a family and each brother has a sister. How many children are in the family?