CONSIDER THE COMMUNICATION PROBLEM $\Rightarrow$ MINIMIZE P_e AND ASSUME $P(\mathcal{H}_0) = P(\mathcal{H}_1)$. WE HAVE AN ML RECEIVER OR DECIDE $\mathcal{H}_1$ IF

$$\frac{p(\underline{x} \mid \mathcal{H}_1)}{p(\underline{x} \mid \mathcal{H}_0)} > \gamma = \frac{P(\mathcal{H}_0)}{P(\mathcal{H}_1)} = 1$$

BUT $p(\underline{x} \mid \mathcal{H}_i) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2} \sum_n (x(n) - s_i(n))^2}$

CHOOSE $\mathcal{H}_i$ FOR WHICH

$$D_i^2 = \sum_{n=0}^{N-1} (x(n) - s_i(n))^2$$

IS MINIMUM. CALLED A <u>MINIMUM DISTANCE</u> RECEIVER.

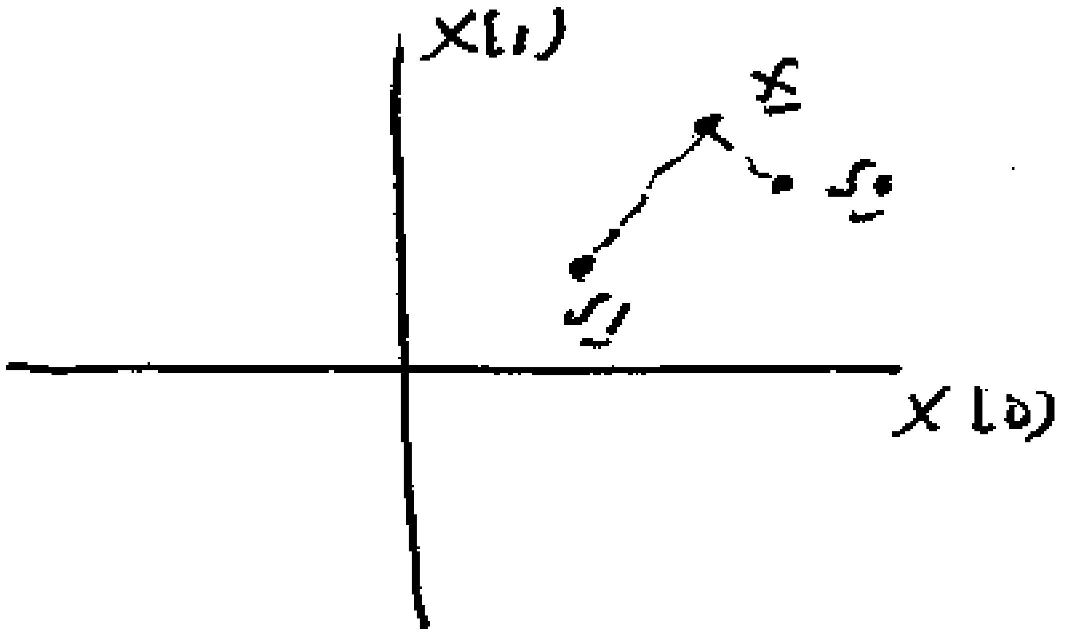
$$D_i^2 = \| \underline{x} - \underline{s}_i \|^2$$

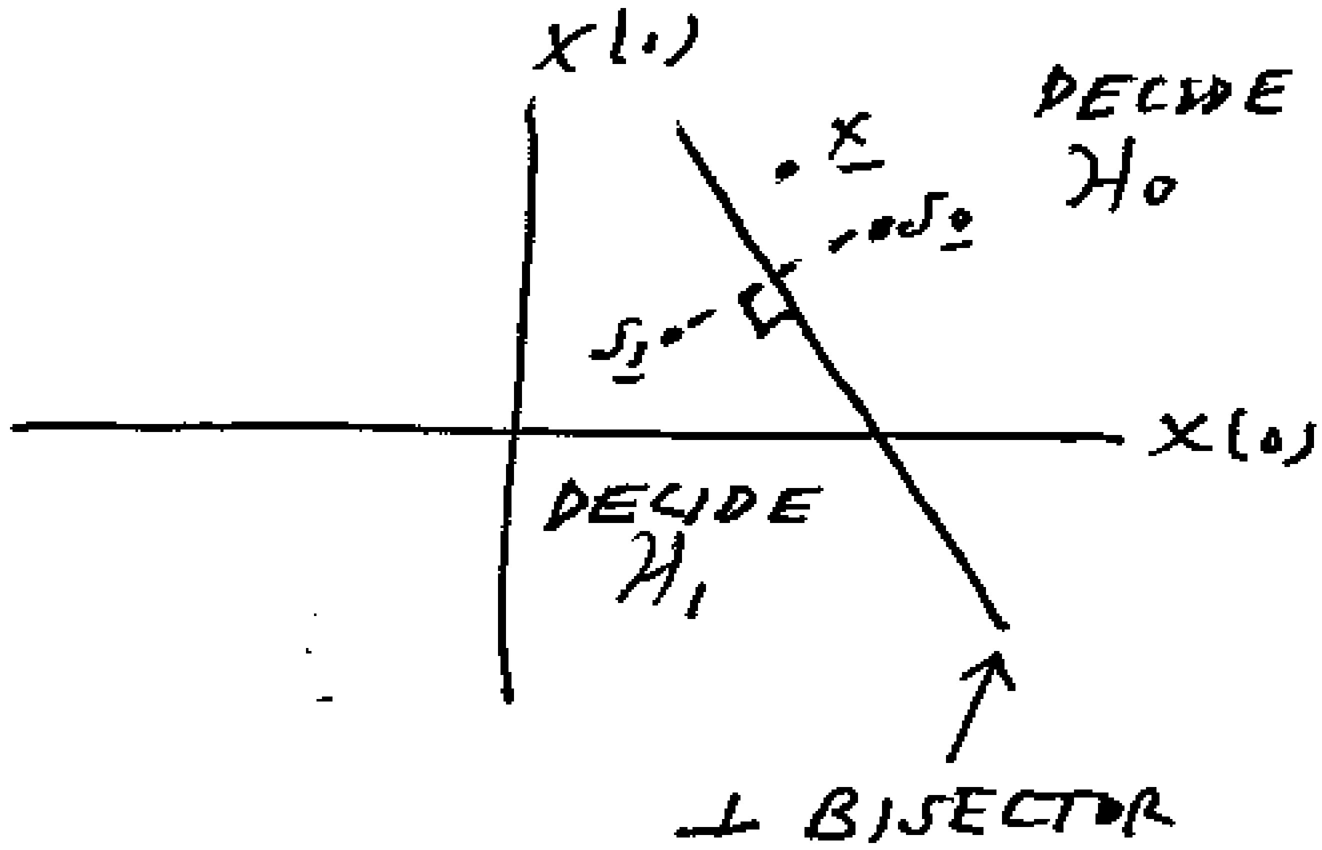
WHERE $\| \underline{\xi} \| = \sqrt{\sum_{i=0}^{N-1} \xi_i^2}$

= EUCLIDEAN DISTANCE

CHOOSE $\underline{s}_0$ ($\mathcal{H}_0$) IF $\underline{s}_0$ IS CLOSER TO $\underline{x}$ AND OTHERWISE $\underline{s}_1$ ($\mathcal{H}_1$).

EXAMPLE : $N = 2$





ALTERNATIVE RECEIVER IS

$$D_i^2 = \sum_n (x[n] - s_i[n])^2$$
$$= \sum_n x^2[n] - 2 \sum_n x[n] s_i[n] + \mathcal{E}_i$$

TO MINIMIZE $D_i^2 \Rightarrow$ MAXIMIZE

$$T_i(\underline{x}) = \underbrace{\sum_{n=0}^{N-1} x[n] s_i[n]}_{\text{CORRELATOR}} \underbrace{- \tfrac{1}{2} \mathcal{E}_i}_{\text{BIAS TERM}}$$

IF $\mathcal{E}_0 = \mathcal{E}_1$, CHOOSE HYPOTHESIS YIELDING LARGER CORRELATION.

PERFORMANCE

WISH TO DETERMINE P_e.

$$P_e = P(\mathcal{H}_1|\mathcal{H}_0)P(\mathcal{H}_0) + P(\mathcal{H}_0|\mathcal{H}_1)P(\mathcal{H}_1)$$
$$= \tfrac{1}{2}\left[P(\mathcal{H}_1|\mathcal{H}_0) + P(\mathcal{H}_0|\mathcal{H}_1)\right]$$
$$= \tfrac{1}{2}\left[P_r\{T_1(\underline{x}) - T_0(\underline{x}) > 0 \mid \mathcal{H}_0\} + P_r\{T_0(\underline{x}) - T_1(\underline{x}) > 0 \mid \mathcal{H}_1\}\right]$$

LET $T = T_1 - T_0$

$$= \Sigma\, x(n)(s_1(n) - s_0(n)) - \tfrac{1}{2}(\mathcal{E}_1 - \mathcal{E}_0)$$

T IS GAUSSIAN

$$E(T|\mathcal{H}_0) = \Sigma\, s_0(n)(s_1(n) - s_0(n)) - \tfrac{1}{2}(\mathcal{E}_1 - \mathcal{E}_0)$$

$$= \Sigma\, s_0(n)s_1(n) - \mathcal{E}_0 - \tfrac{1}{2}\mathcal{E}_1 + \tfrac{1}{2}\mathcal{E}_0$$

$$= -\tfrac{1}{2}\,\Sigma(s_1(n) - s_0(n))^2$$

$$= -\tfrac{1}{2}\,\|\underline{s}_1 - \underline{s}_0\|^2$$

SIMILARLY $E(T|\mathcal{H}_1) = \tfrac{1}{2}\|\underline{s}_1 - \underline{s}_0\|^2$

$$\mathrm{VAR}(T \mid \mathcal{H}_0) = \mathrm{VAR}\left(\sum x[n](s_1[n] - s_0[n]) \mid \mathcal{H}_0 \right)$$

$$= \sum \mathrm{VAR}(x[n]) (s_1[n] - s_0[n])^2$$

$$= \sigma^2 \| \underline{s}_1 - \underline{s}_0 \|^2 = \mathrm{VAR}(T \mid \mathcal{H}_1)$$

$$\Rightarrow \quad T \sim \begin{array}{ll} N\left(-\tfrac{1}{2}\|\underline{s}_1 - \underline{s}_0\|^2, \ \sigma^2 \|\underline{s}_1 - \underline{s}_0\|^2\right) & \mathcal{H}_0 \\ N\left(\tfrac{1}{2}\|\underline{s}_1 - \underline{s}_0\|^2, \ \sigma^2 \|\underline{s}_1 - \underline{s}_0\|^2\right) & \mathcal{H}_1 \end{array}$$

$$P_e = \tfrac{1}{2} \Pr\{ T > 0 \mid \mathcal{H}_0 \} + \tfrac{1}{2} \Pr\{ T < 0 \mid \mathcal{H}_1 \}$$

$$= \tfrac{1}{2} Q\left(\frac{\tfrac{1}{2}\|\underline{s}_1 - \underline{s}_0\|^2}{\sqrt{\sigma^2 \|\underline{s}_1 - \underline{s}_0\|^2}} \right)$$

$$+ \tfrac{1}{2} \left[1 - Q\left(\frac{-\tfrac{1}{2}\|\underline{s}_1 - \underline{s}_0\|^2}{\sqrt{\sigma^2 \|\underline{s}_1 - \underline{s}_0\|^2}} \right) \right]$$

$$= Q\left(\frac{\tfrac{1}{2}\|\underline{s}_1 - \underline{s}_0\|^2}{\sqrt{\sigma^2 \|\underline{s}_1 - \underline{s}_0\|^2}} \right)$$

SINCE $Q(-x) = 1 - Q(x)$

OR $\quad P_e = Q\left(\tfrac{1}{2} \sqrt{\frac{\|\underline{s}_1 - \underline{s}_0\|^2}{\sigma^2}} \right)$

P_e DECREASES AS SEPARATION BETWEEN $\underline{s}_0$ AND $\underline{s}_1$ IN R^N INCREASES.

PRACTICAL CONSTRAINT IS TO CONSTRAIN ENERGY OR AVERAGE ENERGY $\overline{E} = \frac{1}{2}(\mathcal{E}_0 + \mathcal{E}_1)$
$(\; P(\mathcal{H}_0) = P(\mathcal{H}_1)\;)$

THEN

$$\|\underline{s}_1 - \underline{s}_0\|^2 = \underline{s}_1^T\underline{s}_1 - 2\underline{s}_1^T\underline{s}_0 + \underline{s}_0^T\underline{s}_0$$

$$= 2\overline{\mathcal{E}} - 2\underline{s}_1^T\underline{s}_0$$

DEFINE $\rho_s = \dfrac{\underline{s}_1^T\underline{s}_0}{\frac{1}{2}(\underline{s}_1^T\underline{s}_1 + \underline{s}_0^T\underline{s}_0)}$

= SIGNAL CORRELATION

$$\rho_s = \begin{cases} \pm 1 & \text{IF } \underline{s}_1 = \pm \underline{s}_0 \\ 0 & \text{IF } \underline{s}_1^T\underline{s}_0 = 0 \end{cases}$$

$$\|\underline{s}_1 - \underline{s}_0\|^2 = 2\overline{\mathcal{E}} - 2\overline{\mathcal{E}}\rho_s$$
$$= 2\overline{\mathcal{E}}(1 - \rho_s)$$

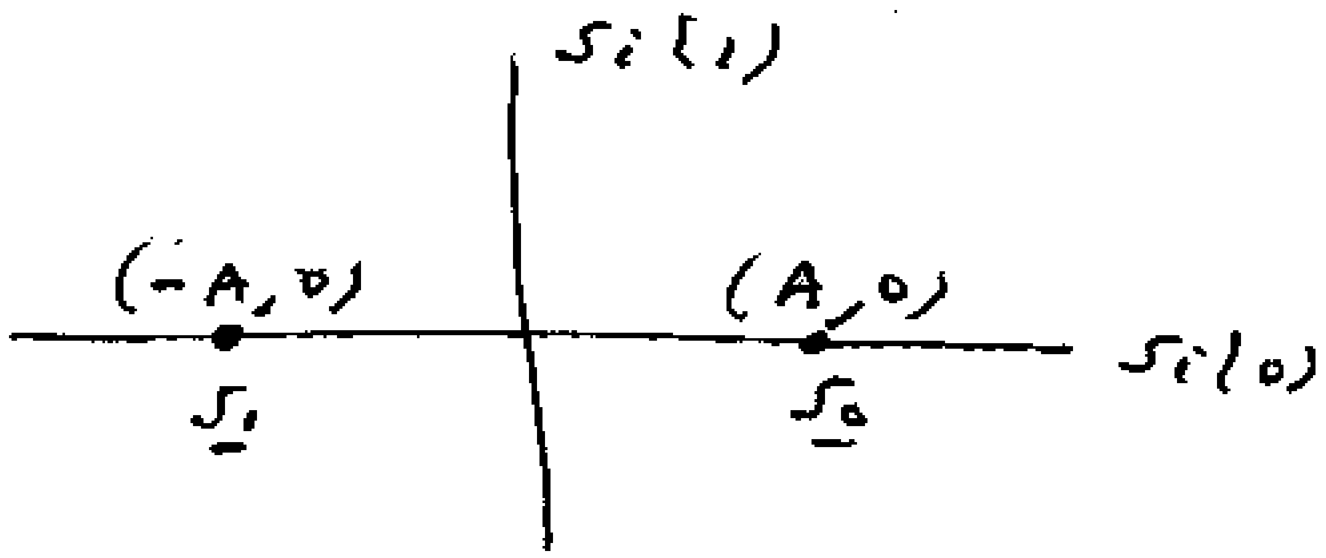
$$\therefore P_e = Q\left(\sqrt{\frac{\bar{\mathcal{E}}(1-\rho_s)}{2\sigma^2}}\right)$$

P_e MINIMIZED (FOR GIVEN $\bar{\mathcal{E}}$) WHEN $\rho_s = -1$ SINCE $|\rho_s| \leq 1$.

<u>EXAMPLES</u>: PHASE SHIFT KEYING (PSK)

$$s_0[n] = A \cos 2\pi f_0 n$$
$$s_1[n] = A \cos(2\pi f_0 n + \pi)$$
$$= -A \cos 2\pi f_0 n$$

$$\Rightarrow \underline{s}_1 = -\underline{s}_0$$



$N = 2$
$f_0 = 0.25$

$\underline{s}_1 = -\underline{s}_0$ (ANTIPODAL)

$\Rightarrow \rho_s = -1 \Rightarrow$ OPTIMAL SIGNALS MINIMIZES P_e

$$P_e = Q\left(\sqrt{\bar{\mathcal{E}}/\sigma^2}\right) \qquad \bar{\mathcal{E}} = \mathcal{E}_1 = \mathcal{E}_0 \approx NA^2/2$$

FOR $P_e = 10^{-8}$ NEED $\frac{\mathcal{E}}{\sigma^2} = 15\,dB$

EXAMPLE : FREQUENCY SHIFT KEYING (FSK)

$$s_0[n] = A\cos 2\pi f_0 n$$
$$s_1[n] = A\cos 2\pi f_1 n$$

FOR $|f_1 - f_0| \gg 1/N$

$$\sum_n s_0[n]s_1[n] \approx 0 \Rightarrow \rho_s \approx 0$$

$$P_e = Q\left(\sqrt{\frac{\mathcal{E}}{2\sigma^2}}\right)$$

FSK 3 dB POORER THAN PSK
NOT USED MUCH IN PRACTICE)

SEE M-ARY CASE

LINEAR MODEL

SEE [KAY 1993] CHAPTER 4.

EXAMPLE :

$$x(n) = a\cos 2\pi f_0 n + b\sin 2\pi f_0 n + \underset{\uparrow \text{ WGN}}{w(n)} \qquad n = 0, 1, \ldots, N-1$$

IN MATRIX FORMAT

$$\underbrace{\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} 1 & 0 \\ \cos 2\pi f_0 & \sin 2\pi f_0 \\ \vdots & \\ \cos 2\pi f_0 (N-1) & \sin 2\pi f_0 (N-1) \end{bmatrix}}_{\underline{H}} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\underline{\theta}} + \underbrace{\begin{bmatrix} w(0) \\ w(1) \\ \vdots \\ w(N-1) \end{bmatrix}}_{\underline{w}}$$

$\underline{H}$ IS $N \times 2$ AND IS KNOWN

$\underline{\theta}$ IS 2×1 AND IS KNOWN

$\underline{w} \sim N(\underline{0}, \sigma^2 \underline{I})$

EXAMPLE : DC LEVEL IN WHITE GAUSSIAN NOISE

$$x[n] = A + w[n] \qquad n = 0, 1, \ldots, N-1$$

$$\underbrace{\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}}_{\underline{H}} \underbrace{A}_{\underline{\theta}} + \underbrace{\begin{bmatrix} w(0) \\ w(1) \\ \vdots \\ w(N-1) \end{bmatrix}}_{\underline{w}}$$

$\underline{w} \sim N(\underline{0}, \sigma^2 \underline{I})$

$\underline{H}, \underline{\theta}$ KNOWN

IN GENERAL

$$\underset{N\times 1}{\underline{x}} = \underset{N\times p}{\underline{H}}\,\underset{p\times 1}{\underline{\theta}} + \underline{w} \leftarrow N\times 1$$

$N > p$

$\underline{w} \sim N(\underline{0}, \sigma^2 \underline{I})$

OR MORE GENERALLY CAN LET

$\underline{w} \sim N(\underline{0}, \underline{C}) \qquad \underline{C} \neq \sigma^2 \underline{I}$

$w[n]$ NOT WGN

WHEN $\underline{\theta}$ IS NOT KNOWN, AN OPTIMAL ESTIMATOR IS

$$\hat{\underline{\theta}} = (\underline{H}^T \underline{C}^{-1} \underline{H})^{-1} \underline{H}^T \underline{C}^{-1} \underline{x}$$

WHICH HAS COVARIANCE MATRIX

$$\underline{C}_{\hat{\theta}} = (\underline{H}^T \underline{C}^{-1} \underline{H})^{-1}$$

(SEE KAY 1993, CHAPTER 4)

WE NOW ASSUME $\underline{\theta}$ IS KNOWN.

<u>EXAMPLE</u> : SINUSOIDAL DETECTION

WISH TO DETECT $s[n] = A\cos(2\pi f_0 n + \phi)$ IN WGN

TO CAST IN LINEAR MODEL FORM:

$$s[n] = \underbrace{A\cos\phi}_{a} \cos 2\pi f_0 n \underbrace{- A\sin\phi}_{b} \sin 2\pi f_0 n$$

$$\mathcal{H}_0 : \; x[n] = w[n]$$
$$\mathcal{H}_1 : \; x[n] = a\cos 2\pi f_0 n + b\sin 2\pi f_0 n + w[n]$$

FOR $n = 0, 1, \ldots, N-1$

OR $\mathcal{H}_0 : \underline{x} = \underline{w}$

$\mathcal{H}_1 : \underline{x} = \underbrace{\underline{H}\underline{\theta}_1}_{\underline{s}} + \underline{w} \qquad \underline{\theta}_1 = \begin{pmatrix} a \\ b \end{pmatrix}$

THIS IS SAME PROBLEM AS BEFORE

$\Rightarrow$ DECIDE $\mathcal{H}_1$ IF $T(\underline{x}) = \underline{x}^T \underline{C}^{-1} \underline{s} > \gamma'$

OR $T(\underline{x}) = \underline{x}^T \underline{C}^{-1} \underline{H}\underline{\theta}_1 > \gamma'$

FOR THIS EXAMPLE $\underline{C} = \sigma^2 \underline{I}$

$$T(\underline{x}) = \frac{1}{\sigma^2} \underline{x}^T \underline{H}\underline{\theta}_1$$

$$= \frac{N}{2\sigma^2} \left(\frac{2}{N} \underline{H}^T \underline{x} \right)^T \underline{\theta}_1$$

OR $T'(\underline{x}) = \left(\frac{2}{N} \underline{H}^T \underline{x} \right)^T \underline{\theta}_1$

BUT $\frac{2}{N} \underline{H}^T \underline{x} = \begin{bmatrix} \frac{2}{N} \sum_{n=0}^{N-1} x(n) \cos 2\pi f_0 n \\ \frac{2}{N} \sum_{n=0}^{N-1} x(n) \sin 2\pi f_0 n \end{bmatrix}$

AND $(\underline{H}^T \underline{H})^{-1} = \frac{2}{N} \underline{I}$

$= \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} =$ FOURIER SERIES COEFFICIENTS

$T'(\underline{x}) = \hat{a} a + \hat{b} b$

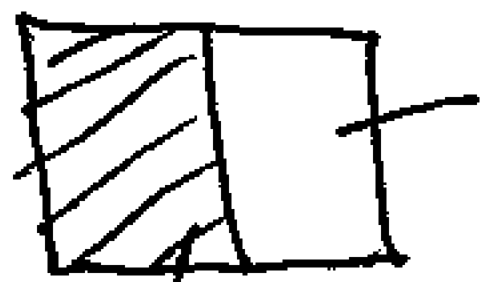
$=$ CORRELATION OF TRUE VALUE WITH ESTIMATE

SIGNAL PROCESSING EXAMPLE

PATTERN RECOGNITION – WISH TO CLASSIFY EACH PIXEL OF IMAGE ACCORDING TO GRAY LEVEL

EXAMPLE – BINARY CLASSIFICATION

IMAGE



COULD HAVE M DIFFERENT GRAY LEVELS

ACTUAL GRAY LEVEL DEPENDS ON LIGHTING, ORIENTATION, ETC

MODEL PIXEL VALUES IN A REGION AS RANDOM WITH $\underline{x} \sim N(\underline{\mu}_i, \sigma^2 \underline{I})$ WHERE ONLY MEAN LEVEL CHANGES

TO MINIMIZE P_e USE MAP RULE
ASSUME $P(\mathcal{H}_i) = 1/M \Rightarrow$ USE ML RULE

$\Rightarrow$ MAXIMIZE $p(\underline{x} \mid \mathcal{H}_i)$ OVER i

WHERE $\underline{x} \sim N(\underline{\mu}_i, \sigma^2 \underline{I})$ $\quad N \times 1$

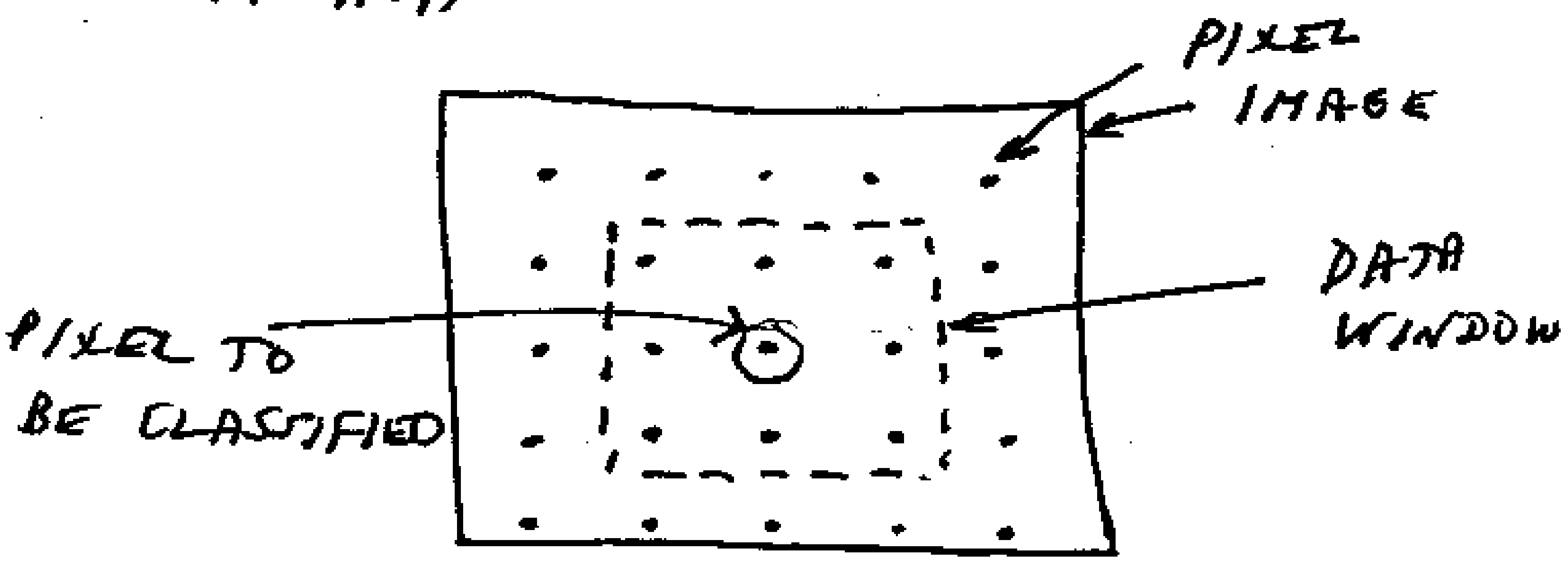
$$p(\underline{x} \mid H_i) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}(\underline{x}-\underline{\mu}_i)^T(\underline{x}-\underline{\mu}_i)}$$

TO MAXIMIZE $\Rightarrow$ MINIMIZE

$$D_i^2 = (\underline{x}-\underline{\mu}_i)^T(\underline{x}-\underline{\mu}_i)$$

$$= \| \underline{x} - \underline{\mu}_i \|^2$$ MINIMUM DISTANCE RECEIVER

TO APPLY TO OUR PROBLEM USE FOLLOWING APPROACH



FOR BINARY CLASSIFICATION
ASSUME $\underline{\mu}_i = \underline{0}$ (BLACK) OR
$\underline{\mu}_i = \underline{1}$ (WHITE)

LET $\underline{x}$ = VECTOR OF ALL DATA (PIXEL) VALUES IN WINDOW

$$\Rightarrow \quad D_i^2 = \| \underline{x} - \underline{\mu}_i \|^2$$

$$= \| \underline{x} \|^2 \qquad \text{BLACK} \qquad (H_0)$$

$$\| \underline{x} - \underline{1} \|^2 \qquad \text{WHITE} \qquad (H_1)$$

AND CHOOSE WHICHEVER IS SMALLER
OR DECIDE H_1 IF

$$\| \underline{x} \|^2 > \| \underline{x} - \underline{1} \|^2$$

$$\underline{x}^T \underline{x} > (\underline{x} - \underline{1})^T (\underline{x} - \underline{1})$$
$$= \underline{x}^T \underline{x} - 2 \underline{1}^T \underline{x} + \underline{1}^T \underline{1}$$

$$\Rightarrow \quad 2 \underline{1}^T \underline{x} > \underline{1}^T \underline{1} = N$$

$$\frac{1}{N} \underline{1}^T \underline{x} > \frac{1}{2}$$

$$\bar{x} > \frac{1}{2}.$$ AVERAGE PIXELS IN WINDOW.

NOTE: WE DO THIS FOR ALL PIXELS
ASSUMES THAT <u>ALL PIXELS WITHIN WINDOW HAVE SAME MEAN VALUE</u>.

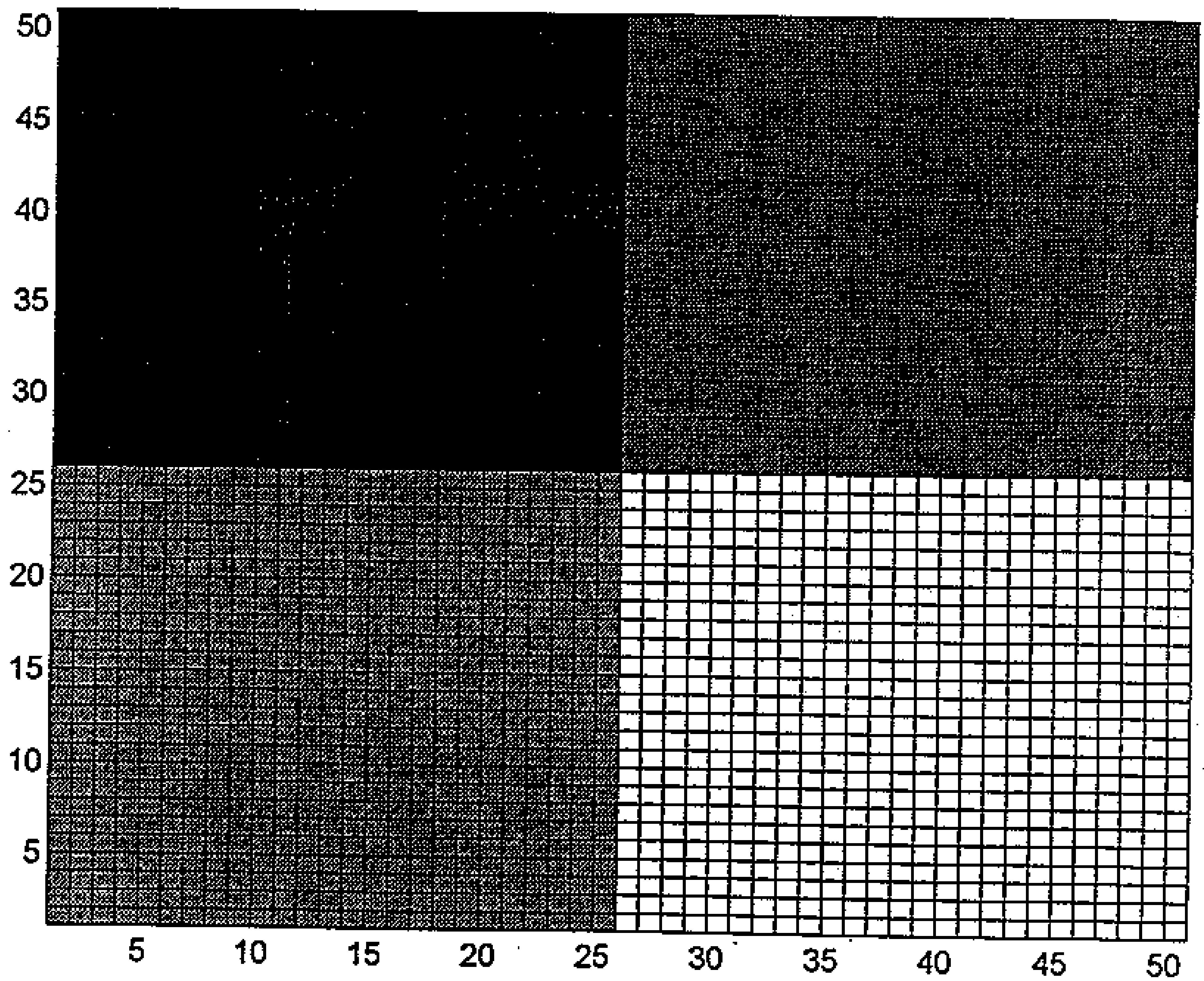


FIGURE 4.17 – SYNTHETIC IMAGE

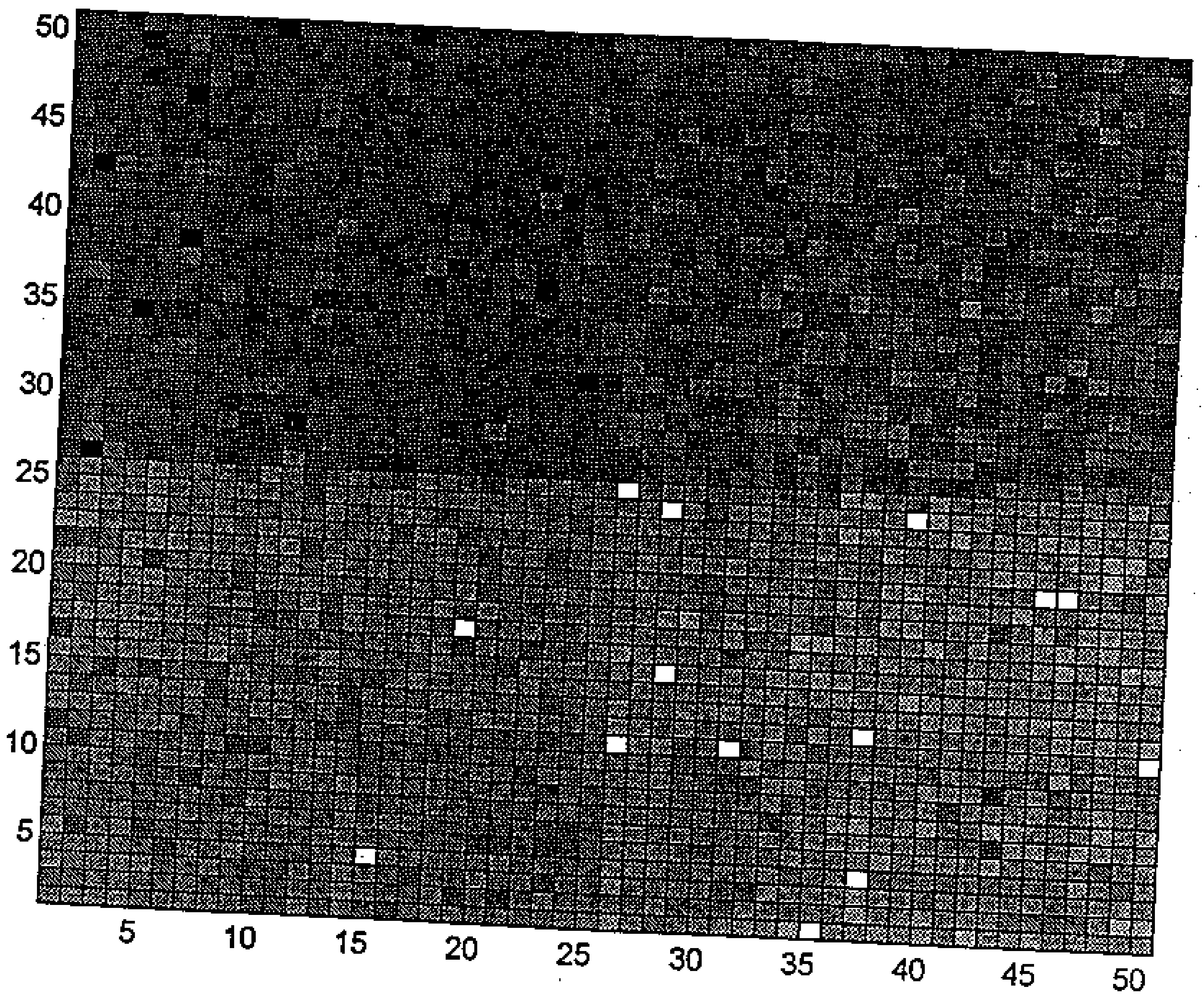


FIGURE 4.18 - NOISE CORRUPTED IMAGE

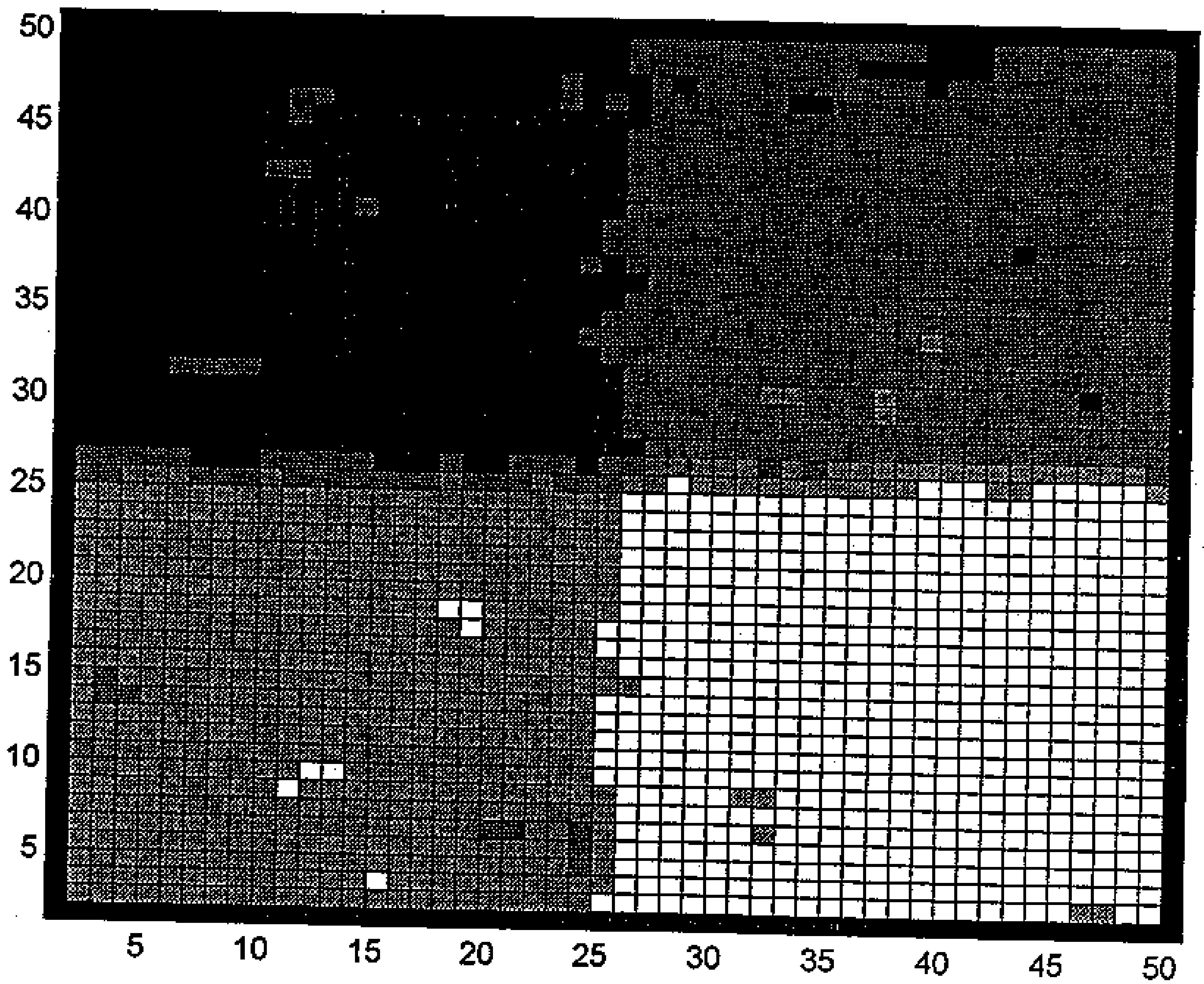


FIGURE 4.19 - CLASSIFICATION USING 3X3 WINDOW

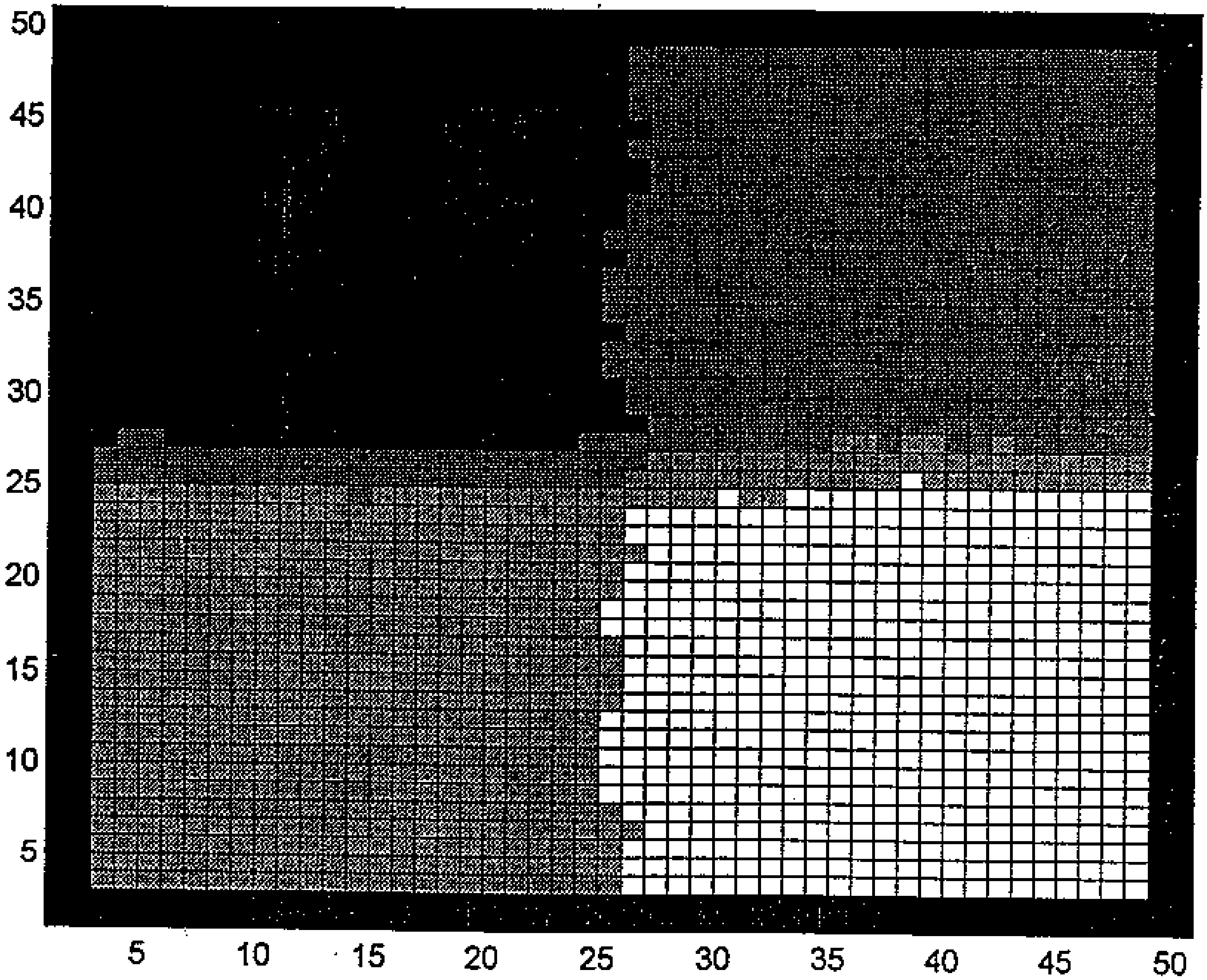


FIGURE 4.20 - CLASSIFICATION USING 5X5 WINDOW

CHAPTER 5 - RANDOM SIGNALS

SOME SIGNALS ARE UNKNOWN IN FORM - CONSIDER A SPEECH SIGNAL

SAME SOUND / SAME SPEAKER $\Rightarrow$
DIFFERENT WAVEFORMS FOR EACH UTTERANCE (SEE CHAPTER 1)

USE <u>RANDOM</u> SIGNAL MODEL

ESTIMATOR - CORRELATOR

<u>EXAMPLE</u>: ENERGY DETECTOR
ASSUME SIGNAL IS $s[n]$ WHERE $s[n]$ IS "WGN" WITH VARIANCE σ_s^2

$$\mathcal{H}_0 : x[n] = w[n]$$
$$\mathcal{H}_1 : x[n] = s[n] + w[n]$$

INDEPENDENT OF EACH OTHER

NP DETECTOR COMPUTES

$$L(\underline{x}) = \frac{p(\underline{x};\mathcal{H}_1)}{p(\underline{x};\mathcal{H}_0)}$$

$$\underline{x} = \underline{w} \sim N(\underline{0}, \sigma^2 \underline{I}) \quad \mathcal{H}_0$$

$$= \underline{s} + \underline{w} \sim N(\underline{0}, (\sigma^2 + \sigma_s^2)\underline{I}) \quad \mathcal{H}_1$$

ONLY DISCRIMINATION IS IN POWER

$$L(\underline{x}) = \frac{\dfrac{1}{[2\pi(\sigma_s^2+\sigma^2)]^{N/2}} e^{-\frac{1}{2(\sigma_s^2+\sigma^2)} \sum_n x^2(n)}}{\dfrac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2} \sum_n x^2(n)}}$$

$$l(\underline{x}) = \ln L(\underline{x})$$

$$= \frac{N}{2} \ln\left(\frac{\sigma^2}{\sigma_s^2+\sigma^2}\right) - \frac{1}{2}\left(\frac{1}{\sigma_s^2+\sigma^2} - \frac{1}{\sigma^2}\right) \cdot \sum_n x^2(n)$$

$$= \frac{N}{2} \ln\left(\frac{\sigma^2}{\sigma_s^2+\sigma^2}\right) + \frac{1}{2} \frac{\sigma_s^2}{\sigma^2(\sigma_s^2+\sigma^2)} \cdot \sum_n x^2(n)$$