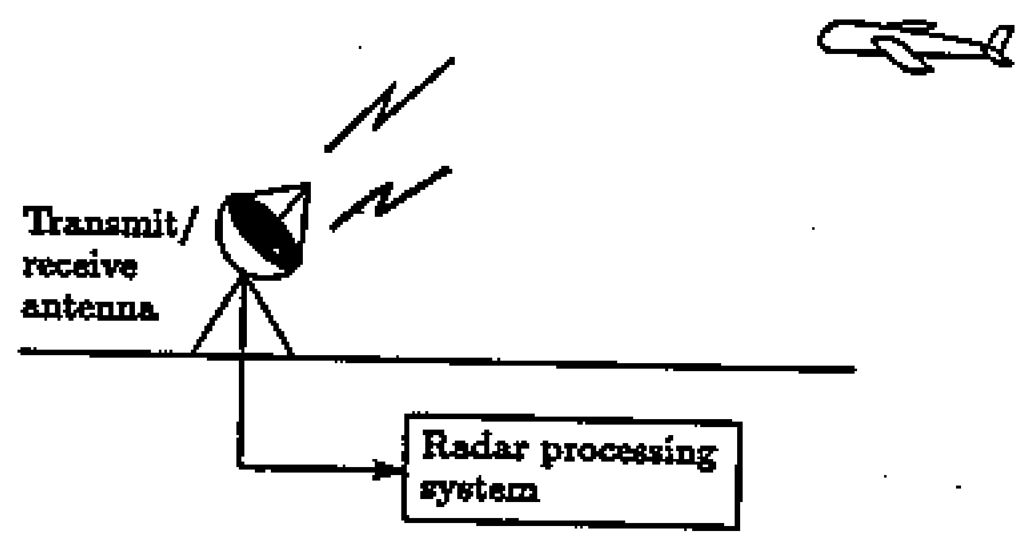
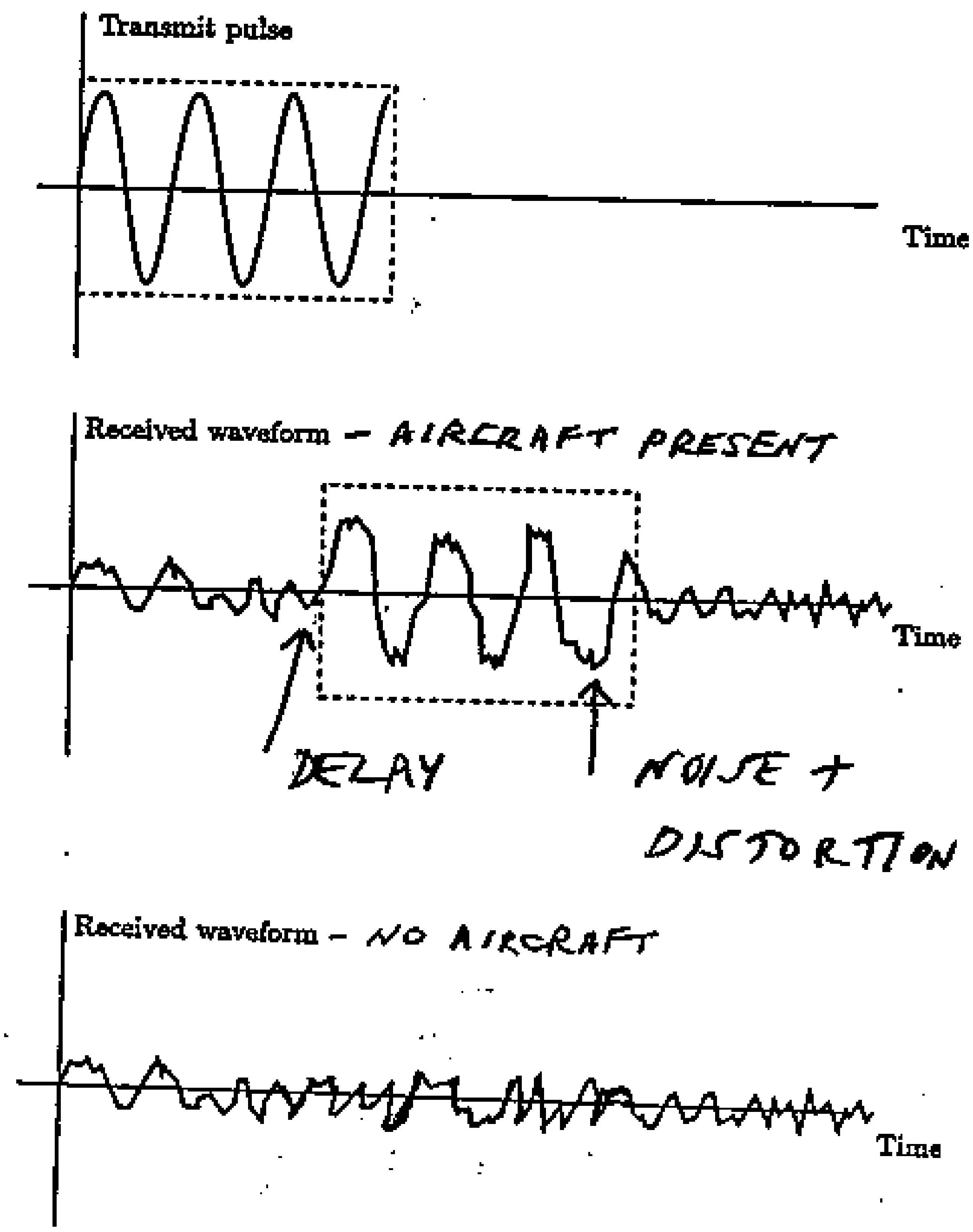
INTRODUCTION

APPLICATION AREAS

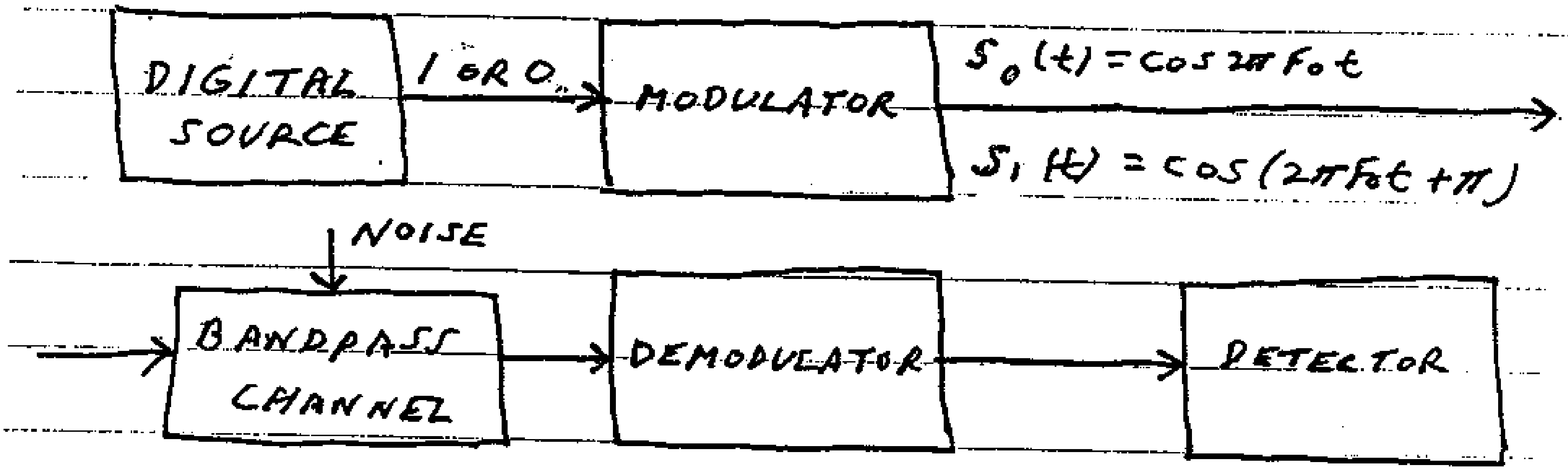
1) RADAR – DETERMINE IF AIRCRAFT IS APPROACHING



(a) Radar

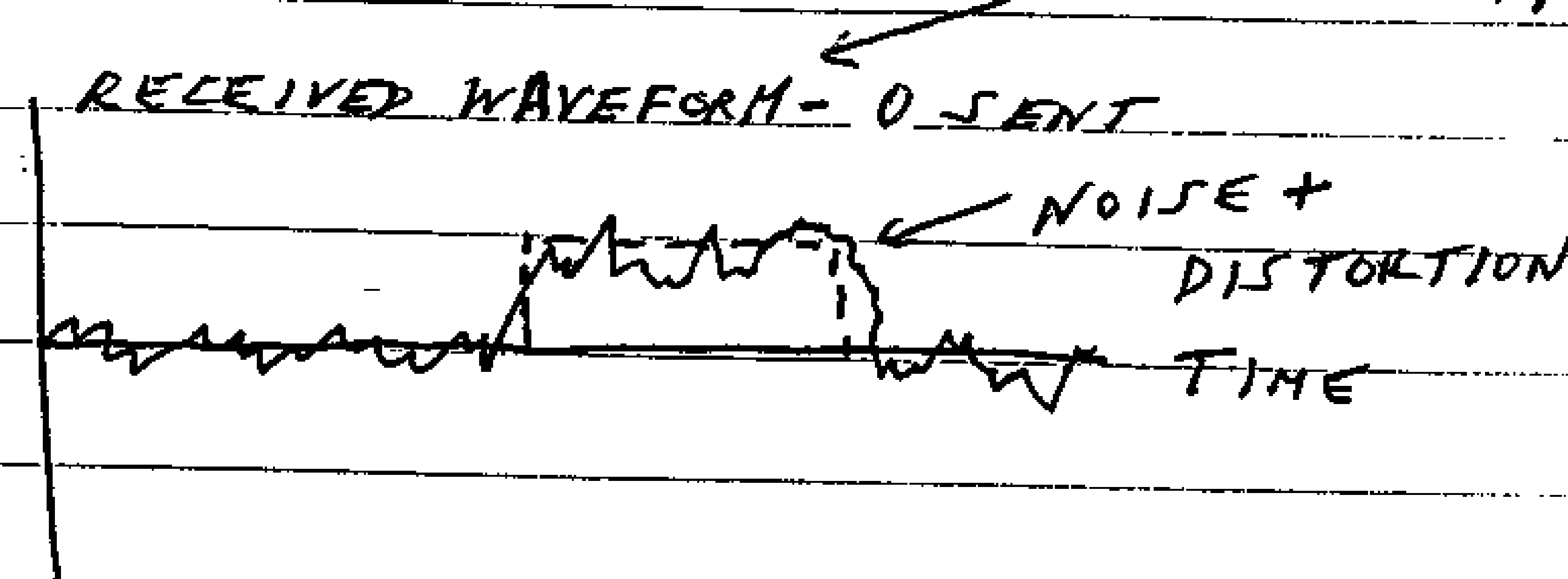


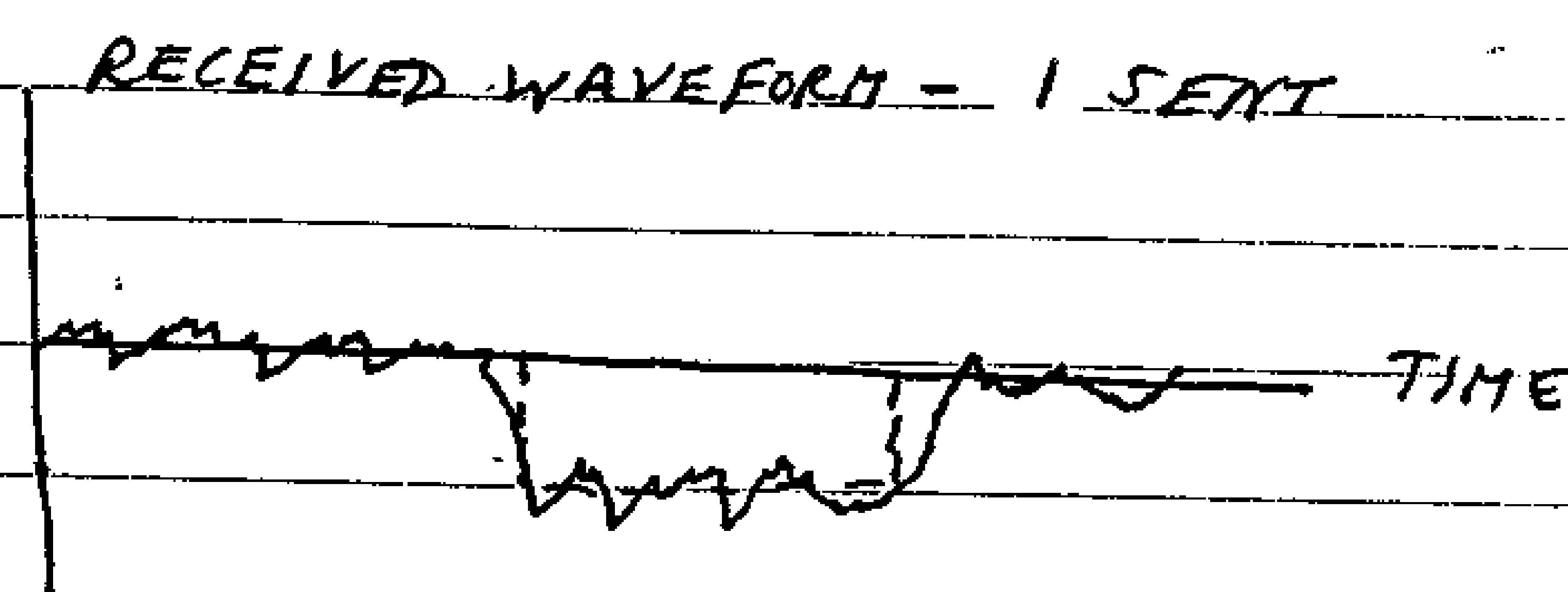
2) COMMUNICATIONS - BPSK



a) BASIC SYSTEM

AT DEMODULATOR OUTPUT





3) SPEECH – RECOGNIZE DIGITS,
$0 \rightarrow 9$

WAVEFORMS DEPEND ON SPEAKER
AND CHANGE FOR A GIVEN DIGIT
SEE FIG. 1.3

4) SONAR – DETECT ENEMY SUBMARINE

5) IMAGE PROCESSING – DETECT HELICOPTER
VIA INFRARED SURVEILLANCE

6) BIOMED – DETECT CARDIAC ARYTHMIA

7) CONTROL – DETECT ABRUPT CHANGE
IN CONTROLLED SYSTEM

8) SEISMIC – DETECT OIL DEPOSIT

ETC.

DETECTION PROBLEM

SIMPLEST – SIGNAL PRESENT VS.
SIGNAL ABSENT (RADAR)

TWO POSSIBILE CHOICES ⇒
BINARY HYPOTHESIS TESTING PROBLEM

ANOTHER EXAMPLE IS BPSK

$$s_0(t) = \cos 2\pi F_0 t$$

VS.

$$s_1(t) = \cos(2\pi F_0 t + \pi)$$

NOW WE HAVE A SIGNAL UNDER EITHER HYPOTHESIS.

EXAMPLE - SPEECH RECOGNITION

CHOICES ARE $0, 1, 2, \ldots, 9$

⇒ MULTIPLE HYPOTHESIS TESTING PROBLEM OR

PATTERN RECOGNITION OR CLASSIFICATION

TO SOLVE ALL THESE PROBLEMS USE STATISTICAL HYPOTHESIS TESTING THEORY

MATHEMATICAL DETECTION PROBLEM

GIVEN DATA SET $\{x[0], x[1], \ldots, x[N-1]\}$
FIND $T(x[0], x[1], \ldots, x[N-1])$ AND
BASED ON T, MAKE A DECISION

EXAMPLE: DC LEVEL IN WHITE GAUSSIAN NOISE (WGN), $N=1$

OBSERVE EITHER $x[0] = w[0]$ (NOISE)
OR $x[0] = A + w[0]$
$= 1 + w[0]$
(SIGNAL + NOISE)

HOW TO CHOOSE BASED ON VALUE OF $x[0]$ OBSERVED?

NOTE: $E(x[0]) =$ 0 NOISE
1 SIGNAL + NOISE

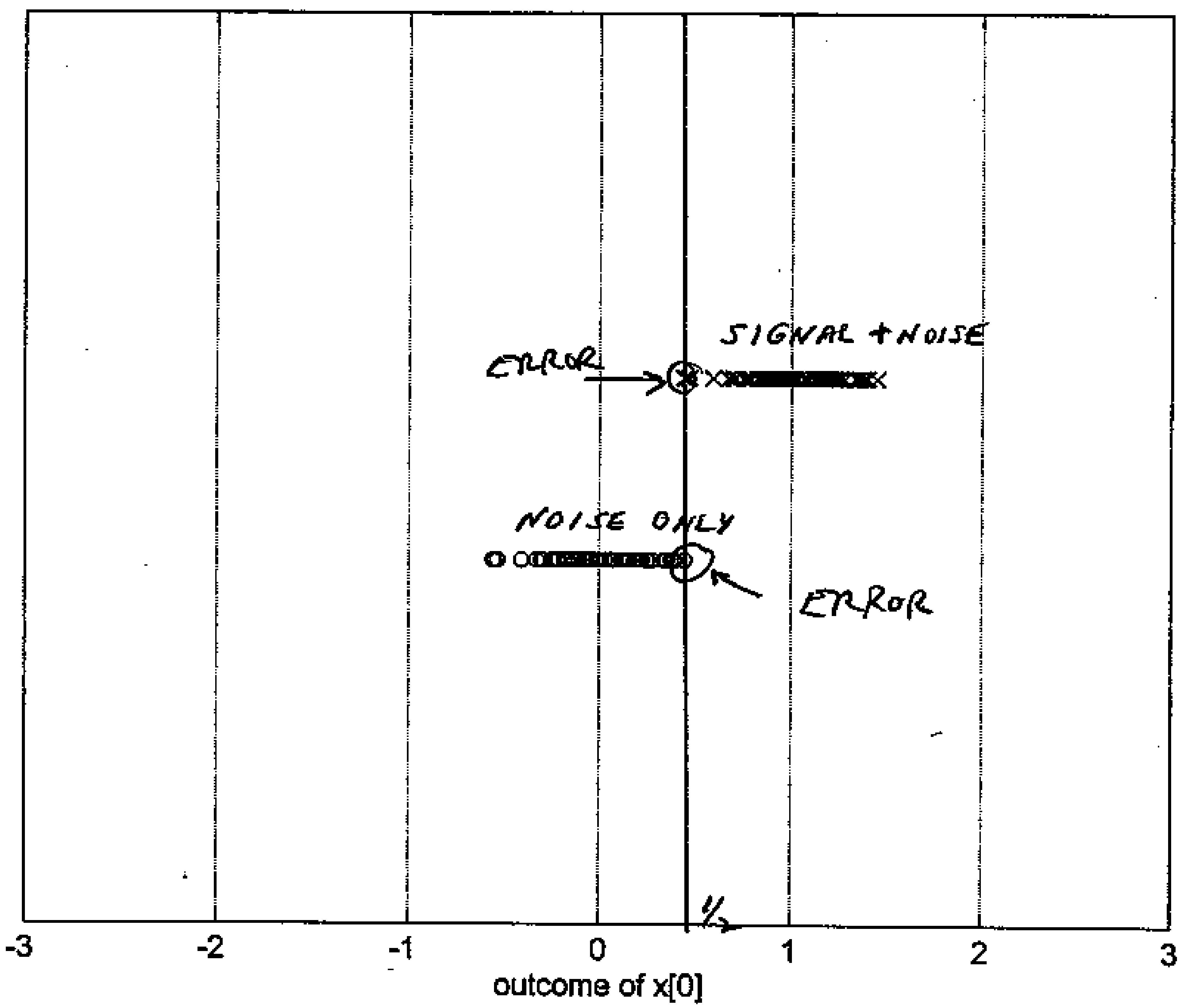
DETECTOR CHOOSES SIGNAL PRESENT IF

$$x[0] > 1/2$$

GOOD DETECTOR? WHEN WILL WE BE WRONG?

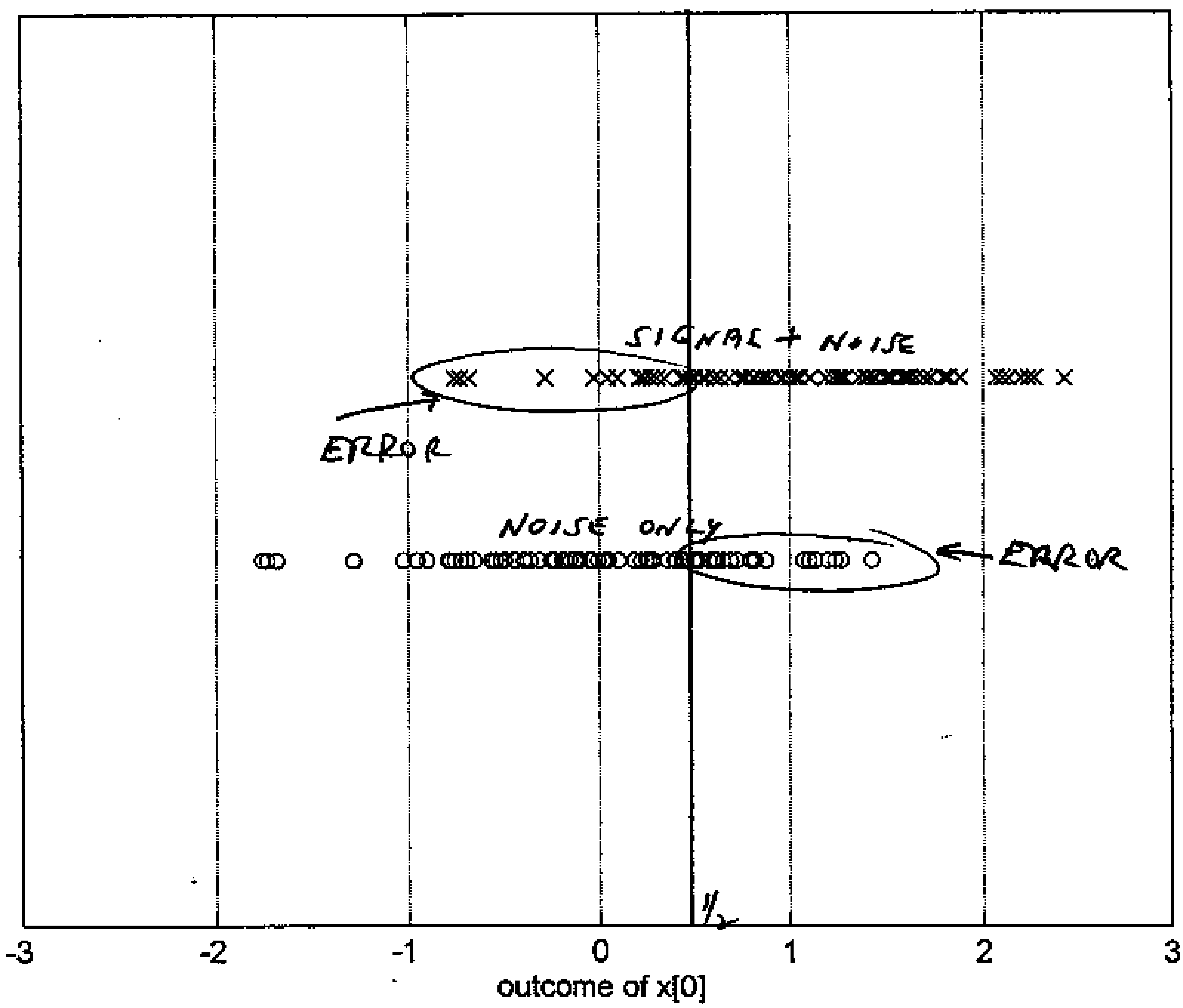
100 REALIZATIONS OF x[0]

$\sigma^2 = 0.05$



100 REALIZATIONS OF x[0]

$\sigma^2 = 0.5$



SEE FIG. 1.5 FOR HISTOGRAMS

PERFORMANCE DEPENDS ON SEPARATION BETWEEN PDFS. ↑ DISCRIMINATION

LET $\mathcal{H}_0$ = NOISE ONLY
$\mathcal{H}_1$ = SIGNAL + NOISE } HYPOTHESES

$\mathcal{H}_0$: $x[0] = w[0]$
$\mathcal{H}_1$: $x[0] = 1 + w[0]$

PDFS ARE

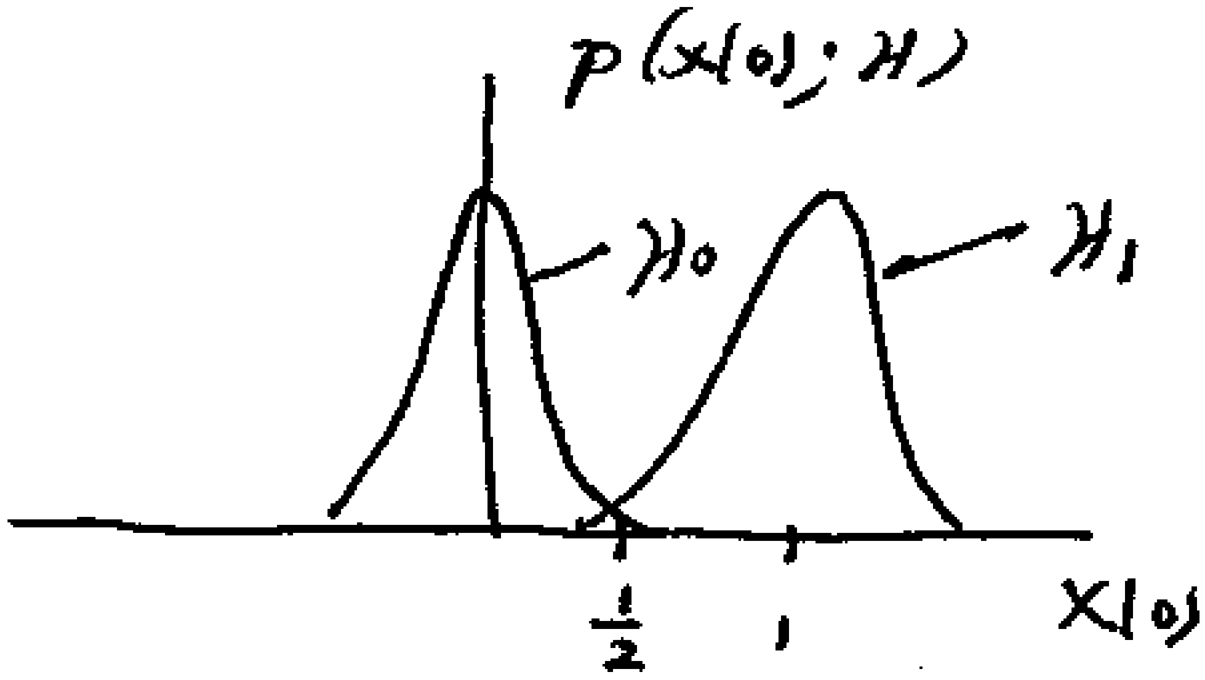
$$p(x[0]; \mathcal{H}_0) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2} x^2[0]}$$

$$p(x[0]; \mathcal{H}_1) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2} (x[0]-1)^2}$$

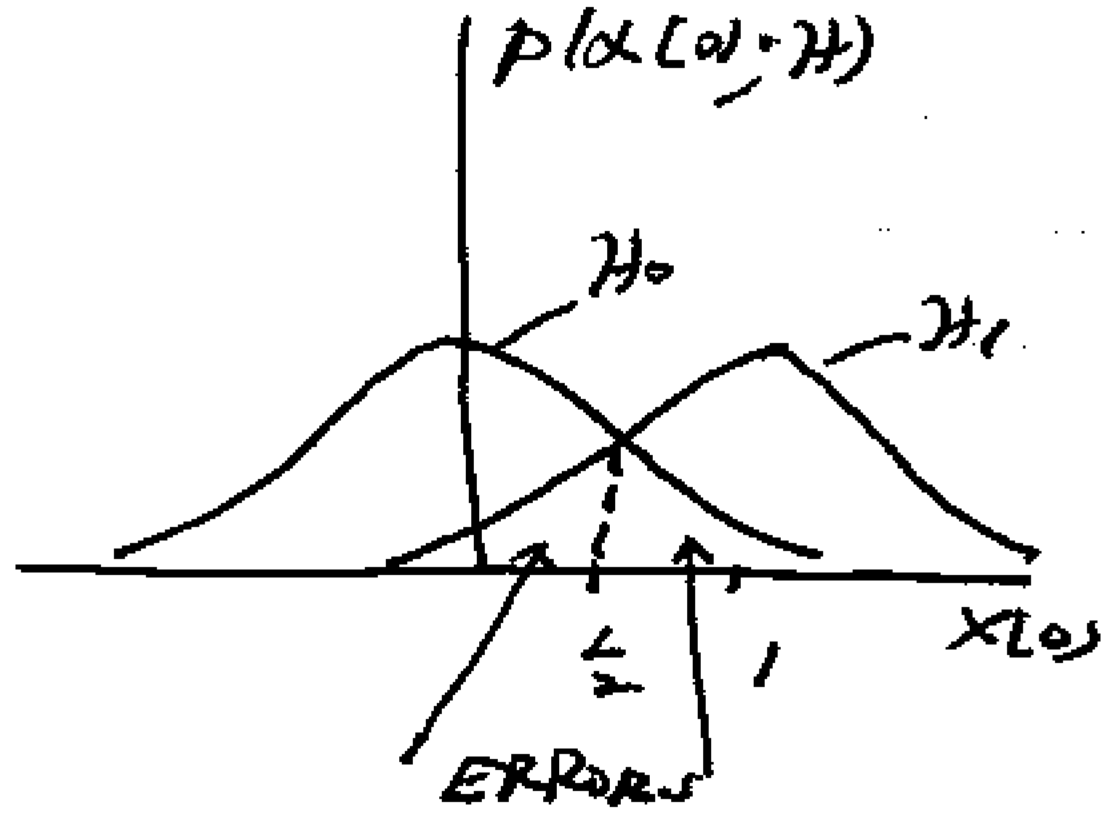
OR

$x[0] \sim N(0, \sigma^2)$ UNDER $\mathcal{H}_0$
$N(1, \sigma^2)$ " $\mathcal{H}_1$

SEE FIG. 1.6



$\sigma^2 = 0.05$



$\sigma^2 = 0.5$

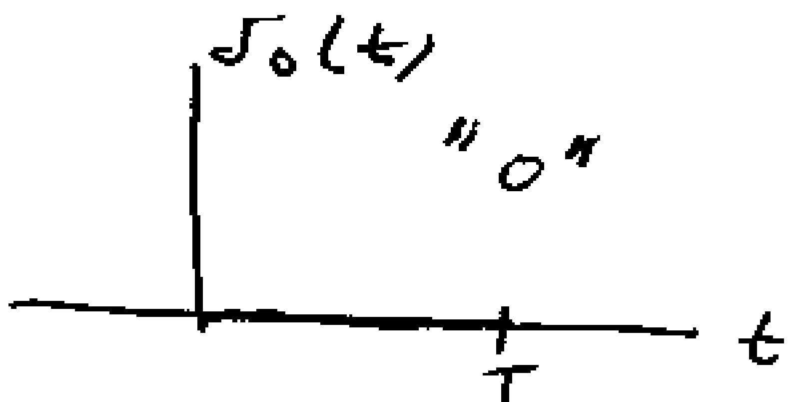
ALTERNATIVE VIEWPOINT: FAMILY OF PDFS

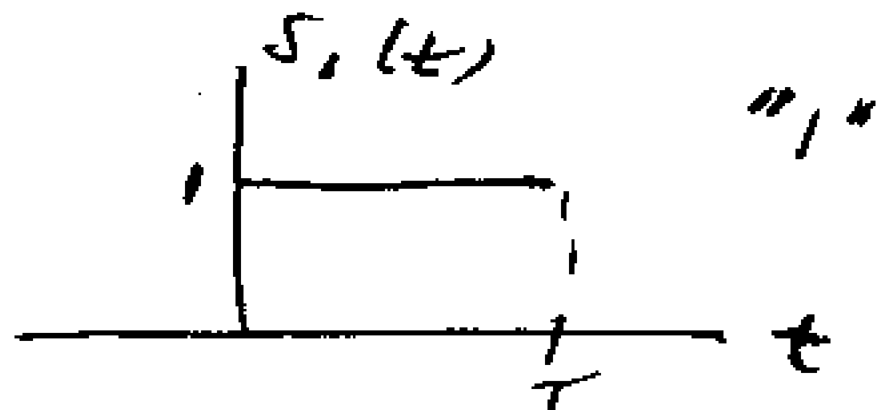
$$p(x[0];A) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x[0]-A)^2} \qquad -\infty < A < \infty$$

$$\left.\begin{aligned} \mathcal{H}_0 &: A = 0 \\ \mathcal{H}_1 &: A = 1 \end{aligned}\right\}$$ TERMED A <u>PARAMETER TEST</u>

SOMETIMES WE THINK OF $\mathcal{H}_0, \mathcal{H}_1$ AS RANDOM EVENTS.

<u>EXAMPLE</u>: ON-OFF KEYED COMMUNICATION SYSTEM





FOR A BIT STREAM ... 0100110101100...
USUALLY HAVE HALF "0"'S AND HALF
"1"'S $\Rightarrow$

$$\mathcal{H}_0 : \text{"0" WITH PROB.} = 1/2$$
$$\mathcal{H}_1 : \text{"1"} \quad \text{"} \quad \text{"} \quad \text{"}$$

OR $\quad Pr\{\mathcal{H}_0\} = 1/2$

$\quad\quad Pr\{\mathcal{H}_1\} = 1/2$

IN THIS CASE PDF DENOTED AS

$$\underset{\uparrow}{p(x[0] \mid \mathcal{H}_0)}, \quad \underset{\uparrow}{p(x[0] \mid \mathcal{H}_1)}$$

CONDITIONAL PDFS

ANALOGOUS TO CLASSICAL VS.
BAYESIAN ESTIMATION

DETECTION HIERARCHY

SEE TABLE 1.1.

NOISE

EASY-GOOD PERFORMANCE	GAUSSIAN KNOWN PDF	GAUSSIAN UNKNOWN PDF	NON GAUSSIAN KNOWN PDF	NON GAUSSIAN UNKNOWN PDF
SIGNAL: DETERMINISTIC-KNOWN	4	9	10	*
SIGNAL: DETERMINISTIC-UNKNOWN	7	9	10	*
SIGNAL: RANDOM-KNOWN PDF	5	9	*	*
SIGNAL: RANDOM-UNKNOWN PDF	8	*	*	*

HARD-POOR PERFORMANCE

* NOT DISCUSSED (BEYOND SCOPE OF TEXT)

TABLE 1.1 - HIERARCHY OF DETECTION PROBLEMS AND CHAPTERS WHERE DISCUSSED

READ CHAPTER 2 - WILL COVER SOME PARTS LATER

CHAPTER 3 - DECISION THEORY

EXAMPLE : OBSERVE $x(0)$ FROM EITHER

$N(0,1)$ OR $N(1,1)$

NORMAL OR GAUSSIAN — MEAN — VARIANCE

EQUIVALENT

$\mathcal{H}_0 : \mu = 0$ PARAMETER TEST
$\mathcal{H}_1 : \mu = 1$

OR
$\mathcal{H}_0 : x(0) = w(0)$
$\mathcal{H}_1 : x(0) = 1 + w(0)$

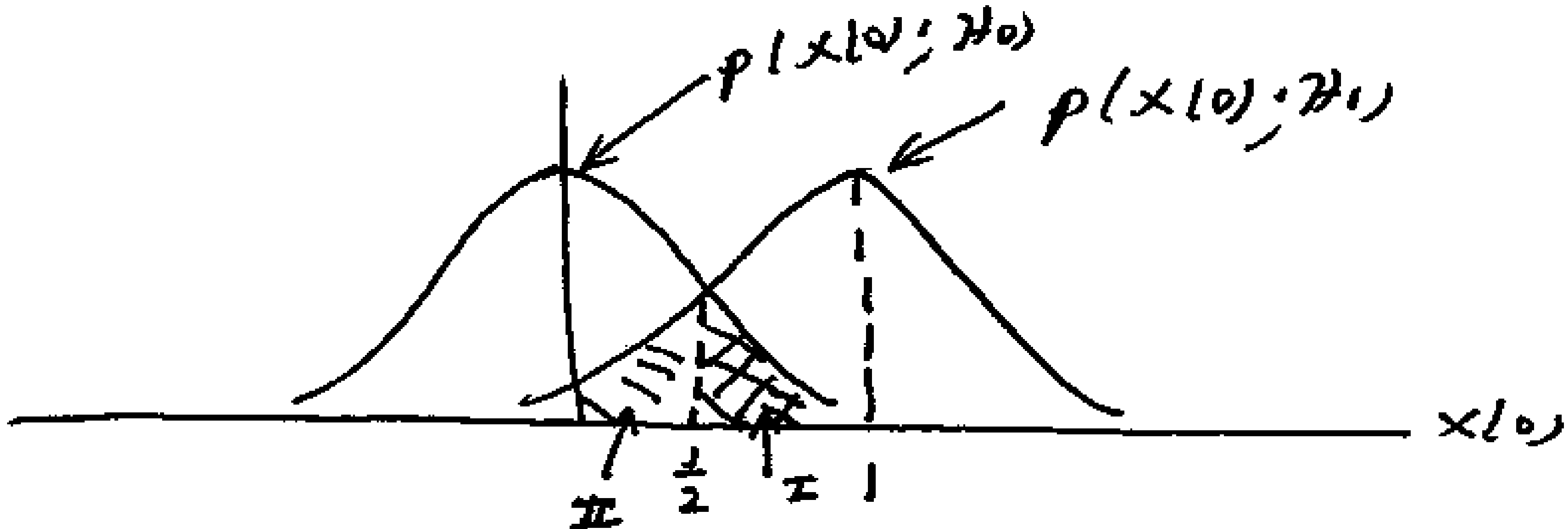
WHERE $w(0) \sim N(0,1)$

$\sim$ DISTRIBUTED ACCORDING TO

IN STATISTICS $\mathcal{H}_0$ = NULL HYPOTHESIS
$\mathcal{H}_1$ = ALTERNATIVE "

A REASONABLE APPROACH DECIDES $\mathcal{H}_1$ IF $x(0) > 1/2$

$1/2$ = THRESHOLD



← TYPE II ERROR → | ← TYPE I ERROR →

ERRORS - DECIDE $\mathcal{H}_1$ BUT $\mathcal{H}_0$ IS TRUE - I
" $\mathcal{H}_0$ BUT $\mathcal{H}_1$ " " - II

TYPE I ERROR HAS PROBABILITY

$$P(\mathcal{H}_1; \mathcal{H}_0) = P_r\{x[0] > \tfrac{1}{2}; \mathcal{H}_0\}$$

↑ DECIDED ↑ TRUE

TYPE II ERROR HAS PROBABILITY

$$P(\mathcal{H}_0; \mathcal{H}_1) = P_r\{x[0] < \tfrac{1}{2}; \mathcal{H}_1\}$$

CAN WE REDUCE ERRORS ?

TO REDUCE TYPE I NOTE THAT WE DECIDE $\mathcal{H}_1$ IF $x[0] > 1/2$. HOW ABOUT IF WE DECIDE $\mathcal{H}_1$ IF $x[0] > 1$?

AS THRESHOLD CHANGES ONE ERROR <u>INCREASES</u> WHILE OTHER <u>DECREASES</u>.

IN RADAR PROBLEM

$\mathcal{H}_0 : x[0] = w[0]$ NO AIRCRAFT
$\mathcal{H}_1 : x[0] = 1 + w[0]$ AIRCRAFT PRESENT

TYPE I ERROR IS TO DECIDE $\mathcal{H}_1$ WHEN $\mathcal{H}_0$ TRUE OR A <u>FALSE ALARM</u>.

$$\Rightarrow \quad P_{FA} = P(\mathcal{H}_1 ; \mathcal{H}_0)$$

WANT THIS SMALL, SAY $\alpha = 10^{-8}$. WHY?

CONSTRAIN $P_{FA} = \alpha$. GIVEN THIS ERROR PROBABILITY, WISH TO MINIMIZE TYPE II OR MAXIMIZE

$$1 - \underbrace{P(\mathcal{H}_0 ; \mathcal{H}_1)}_{\text{TYPE II}}$$

TYPE II ERROR IS TO DECIDE $\mathcal{H}_0$ (NO AIRCRAFT) WHEN $\mathcal{H}_1$ IS TRUE (AIRCRAFT PRESENT) $\Rightarrow$ THIS IS A <u>MISS</u>

$$P_M = P(\mathcal{H}_0; \mathcal{H}_1)$$

$$\begin{aligned} 1 - P_M &= 1 - P(\mathcal{H}_0; \mathcal{H}_1) \\ &= \text{PROBABILITY OF } \underline{\text{DETECTION}} \\ &= P_D \end{aligned}$$

<u>SUMMARY</u> : MAKES SENSE TO <u>MAXIMIZE</u> P_D SUBJECT TO CONSTRAINT $P_{FA} = \alpha$.
(NEYMAN - PEARSON APPROACH)

FOR PREVIOUS PROBLEM WE CAN ADJUST THRESHOLD OR DECIDE $\mathcal{H}_1$ IF

$$x[0] > \gamma$$

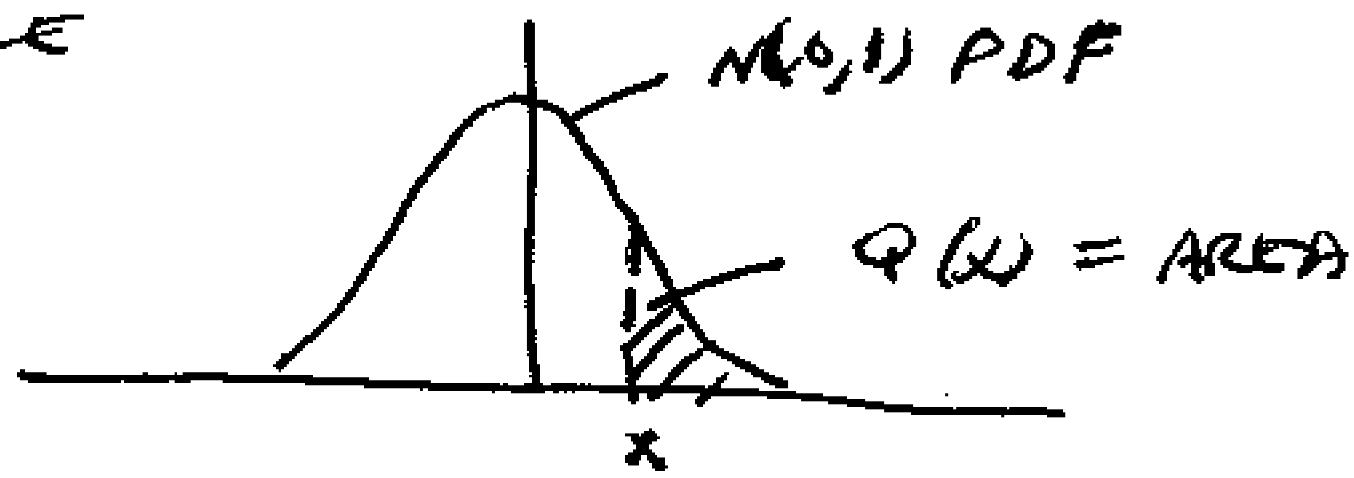
TO CONSTRAIN P_{FA}.

$$\begin{aligned} P_{FA} &= P(\mathcal{H}_1; \mathcal{H}_0) \\ &= P_r\{x[0] > \gamma; \mathcal{H}_0\} \end{aligned}$$

$$= \underbrace{\int_{\gamma}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}}_{\text{UNDER } \mathcal{H}_0 \text{ OR } x[0] \sim \mathcal{N}(0,1)} dt$$

DEFINE $\quad Q(x) = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$

RIGHT-TAIL PROBABILITY FOR $\mathcal{N}(0,1)$ RANDOM VARIABLE



$\Rightarrow \quad P_{FA} = Q(\gamma)$

FOR $P_{FA} = 10^{-3} \Rightarrow \gamma = 3$ (FROM TABLE)

DECIDE $\mathcal{H}_1$ IF $x[0] > 3$. TO DETERMINE P_D

$$P_D = P(\mathcal{H}_1 ; \mathcal{H}_1)$$

$$= P_r\{x[0] > 3 ; \mathcal{H}_1\}$$

$$= \int_{\gamma}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(t-1)^2} dt$$

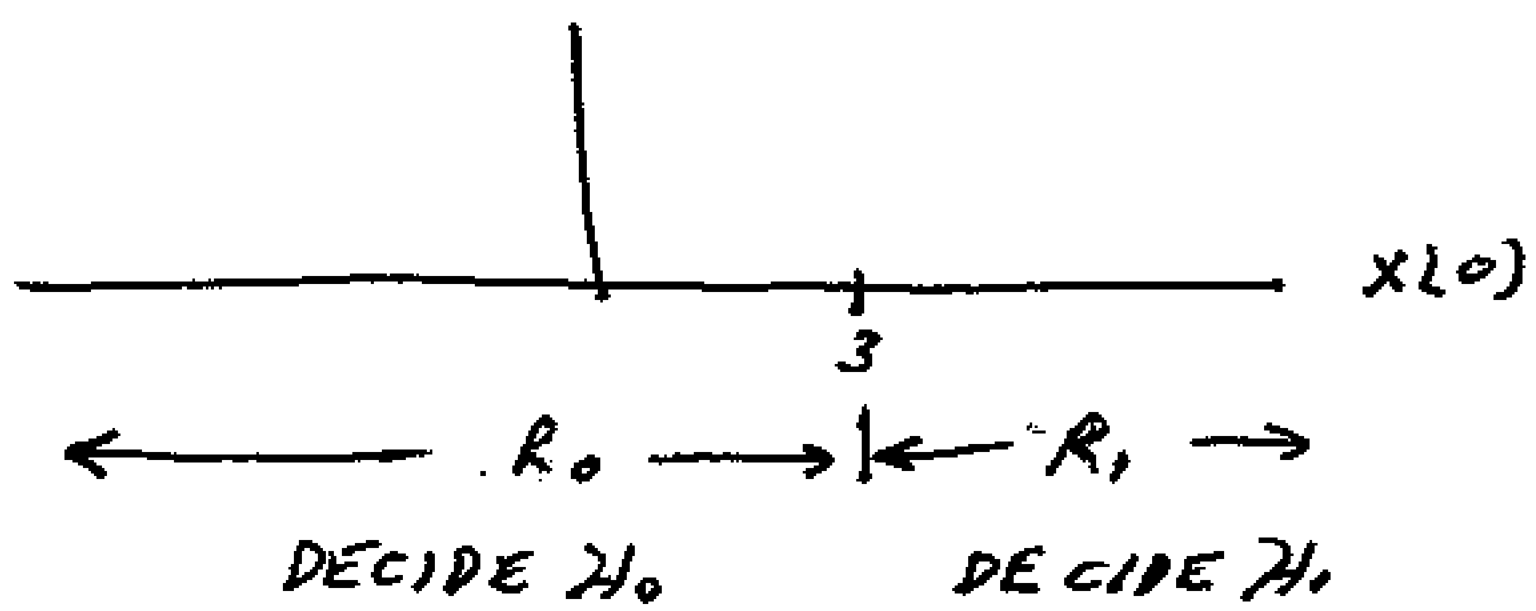
$$= Q(\gamma - 1) = Q(2) = 0.023$$

NOT TOO GOOD — USUALLY WANT $P_D > 0.9$. BETTER DETECTOR?

NEYMAN-PEARSON LEMMA

GENERAL OPERATION OF DETECTOR — GIVEN $\{x[0], x[1], \ldots, x[N-1]\}$ MAP INTO A DECISION

PREVIOUS EXAMPLE



IN GENERAL

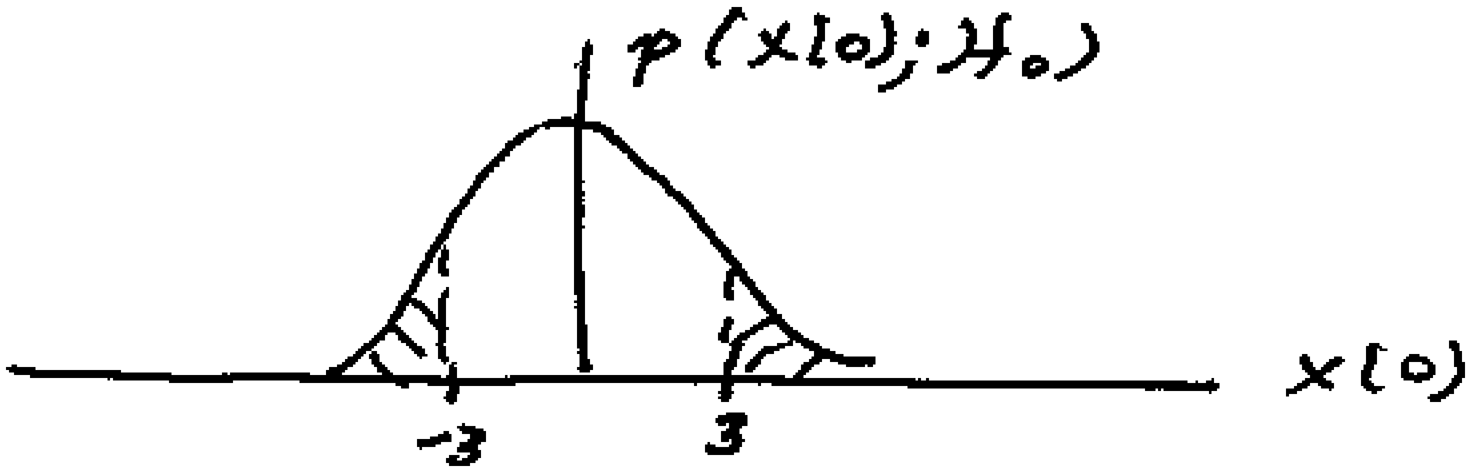
$$R_1 = \{\mathbf{x} : \text{DECIDE } \mathcal{H}_1\}$$

CALLED <u>CRITICAL REGION</u>

HENCE $P_{FA} = Pr\{\text{DECIDE } \mathcal{H}_1 ; \mathcal{H}_0\}$

$$= \int_{R_1} p(\underline{x} ; \mathcal{H}_0)\, d\underline{x}$$

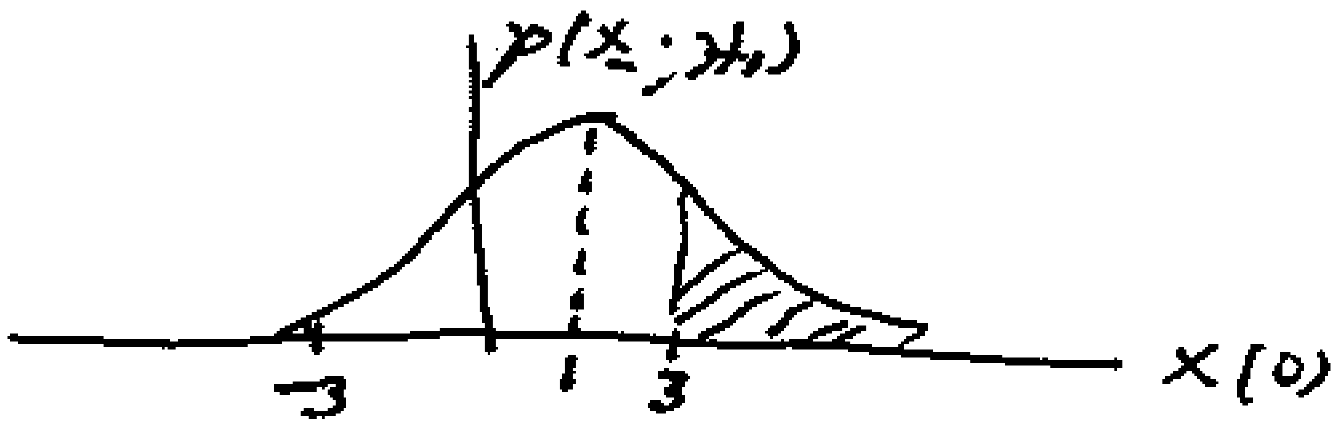
ONCE R_1 IS CHOSEN P_{FA} IS FIXED. MANY CRITICAL REGIONS EXIST WITH SAME P_{FA}. CHOOSE BEST ONE.



← R_1' →| |← R_1 → ⇒ SAME P_{FA}

CLEARLY R_1 IS BETTER ⇒ LARGER P_D

$$P_D = \int_{R_1} p(\underline{x} ; \mathcal{H}_1)\, d\underline{x}$$



← R_1' →| |← R_1 → ⇒ P_D LARGER FOR R_1

NEYMAN-PEARSON TELLS US HOW TO CHOOSE BEST R_1.