TO MAXIMIZE P_D FOR A GIVEN $P_{FA} = \alpha$ DECIDE $\mathcal{H}_1$ IF

$$L(\underline{x}) = \frac{p(\underline{x};\mathcal{H}_1)}{p(\underline{x};\mathcal{H}_0)} > \gamma$$

THRESHOLD

LIKELIHOOD RATIO

RATIO OF "PROBABILITIES" THAT $\underline{x}$ COMES FROM $\mathcal{H}_1$ vs $\mathcal{H}_0$

TO FIND THRESHOLD USE $P_{FA} = \alpha$ CONSTRAINT.

NOTE THAT

$$R_1 = \{ \underline{x} : L(\underline{x}) > \gamma \}$$

<u>EXAMPLE</u>: PREVIOUS EXAMPLE

$$\mathcal{H}_0 : \; x[0] = w[0]$$
$$\mathcal{H}_1 : \; x[0] = 1 + w[0]$$

$w[0] \sim \mathcal{N}(0, 1)$. WANT $P_{FA} = 10^{-3}$.

NEYMAN - PEARSON LEMMA SAYS
DECIDE $\mathcal{H}_1$ IF

$$\frac{p(x[0];\mathcal{H}_1)}{p(x[0];\mathcal{H}_0)} > \gamma$$

$$\frac{\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(x[0]-1)^2}}{\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2[0]}} > \gamma$$

$$e^{x[0]-1/2} > \gamma$$

LET $\gamma = e^{\beta}$ (WHY CAN'T γ BE NEGATIVE?)

$$e^{x[0]-1/2} > e^{\beta}$$

OR $x[0] > \beta + \frac{1}{2} = \underbrace{\ln\gamma + 1/2}_{\gamma'}$

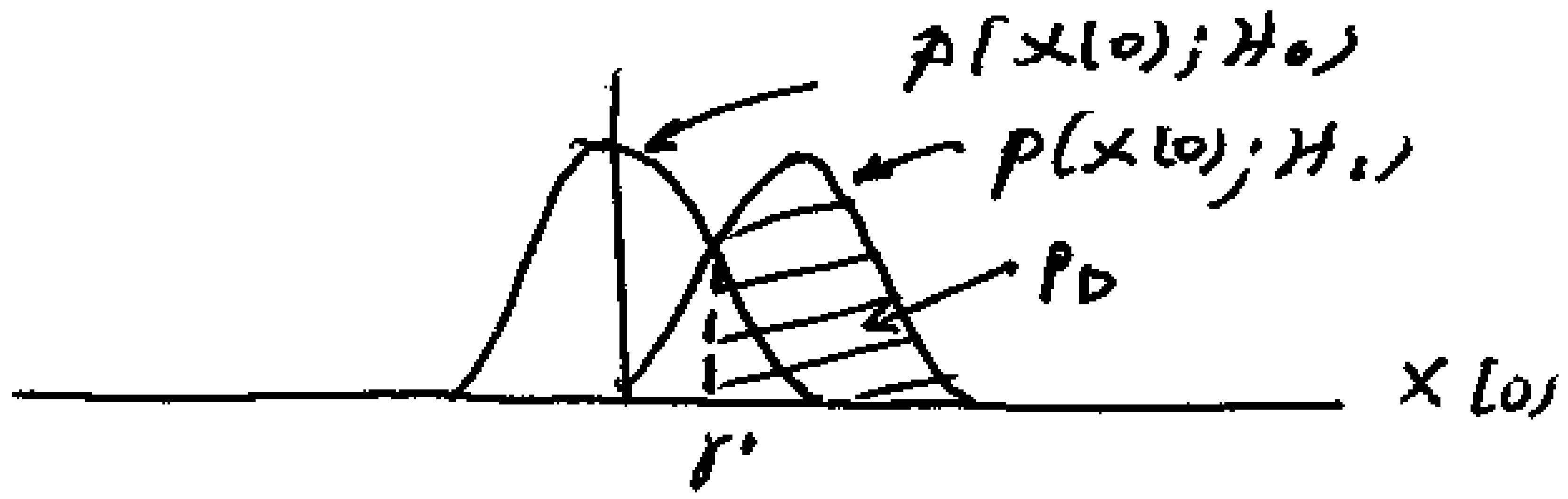
DECIDE $\mathcal{H}_1$ IF $x[0] > \gamma'$. TO FIND NEW THRESHOLD USE $P_{FA} = 10^{-3}$

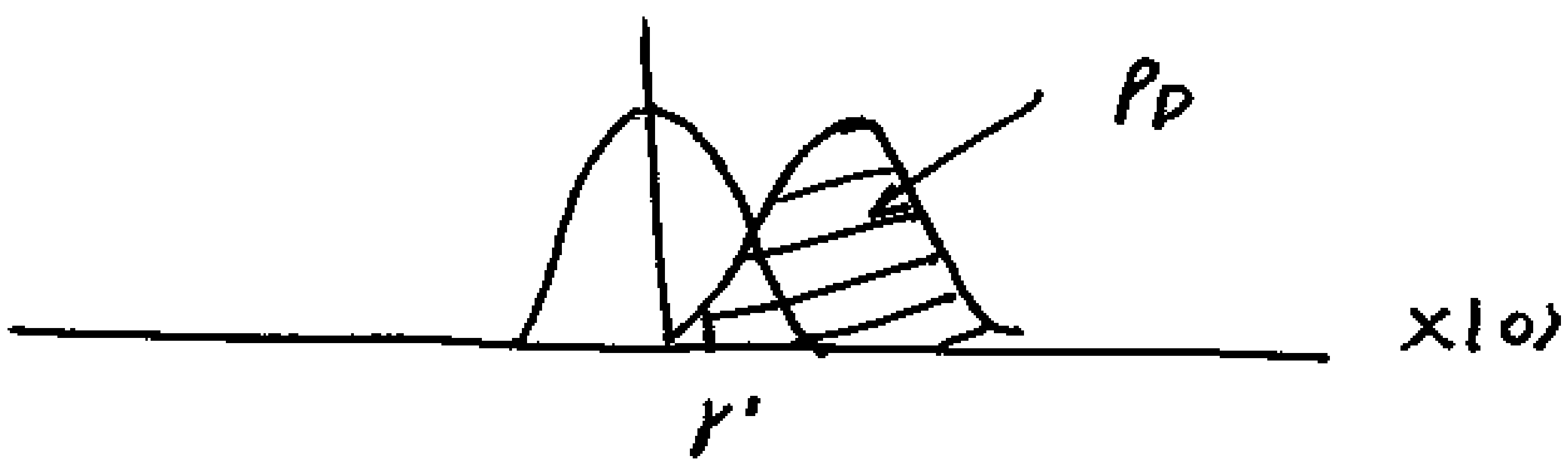
$$P_{FA} = \int_{\gamma'}^{\infty} p(x[0];\mathcal{H}_0)\, dx[0]$$

$\Rightarrow \quad \gamma' = 3.$ SAME DETECTOR AS BEFORE.

$\Rightarrow$ OPTIMAL IN N-P SENSE (LARGEST P_D FOR GIVEN P_{FA})

HOW CAN WE INCREASE P_D?





PENALTY PAID?

<u>EXAMPLE</u>: DC LEVEL IN WGN

$$\mathcal{H}_0 : x[n] = w[n] \qquad n = 0, 1, \ldots, N-1$$
$$\mathcal{H}_1 : x[n] = A + w[n] \qquad n = 0, 1, \ldots, N-1$$

SIGNAL IS $s[n] = A > 0$

W(n) IS WHITE GAUSSIAN NOISE

$E(w(n)) = 0$

$$E(w(m)\,w(n)) = \begin{cases} \sigma^2 & m = n \\ 0 & m \neq n \end{cases}$$

$w(n) \sim N(0, \sigma^2)$

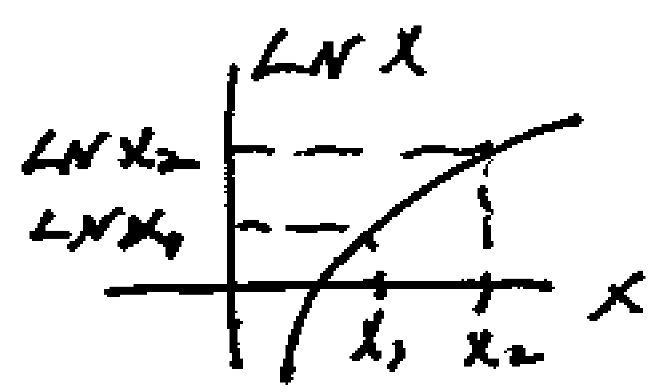
DESIGN A N-P DETECTOR.

SOLUTION : TO MAXIMIZE P_D FOR GIVEN $P_{FA} = \alpha$ DECIDE $\mathcal{H}_1$ IF

$$\frac{p(\underline{x};\mathcal{H}_1)}{p(\underline{x};\mathcal{H}_0)} > \gamma$$

$$\frac{\dfrac{1}{(2\pi\sigma^2)^{N/2}}\, e^{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x(n)-A)^2}}{\dfrac{1}{(2\pi\sigma^2)^{N/2}}\, e^{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2(n)}} > \gamma$$

TAKE LN BOTH SIDES



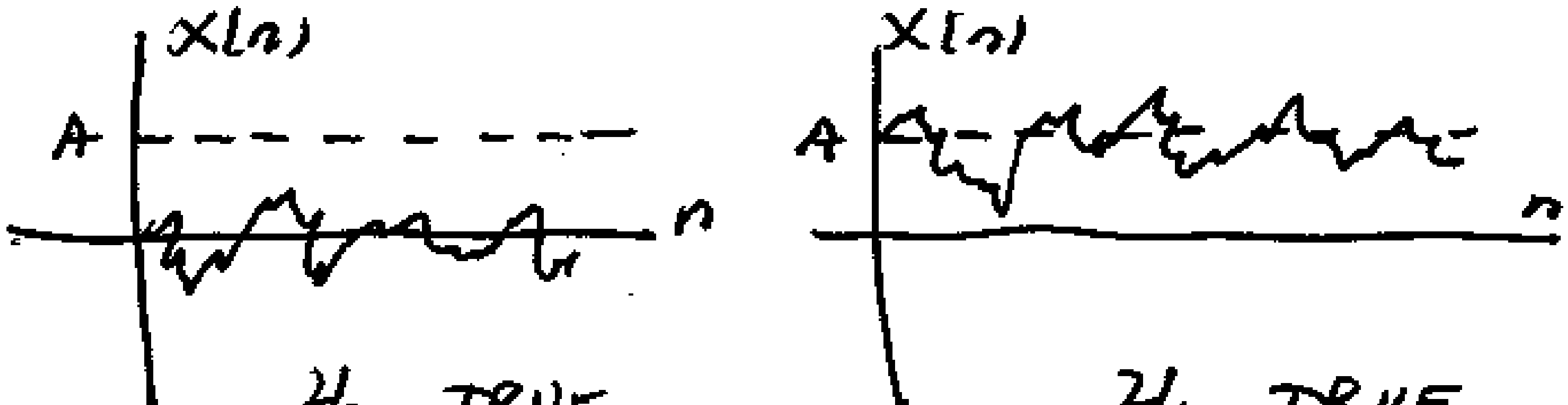
$$-\frac{1}{2\sigma^2}\left(-2A\sum_n x(n) + NA^2\right) > \ln\gamma$$

$$\frac{A}{\sigma^2}\sum_n x(n) > \ln\gamma + \frac{NA^2}{2\sigma^2}$$

SINCE $A > 0$ (INEQUALITY DOESN'T CHANGE DIRECTION)

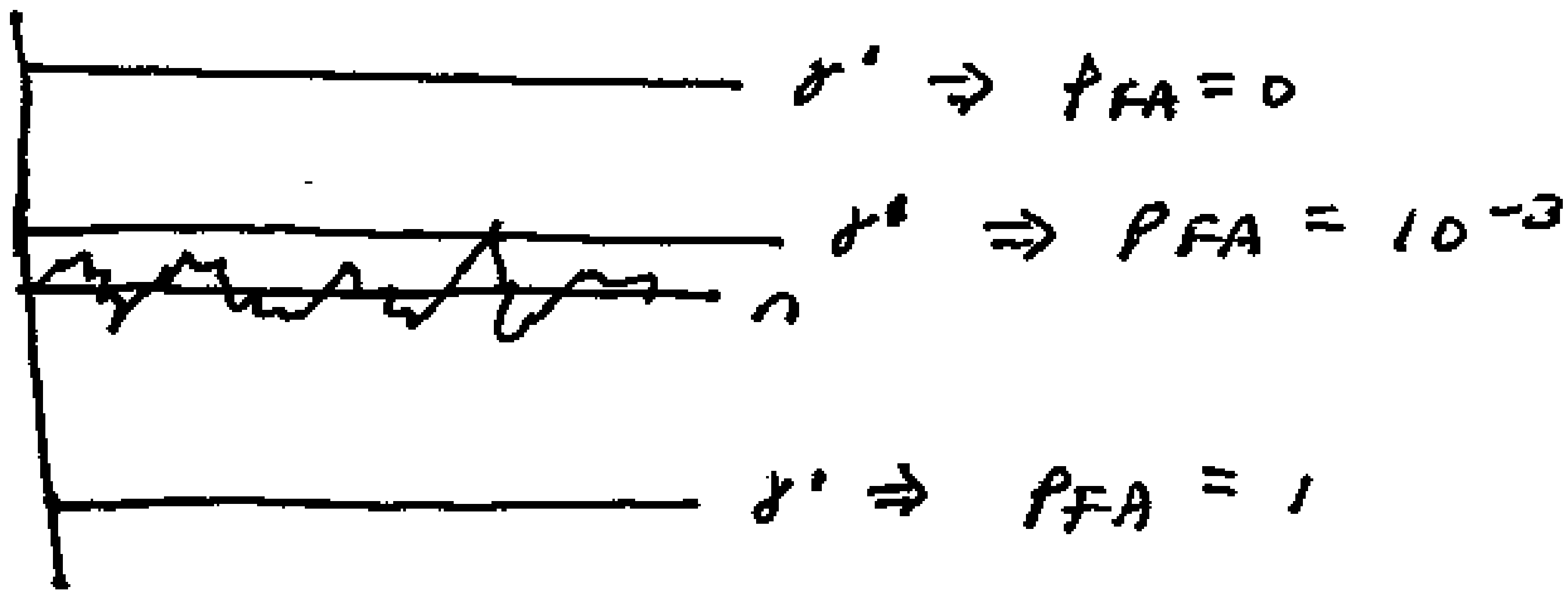
$$\frac{1}{N}\sum_{n=0}^{N-1} x[n] > \frac{\sigma^2}{NA}\ln\gamma + \frac{A}{2} = \gamma'$$

COMPARE SAMPLE MEAN TO THRESHOLD



$\mathcal{H}_0$ TRUE $\mathcal{H}_1$ TRUE

CHOICE OF γ' DEPENDS UPON P_{FA}



<u>PERFORMANCE EVALUATION</u>

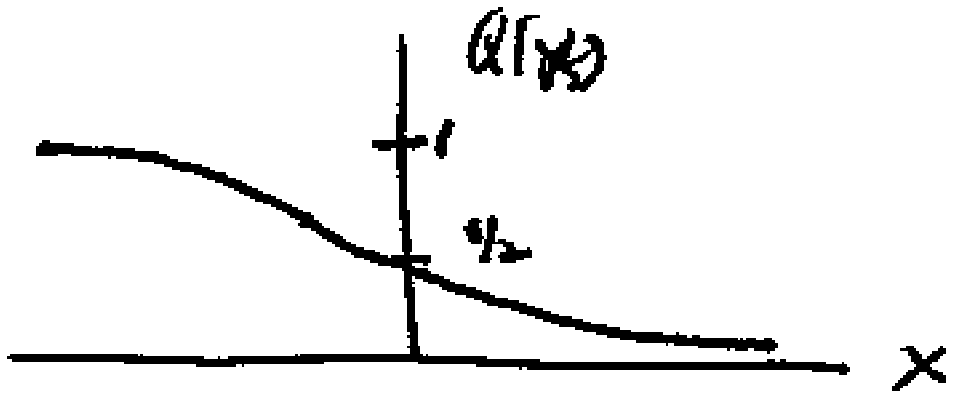
LET $T(\underline{x}) = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$

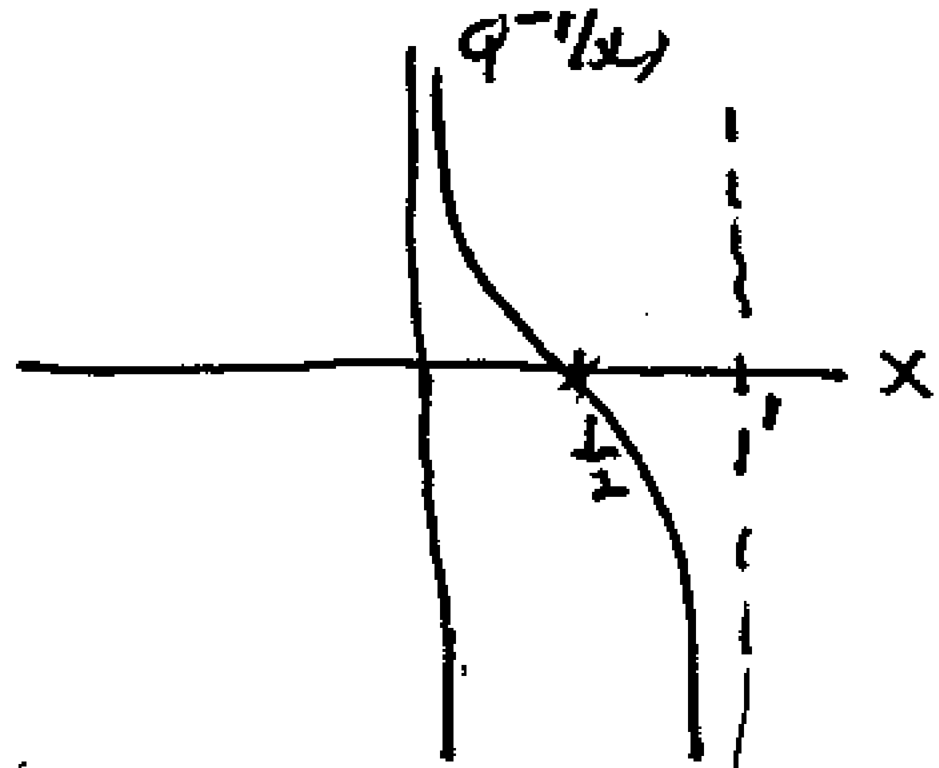
UNDER $\mathcal{H}_0$ $\quad T(\underline{x}) \sim \mathcal{N}(0, \sigma^2/N)$

$$P_{FA} = P_r\{T(\underline{x}) > \gamma' ; \mathcal{H}_0\}$$

$$= Q\left(\frac{\gamma'}{\sqrt{\sigma^2/N}}\right)$$

OR $\quad \gamma' = \sqrt{\sigma^2/N}\; Q^{-1}(P_{FA})$





UNDER $\mathcal{H}_1 \quad T(\underline{x}) \sim \mathcal{N}(A, \sigma^2/N)$

$$P_D = P_r\{T(\underline{x}) > \gamma' ; \mathcal{H}_1\}$$

$$= Q\left(\frac{\gamma' - A}{\sqrt{\sigma^2/N}}\right)$$

$$= Q\left(Q^{-1}(P_{FA}) - \sqrt{\frac{A^2}{\sigma^2/N}}\right)$$

$$= Q\left(Q^{-1}(P_{FA}) - \sqrt{\underbrace{NA^2/\sigma^2}_{\text{ENERGY-TO-NOISE RATIO}}}\right)$$

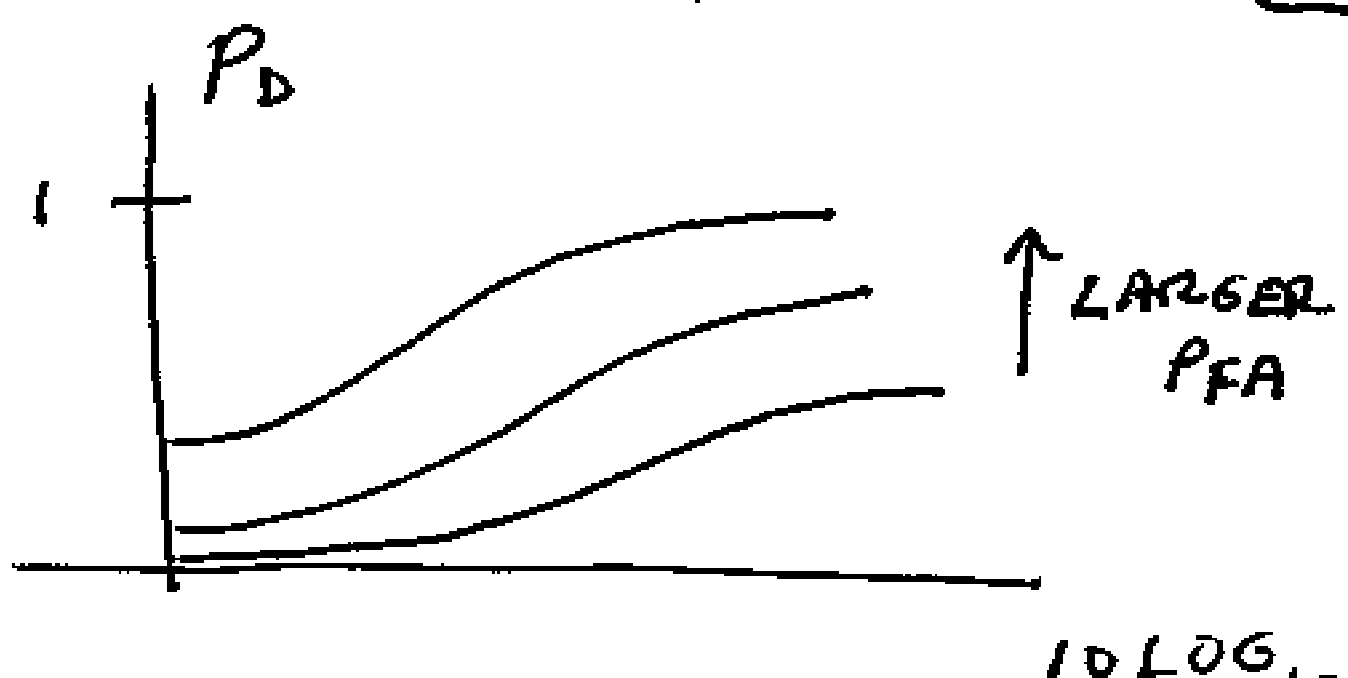


FIG. 3.5

GIVEN P_{FA}, P_D DEPENDS ONLY ON

$$d^2 = \frac{NA^2}{\sigma^2}$$

$$= \frac{\left[E(T; \mathcal{H}_1) - E(T; \mathcal{H}_0)\right]^2}{VAR(T; \mathcal{H}_0)}$$

CALLED DEFLECTION COEFFICIENT – MEASURES DIFFERENCE BETWEEN $p(\underline{x}; \mathcal{H}_0)$ AND $p(\underline{x}; \mathcal{H}_1)$

P_D MONOTONICALLY INCREASES WITH d^2.

NOTE: $T(\underline{x})$ CALLED <u>TEST STATISTIC</u>
RECALL $\frac{1}{N} \sum x[n]$ IS SUFFICIENT STATISTIC. IF S.S. EXISTS THEN $T(\underline{x})$ WILL BE A FUNCTION OF IT.

<u>PARAMETER TESTS</u>

PREVIOUS PROBLEM IS ACTUALLY A <u>PARAMETER</u> <u>TESTING</u> PROBLEM.

UNDER $\mathcal{H}_0$ $\quad \underline{x} \sim \mathcal{N}(\underline{0}, \sigma^2 \underline{I})$
UNDER $\mathcal{H}_1$ $\quad \underline{x} \sim \mathcal{N}(A\underline{1}, \sigma^2 \underline{I})$

$\underline{1} = [1\,1 \ldots 1]^T$. CONSIDER THE FAMILY OF PDFS

$$p(\underline{x}; A) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} (x[n]-A)^2}$$

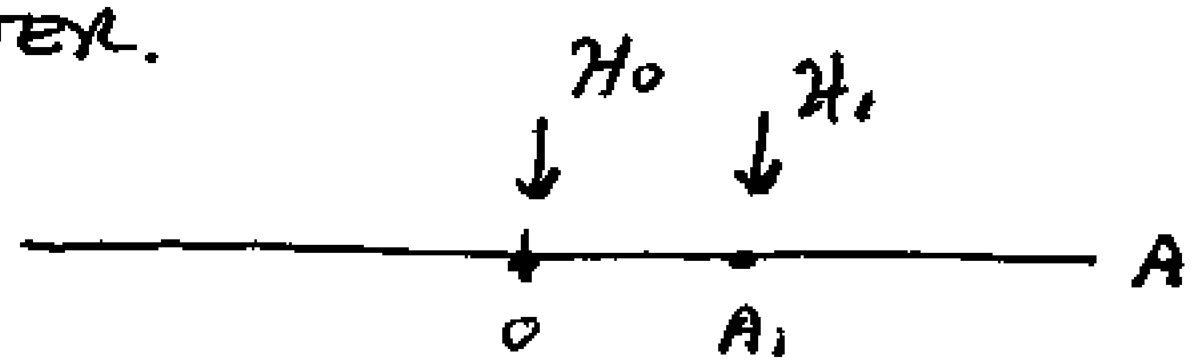
DETECTION PROBLEM ASKS IF $A=0$ (NO SIGNAL) OR $A = A_1 > 0$ (SIGNAL PRESENT) OR

$$\begin{aligned} \mathcal{H}_0 &: \quad A = 0 \\ \mathcal{H}_1 &: \quad A = A_1 \end{aligned}$$

N-P SOLUTION IS TO DECIDE $\mathcal{H}_1$ IF

$$\frac{p(\underline{x}; A=A_1)}{p(\underline{x}; A=0)} > \gamma$$

THIS VIEWPOINT WILL BE USEFUL LATER.



RECEIVER OPERATING CHARACTERISTICS (ROCS)

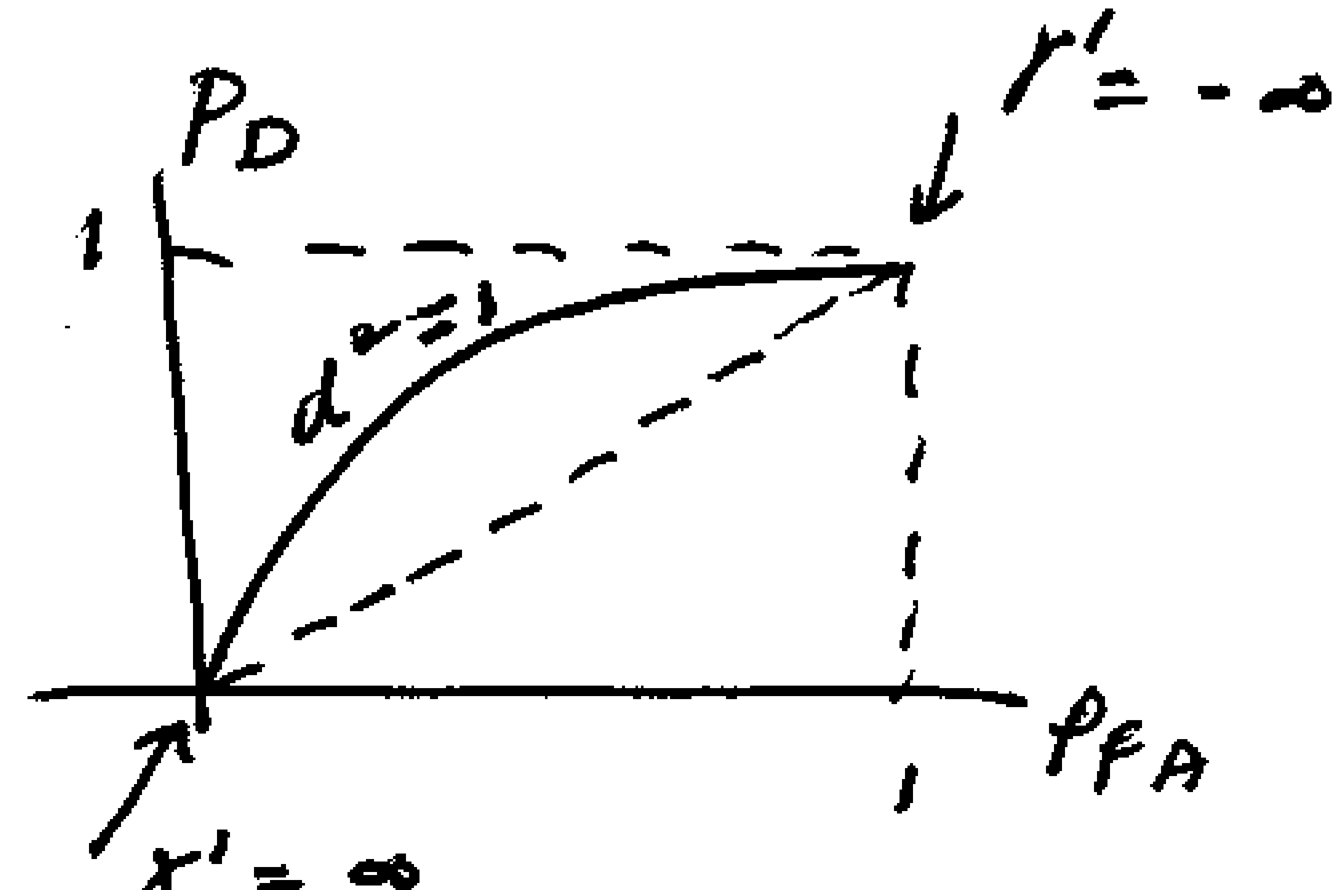
TO SUMMARIZE PERFORMANCE

$$P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{d^2}\right)$$

WHERE $d^2 = NA^2/\sigma^2$

SEE Q.m, Qinv.m CHAPTER 2

P_{FA}	P_D
10^{-3}	0.0183
10^{-2}	0.0924
10^{-1}	0.3891
1	1



AS WE RAISE THRESHOLD P_{FA} GOES DOWN BUT ALSO P_D.

ROC SHOULD BE ABOVE 45° LINE – ELSE WE CAN DO BETTER BY FLIPPING COIN

<u>EXAMPLE</u>: FLIP FAIR COIN

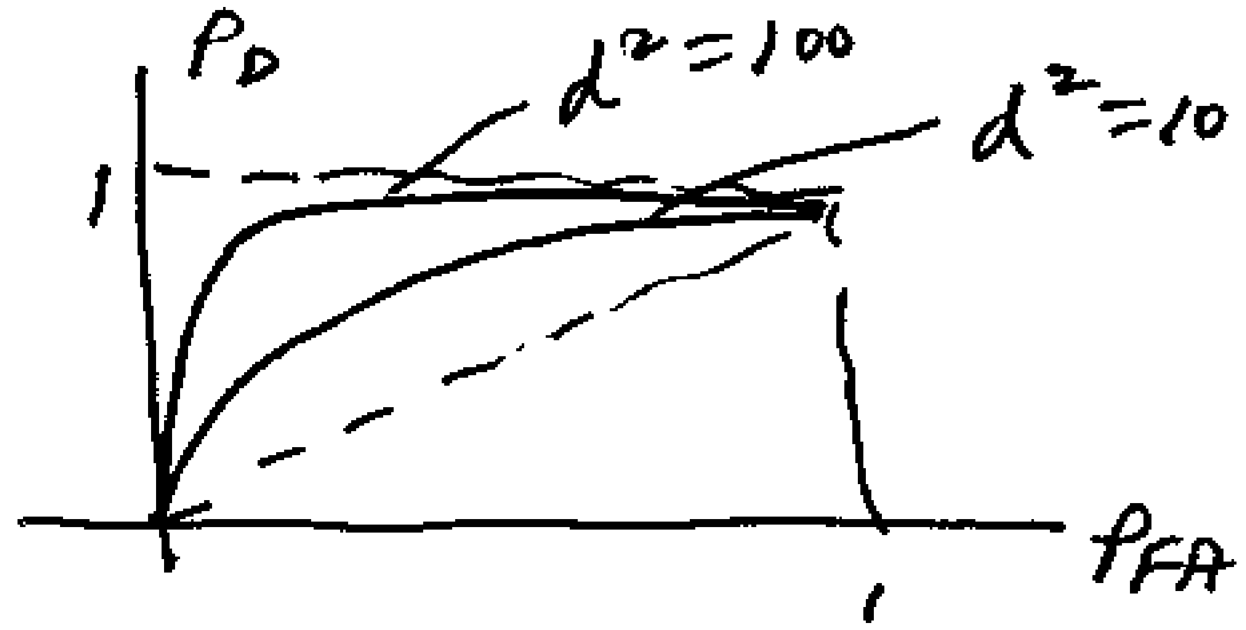
DECIDE $\mathcal{H}_1$ IF HEADS

DECIDE $\mathcal{H}_0$ IF TAILS

$\Rightarrow \quad P_{FA} = P_r\{\text{TAILS}\} = 1/2$

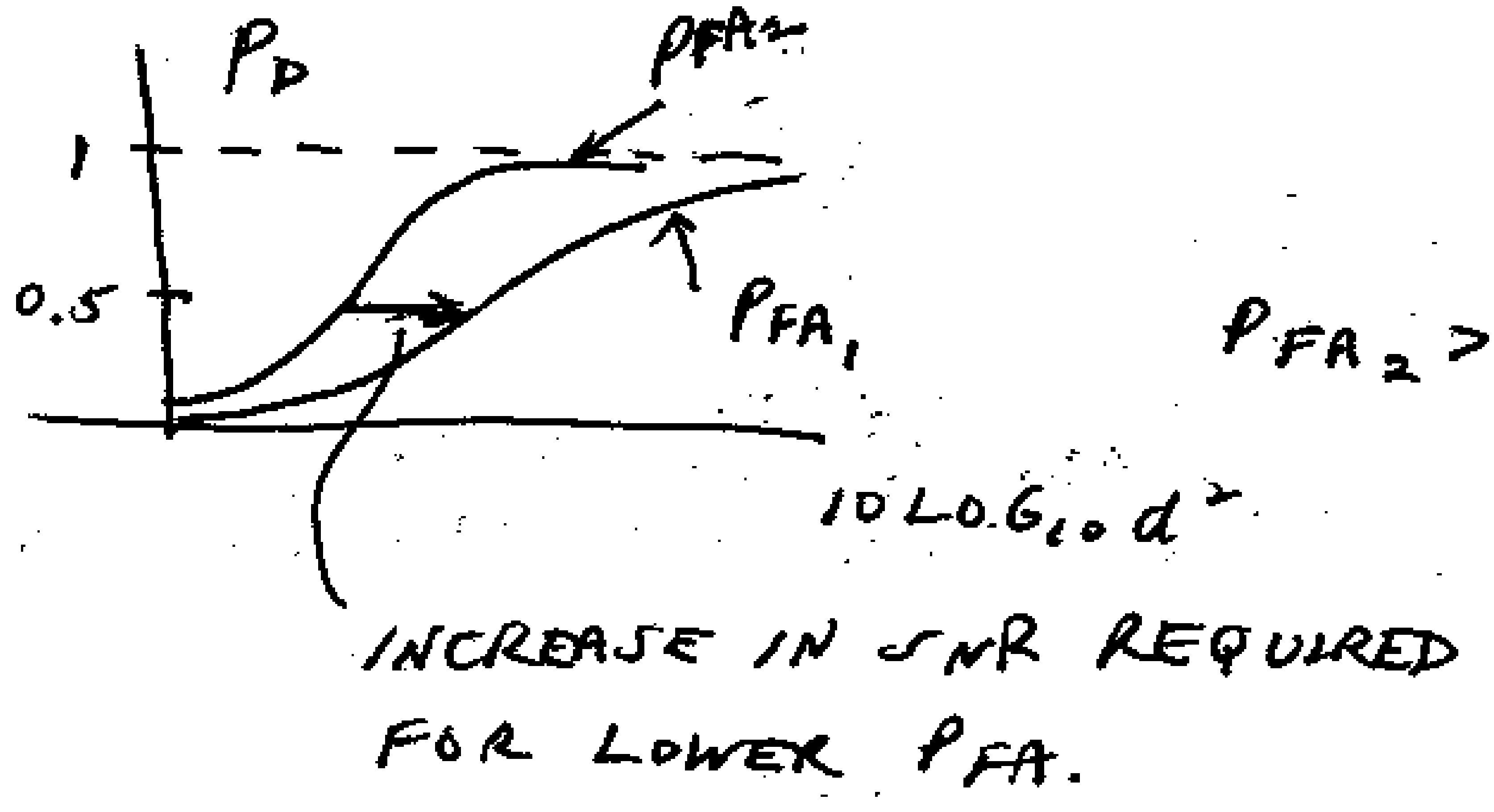
$\quad\quad P_D = P_r\{\text{HEADS}\} = 1/2$

PERFORMANCE IMPROVES WITH d^2



FOR GIVEN P_{FA}, P_D INCREASES.

ANOTHER WAY TO DISPLAY PERFORMANCE

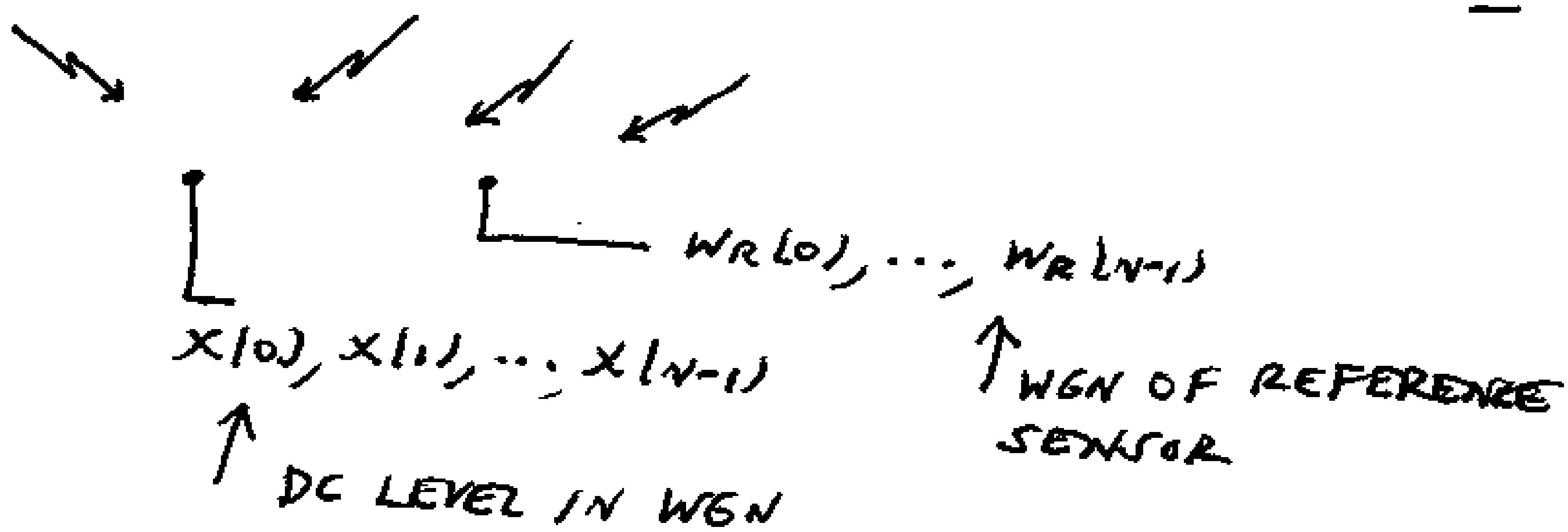


$P_{FA_2} > P_{FA_1}$

IRRELEVANT DATA

WHEN CAN WE DISCARD DATA WITHOUT REDUCING EFFECTIVENESS OF N-P DETECTOR?

EXAMPLE: ARRAY PROCESSING



DEPENDS ON STATISTICAL RELATIONSHIP OF $w_R[n]$ TO $x[n]$. IF $w_R[n]$ IS INDEPENDENT OF $w[n]$, CAN DISCARD.

HOW ABOUT IF $w_R[n] = w[n]$?

CONSIDER $\quad T'(\underline{x}) = \frac{1}{N} \sum_{n=0}^{N-1} (x[n] - w_R[n])$

UNDER $\mathcal{H}_0 \quad T'(\underline{x}) = 0$

UNDER $\mathcal{H}_1 \quad T'(\underline{x}) = A$

$\Rightarrow$ IF $\gamma' = A/2$ PERFECT DETECTION FOR ANY P_{FA}.

IN GENERAL CONSIDER TWO SETS OF DATA $\underline{x}_1, \underline{x}_2$. THE LIKELIHOOD RATIO IS

$$L(\underline{x}_1, \underline{x}_2) = \frac{p(\underline{x}_1, \underline{x}_2; \mathcal{H}_1)}{p(\underline{x}_1, \underline{x}_2; \mathcal{H}_0)}$$

$$= \frac{p(\underline{x}_2 | \underline{x}_1; \mathcal{H}_1)\, p(\underline{x}_1; \mathcal{H}_1)}{p(\underline{x}_2 | \underline{x}_1; \mathcal{H}_0)\, p(\underline{x}_1; \mathcal{H}_0)}$$

IF $= 1 \Rightarrow L(\underline{x}_1, \underline{x}_2)$ DOESN'T DEPEND ON $\underline{x}_2 \Rightarrow$ $\underline{x}_2$ IS IRRELEVANT

$$p(\underline{x}_2 | \underline{x}_1; \mathcal{H}_0) = p(\underline{x}_2 | \underline{x}_1; \mathcal{H}_1)$$

GIVEN $\underline{x}_1$, PDF OF $\underline{x}_2$ DOESN'T DEPEND ON WHICH HYPOTHESIS IS TRUE.

<u>EXAMPLE</u> : SPECIAL CASE - DC LEVEL IN WGN - EXTRA NOISE SAMPLES

$$p(\underline{x}_2|\underline{x}_1; \mathcal{H}_0) = p(\underline{x}_2; \mathcal{H}_0) = p(\underline{x}_2; \mathcal{H}_1) = p(\underline{x}_2|\underline{x}_1; \mathcal{H}_1)$$

$$\mathcal{H}_0 : \; x[n] = w[n] \qquad n = 0, 1, \ldots, 2N-1$$

$$\mathcal{H}_1 : x[n] = \begin{cases} A + w[n] & n = 0, 1, \ldots, N-1 \\ w[n] & n = N, \ldots, 2N-1 \end{cases}$$

$w[n]$ IS WGN $\Rightarrow$ $\{w[0], \ldots, w[N-1]\}$ IS INDEPENDENT OF $\{w[N], \ldots, w[2N-1]\}$

$$\Rightarrow \quad \underline{x}_2 = [w[N] \ldots, w[2N-1]]^T$$

$$p(\underline{x}_2; \mathcal{H}_0) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} x_2^2[n]}$$

$$= p(\underline{x}_2; \mathcal{H}_1)$$

WHAT IF $w[n]$ IS COLORED NOISE?

<u>CONCLUSION</u> : FOR DETECTION IN <u>WGN</u> LIMIT OBSERVATION INTERVAL TO SIGNAL INTERVAL

<u>MINIMUM PROB. OF ERROR</u>

MAKES SENSE AT TIMES TO CONSIDER $\mathcal{H}_0, \mathcal{H}_1$ AS RANDOM EVENTS AND ASSIGN PRIOR PROBABILITIES

<u>EXAMPLE</u> : RADAR DETECTION

$$P(\mathcal{H}_0) = P_r\{\text{NO TARGET}\} = ?$$

DIGITAL COMMUNICATIONS - EQUALLY LIKELY FOR "0" OR "1" TO BE SENT

$\mathcal{H}_0$: "0" SENT $\quad P(\mathcal{H}_0) = 1/2$
$\mathcal{H}_1$: "1" SENT $\quad P(\mathcal{H}_1) = 1/2$

ALLOWS US TO DEFINE PROBABILITY OF ERROR P_e

BAYESIAN APPROACH - MUST BE CAREFUL

IF WE ASSIGN $P(\mathcal{H}_1) = 0.99$, WHAT WILL DETECTOR DO?

DEFINE P_e AS

$$P_e = P_r\{\text{DECIDE } \mathcal{H}_0, \mathcal{H}_1 \text{ TRUE}\} + P_r\{\text{DECIDE } \mathcal{H}_1, \mathcal{H}_0 \text{ TRUE}\}$$

$$= \underbrace{P_r\{\text{DECIDE } \mathcal{H}_0 \mid \mathcal{H}_1 \text{ TRUE}\}}_{P(\mathcal{H}_0 \mid \mathcal{H}_1)} \underbrace{P_r\{\mathcal{H}_1 \text{ TRUE}\}}_{P(\mathcal{H}_1)} + \underbrace{P_r\{\text{DECIDE } \mathcal{H}_1 \mid \mathcal{H}_0 \text{ TRUE}\}}_{P(\mathcal{H}_1 \mid \mathcal{H}_0)} \underbrace{P_r\{\mathcal{H}_0 \text{ TRUE}\}}_{P(\mathcal{H}_0)}$$

CAN BE SHOWN THAT TO MINIMIZE P_e SHOULD DECIDE $\mathcal{H}_1$ IF

$$\frac{p(\underline{x} \mid \mathcal{H}_1)}{p(\underline{x} \mid \mathcal{H}_0)} > \frac{P(\mathcal{H}_0)}{P(\mathcal{H}_1)} = \gamma$$

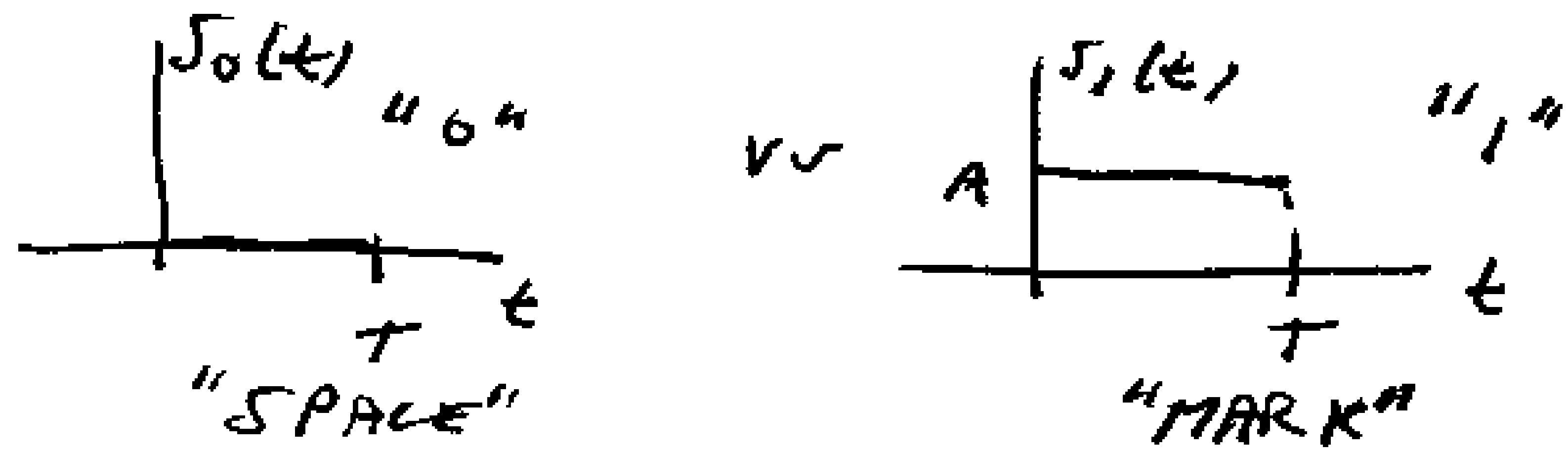
CONDITIONAL LIKELIHOOD RATIO

EXAMPLE: $P(\mathcal{H}_0) = P(\mathcal{H}_1) \Rightarrow$

CHOOSE $\mathcal{H}_1$ IF $p(\underline{x} \mid \mathcal{H}_1) > p(\underline{x} \mid \mathcal{H}_0)$
OR MORE LIKELY HYPOTHESIS

CALLED A MAXIMUM LIKELIHOOD (ML) DETECTOR

EXAMPLE: DC LEVEL IN WGN OR ON-OFF COMM. SYSTEM



AFTER SAMPLING RECEIVED WAVEFORM:

$$\mathcal{H}_0 : x[n] = w[n] \qquad n = 0, 1, \ldots, N-1$$
$$\mathcal{H}_1 : x[n] = A + w[n] \qquad n = 0, 1, \ldots, N-1$$

$A > 0$, $w[n]$ IS WGN, $P(\mathcal{H}_0) = P(\mathcal{H}_1)$

THUS, $\gamma = 1$ AND WE DECIDE $\mathcal{H}_1$ IF

$$\frac{p(\underline{x} \mid \mathcal{H}_1)}{p(\underline{x} \mid \mathcal{H}_0)} > 1$$

$$\frac{\frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2} \sum_n (x[n]-A)^2}}{\frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2} \sum_n x^2[n]}} > 1$$

TAKE LN $\Rightarrow$

$$-\frac{1}{2\sigma^2}\left(-2A \sum_n x[n] + NA^2\right) > 0$$

$$\Rightarrow \quad \bar{x} > A/2 \qquad \bar{x} = \frac{1}{N} \sum_{n=0}^{N-1} x[n]$$

(SAMPLE MEAN)

SAME AS FOR N-P EXCEPT FOR THRESHOLD (CHOSEN TO MINIMIZE P_e)

WHAT IS P_e?

$$\bar{x} \sim N(0, \sigma^2/N) \quad \text{COND. ON } \mathcal{H}_0$$
$$\sim N(A, \sigma^2/N) \quad \text{COND. ON } \mathcal{H}_1$$

$$P_e = \tfrac{1}{2}\left[P(\mathcal{H}_0|\mathcal{H}_1) + P(\mathcal{H}_1|\mathcal{H}_0)\right]$$
$$= \tfrac{1}{2}\left(P_r\{\bar{x} < \tfrac{A}{2} | \mathcal{H}_1\} + P_r\{\bar{x} > \tfrac{A}{2} | \mathcal{H}_0\}\right)$$
$$= \tfrac{1}{2}\left[1 - Q\left(\frac{A/2 - A}{\sqrt{\sigma^2/N}}\right) + Q\left(\frac{A/2}{\sqrt{\sigma^2/N}}\right)\right]$$
$$= Q\left(\sqrt{\frac{NA^2}{4\sigma^2}}\right)$$

↑ NOTE NA^2/σ^2

NEED $\underline{NA^2/\sigma^2 = 108}$ OR ≈ 20 dB FOR $P_e = 10^{-7}$

ALTERNATIVE FORM OF MIN P_e DETECTOR:

DECIDE $\mathcal{H}_1$ IF

$$p(\underline{x}|\mathcal{H}_1)P(\mathcal{H}_1) > p(\underline{x}|\mathcal{H}_0)P(\mathcal{H}_0)$$

BUT $$P(\mathcal{H}_i|\underline{x}) = \frac{p(\underline{x}|\mathcal{H}_i)P(\mathcal{H}_i)}{p(\underline{x})}$$ (BAYES' RULE)

WHERE $$p(\underline{x}) = p(\underline{x}|\mathcal{H}_0)P(\mathcal{H}_0) + p(\underline{x}|\mathcal{H}_1)P(\mathcal{H}_1)$$

DOESN'T DEPEND ON $\mathcal{H}_i$ $\Rightarrow$

DECIDE $\mathcal{H}_1$ IF

$$P(\mathcal{H}_1 | \underline{x}) > P(\mathcal{H}_0 | \underline{x})$$

GIVEN DATA CHOOSE HYPOTHESIS WITH GREATER PROBABILITY.

MAXIMUM A POSTERIORI (MAP) DETECTOR

AFTER DATA OBSERVED

WHAT IS EFFECT OF $P(\mathcal{H}_0)$, $P(\mathcal{H}_1)$ ON DECISION REGIONS? SEE FIG. 3.10.

READ ABOUT BAYES RISK

MULTIPLE HYPOTHESES

N-P DOESN'T EASILY EXTEND – HAVE TO ASSIGN TOO MANY ERROR PROBABILITIES

	$\mathcal{H}_0$	$\mathcal{H}_1$	$\mathcal{H}_2$
$\mathcal{H}_0$		?	?
$\mathcal{H}_1$	?		?
$\mathcal{H}_2$	?	?	

$P(\mathcal{H}_i; \mathcal{H}_j)$

MIN. Pe DOES

MULTIPLE HYPOTHESIS TESTING ≡

DISCRIMINATION ≡ CLASSIFICATION

$$Pe = \sum_{\substack{i=0 \\ i \neq j}}^{M-1} \sum_{j=0}^{M-1} P(\mathcal{H}_i, \mathcal{H}_j)$$

$$= \sum_{\substack{i=0 \\ i \neq j}}^{M-1} \sum_{j=0}^{M-1} P(\mathcal{H}_i | \mathcal{H}_j) P(\mathcal{H}_j)$$

TO MINIMIZE Pe DECIDE $\mathcal{H}_k$ IF

$$P(\mathcal{H}_k | \underline{x}) > P(\mathcal{H}_i | \underline{x}) \quad i \neq k$$

MAP RULE.

<u>PROOF</u> :

$$P(\mathcal{H}_i | \mathcal{H}_j) = \int_{R_i} p(\underline{x} | \mathcal{H}_j) \, d\underline{x}$$

$$Pe = \sum_{\substack{i \\ i \neq j}} \sum_{j} \int_{R_i} p(\underline{x} | \mathcal{H}_j) P(\mathcal{H}_j) \, d\underline{x}$$

$$= \sum_i \int_{R_i} \underbrace{\sum_{\substack{j \\ j \neq i}} p(\underline{x} \mid H_j) P(H_j)}_{\geq 0} \, d\underline{x}$$

TO MINIMIZE P_e NOTE THAT WE HAVE TO ASSIGN EACH $\underline{x}$ TO ONE AND ONLY ONE DECISION REGION.

CONSIDER $\underline{X} = \underline{x}_0$ AND ASSIGN IT TO R_2. THEN, P_e INCREASES BY

$$\sum_{\substack{j \\ j \neq 2}} p(\underline{x}_0 \mid H_j) P(H_j) \, d\underline{x}$$

IF WE ASSIGN IT TO R_3 WE INCUR

$$\sum_{\substack{j \\ j \neq 3}} p(\underline{x}_0 \mid H_j) P(H_j) \, d\underline{x}$$

THUS ASSIGN $\underline{x}$ TO R_k IF

$$\sum_{\substack{j \\ j \neq k}} p(\underline{x} \mid H_j) P(H_j) \text{ IS MINIMUM}$$