2.1 The data $\{x[0], x[1], \ldots, x[N-1]\}$ are observed where the $x[n]$'s are independent and identically distributed (IID) as $\mathcal{N}(0, \sigma^2)$. We wish to estimate the variance σ^2 as

$$\hat{\sigma^2} = \frac{1}{N} \sum_{n=0}^{N-1} x^2[n].$$

Is this an unbiased estimator? Find the variance of $\hat{\sigma^2}$ and examine what happens as $N \to \infty$.

2.4 The heart rate h of a patient is automatically recorded by a computer every 100 ms. In 1 s the measurements $\{\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_{10}\}$ are averaged to obtain $\hat{h}$. If $E(\hat{h}_i) = \alpha h$ for some constant α and $\mathrm{var}(\hat{h}_i) = 1$ for each i, determine whether averaging improves the estimator if $\alpha = 1$ and $\alpha = 1/2$. Assume each measurement is uncorrelated.

2.10 In Example 2.1 assume now that in addition to A, the value of σ^2 is also unknown. We wish to estimate the vector parameter

$$\hat{\boldsymbol{\theta}} = \begin{bmatrix} A \\ \sigma^2 \end{bmatrix}.$$

Is the estimator

$$\hat{\boldsymbol{\theta}} = \begin{bmatrix} \hat{A} \\ \hat{\sigma^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{N} \sum_{n=0}^{N-1} x[n] \\ \frac{1}{N-1} \sum_{n=0}^{N-1} \left(x[n] - \hat{A}\right)^2 \end{bmatrix}$$

unbiased?