Hw 2 Solution

2.1

$$E(\hat{\sigma}^2) = E\left(\frac{1}{N}\sum_{n=0}^{N-1} x^2[n]\right) = \frac{1}{N}\sum_{n=0}^{N-1} E(x^2[n])$$

$$= \sigma^2 \text{ for all } \sigma^2 > 0 \text{ (allowable values)}$$

$\Rightarrow$ unbiased

$$\mathrm{var}(\hat{\sigma}^2) = \frac{1}{N^2}\,\mathrm{var}\left(\sum_n x^2[n]\right)$$

$$= \frac{1}{N^2} N\,\mathrm{var}(x^2[n]) \quad (x[n]\text{'s are IID} \Rightarrow x^2[n] \text{ are IID})$$

$$= \frac{1}{N}\,\mathrm{var}(x^2[n])$$

$$\mathrm{var}(x^2[n]) = E(x^4[n]) - E(x^2[n])^2$$

$$= 3\sigma^4 - \sigma^4 = 2\sigma^4$$

$\Rightarrow \mathrm{var}(\hat{\sigma}^2) = 2\sigma^4/N \to 0$ as $N \to \infty$

Hence, the PDF of $\hat{\sigma}^2$ collapses about the true value as $N \to \infty$.

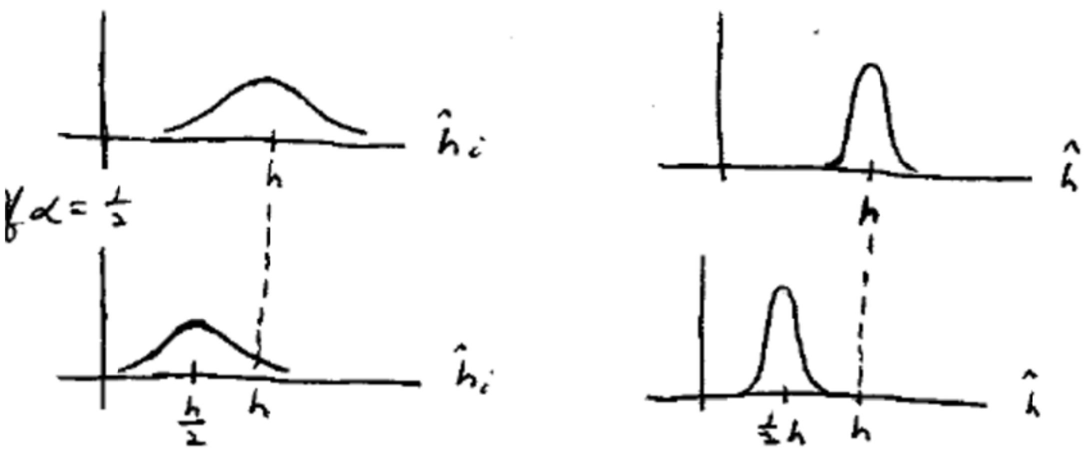
2.4

$$\hat{h} = \frac{1}{10} \sum_{i=1}^{10} \hat{h}_i \qquad E(\hat{h}) = \frac{1}{10} \sum_{i=1}^{10} E(\hat{h}_i)$$

$$= \alpha h$$

$$\mathrm{var}(\hat{h}) = \mathrm{var}(\hat{h}_i)/10 = 1/10$$

If $\alpha = 1$, we have



Before averaging After averaging

In second case ($\alpha = \frac{1}{2}$), averaging causes the PDF to be more heavily concentrated about the wrong value of h. The probability

of $\hat{h}$ being close to h actually decreases due to averaging. For $\alpha = 1$, averaging, of course, is beneficial.

2.10

10) Clearly, $E(\hat{A}) = A$. To find $E(\hat{\sigma}^2)$

$$E(\hat{\sigma}^2) = \frac{1}{N-1} \sum_{n=0}^{N-1} E\{(x[n] - \hat{A})^2\}$$

But $E\{(x[n]-\hat{A})^2\} = E\left[\left(x[n] - \frac{1}{N}\sum_{m=0}^{N-1} x[m]\right)^2\right]$

$$= E\left[\left(x[n](1-\frac{1}{N}) - \frac{1}{N}\sum_{\substack{m=0\\ m\neq n}}^{N-1} x[m]\right)^2\right]$$

$$= \left(\frac{N-1}{N}\right)^2 E\{x^2[n]\} - \frac{2(N-1)}{N^2} E\left[x[n] \sum_{\substack{m=0\\ m\neq n}}^{N-1} x[m]\right]$$

$$+ \frac{1}{N^2} E\left[\left(\sum_{\substack{m=0\\ m\neq n}}^{N-1} x[m]\right)^2\right]$$

$$= \left(\frac{N-1}{N}\right)^2 (\sigma^2 + A^2) - \frac{2(N-1)}{N^2} E\{x[n]\} E\left(\sum_{\substack{m=0\\ m\neq n}}^{N-1} x[m]\right)$$

$$+ \frac{1}{N^2}\left(\mathrm{var}\left(\sum_{\substack{m=0\\ m\neq n}}^{N-1} x[m]\right) + E\left[\sum_{\substack{m=0\\ m\neq n}}^{N-1} x[m]\right]^2\right)$$

$$= \left(\frac{N-1}{N}\right)^2 (\sigma^2 + A^2) - 2\frac{(N-1)}{N^2} A(N-1)A$$

$$+ \frac{1}{N^2}(N-1)\sigma^2 + \frac{1}{N^2}[(N-1)A]^2$$

$$= \sigma^2\left[\frac{N^2 - 2N + 1 + N - 1}{N^2}\right] = \sigma^2 \frac{N-1}{N}$$

$$\Rightarrow E(\hat{\sigma}^2) = \frac{1}{N-1} \sum_{n=0}^{N-1} \sigma^2 \frac{N-1}{N} = \sigma^2$$

$\Rightarrow$ $\hat{\underline{\theta}}$ is unbiased.