Hw 4 Solution

Problem 4.1

1) This fits linear model form.

$$\underline{x} = \underline{H}\,\underline{\theta} + \underline{w}$$

$$\underline{H} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_p \\ \vdots & \vdots & & \\ r_1^{N-1} & r_2^{N-1} & \cdots & r_p^{N-1} \end{bmatrix} \quad \underline{\theta} = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_p \end{bmatrix}$$

$$\hat{\underline{\theta}} = (\underline{H}^T\underline{H})^{-1}\underline{H}^T\underline{x} \qquad \underline{C}_{\hat{\theta}} = \sigma^2(\underline{H}^T\underline{H})^{-1}$$

For $p=2$, $r_1 = 1$, $r_2 = -1$ and N even

$$\underline{H} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ \vdots & \vdots \\ 1 & -1 \end{bmatrix} \Rightarrow \underline{H}^T\underline{H} = \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix} = N\underline{I}$$

since columns are orthogonal

$$\hat{\underline{\theta}} = \frac{1}{N}\underline{H}^T\underline{x} = \begin{bmatrix} \frac{1}{N}\sum_{n=0}^{N-1} x[n] \\ \frac{1}{N}\sum_{n=0}^{N-1} (-1)^n x[n] \end{bmatrix}$$

$$\underline{C}_{\hat{\theta}} = \frac{\sigma^2}{N}\underline{I}$$

Problem 4.2

2) First assume columns of $\underline{H}$ are linearly independent

Then, $\underline{H}\underline{x} = \sum_{i=1}^{p} x_i \underline{h}_i \neq \underline{0}$ if $\underline{x} \neq \underline{0}$

$\Rightarrow \quad \underline{x}^T \underline{H}^T \underline{H}\, \underline{x} = \| \underline{H}\underline{x} \|^2 > 0$ for $\underline{x} \neq \underline{0}$

$\Rightarrow \quad \underline{H}^T \underline{H}$ is positive definite

Now assume $\underline{H}^T\underline{H}$ is positive definite or

$\underline{x}^T \underline{H}^T \underline{H}\underline{x} > 0$ for all $\underline{x} \neq \underline{0}$

or $\quad \| \underline{H}\underline{x} \|^2 > 0$ for all $\underline{x} \neq \underline{0}$

$\Rightarrow \quad \underline{H}\underline{x} \neq \underline{0}$ for all $\underline{x} \neq \underline{0}$

$\Rightarrow$ columns of $\underline{H}$ are linearly independent

It can further be shown that for matrices of the form $\underline{H}^T\underline{H}$, invertibility is equivalent to being positive definite.