Solution Hw 6

Problem 8.1

1) Nonlinear LS due to f_0 and r.
Yes, quadratic in A, B.
Analytically, we could find the values of A and B that minimize J for given f_0 and r. Then, plug these into J, which will now be a nonquadratic function of f_0 and r. Next, use a grid search over $0 < r < 1$ and $0 \le f_0 \le \frac{1}{2}$.

Problem 8.5

5)

$$\underline{s} = \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \cos 2\pi f_1 & \cos 2\pi f_2 & \cdots & \cos 2\pi f_p \\ \vdots & & & \\ \cos 2\pi f_1(N-1) & \cos 2\pi f_2(N-1) & \cdots & \cos 2\pi f_p(N-1) \end{bmatrix}}_{\underline{H}} \underbrace{\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_p \end{bmatrix}}_{\underline{\theta}}$$

$\underline{H}^T \underline{H} \hat{\underline{\theta}} = \underline{H}^T \underline{x}$ are normal equations

If $f_i = i/N$, the column vectors of $\underline{H}$ are orthogonal (see (4.13)). Thus,

$$\underline{H}^T \underline{H} = \frac{N}{2} \underline{I}$$

$$\Rightarrow \quad \hat{\underline{\theta}} = \frac{2}{N} \underline{H}^T \underline{x} \Rightarrow \hat{A}_i = \frac{2}{N} \sum_{n=0}^{N-1} x[n] \cos 2\pi f_i n$$

$$J_{MIN} = \underline{x}^T(\underline{I} - \underline{H}(\underline{H}^T\underline{H})^{-1}\underline{H}^T)\underline{x}$$

$$= \underline{x}^T(\underline{I} - \frac{2}{N}\underline{H}\underline{H}^T)\underline{x}$$

$$= \underline{x}^T\underline{x} - 2/N \,||\underline{H}^T\underline{x}||^2$$

$$= \underline{x}^T\underline{x} - \frac{2}{N}\left(\frac{N}{2}\right)^2 ||\hat{\underline{\theta}}||^2$$

$$= \sum_{n=0}^{N-1} x^2[n] - \frac{N}{2}\sum_{i=1}^{p}\hat{A}_i^2$$

For $w[n]$ WGN the PDF is

$$\hat{\underline{\theta}} \sim N(\underline{\theta}, \sigma^2(\underline{H}^T\underline{H})^{-1}) \quad \text{since}$$

$$E(\hat{\underline{\theta}}) = (\underline{H}^T\underline{H})^{-1}\underline{H}^T E(\underline{x}) = (\underline{H}^T\underline{H})^{-1}\underline{H}^T\underline{H}\underline{\theta} = \underline{\theta}$$

$$\underline{C}_{\hat{\theta}} = E[(\hat{\underline{\theta}}-\underline{\theta})(\hat{\underline{\theta}}-\underline{\theta})^T]$$

$$= E[(\underline{H}^T\underline{H})^{-1}\underline{H}^T\underbrace{(\underline{x}-\underline{H}\underline{\theta})}_{\underline{w}}\underbrace{(\underline{x}-\underline{H}\underline{\theta})^T}_{\underline{w}^T}\underline{H}(\underline{H}^T\underline{H})^{-1}]$$

$$= (\underline{H}^T\underline{H})^{-1}\underline{H}^T \sigma^2 \underline{I}\, \underline{H}(\underline{H}^T\underline{H})^{-1} = \sigma^2(\underline{H}^T\underline{H})^{-1}$$

or $\underline{C}_{\hat{\theta}} = 2\sigma^2/N \; \underline{I}$

and since $\hat{\underline{\theta}}$ is a linear function of $\underline{x}$, we have a Gaussian PDF or

$$\hat{\underline{\theta}} \sim N(\underline{\theta}, 2\sigma^2/N \; \underline{I})$$

Problem 8.27

27) From (8.61)

$$\theta_{k+1} = \theta_k + \left(\underline{H}^T(\theta_k)\underline{H}(\theta_k) - \sum_{n=0}^{N-1} G_n(\theta_k)(x[n] - e^{\theta_k}) \right)^{-1} \cdot \underline{H}^T(\theta_k)(\underline{x} - e^{\theta_k}\underline{1})$$

$$[\underline{H}(\theta)]_i = \frac{\partial s[i]}{\partial \theta} = e^{\theta} \qquad i = 0, 1, \ldots, N-1$$

$$[\underline{G}_n(\theta)]_{11} = \frac{\partial^2 s[n]}{\partial \theta^2} = e^{\theta}$$

$$\Rightarrow \quad \underline{H}(\theta) = e^{\theta}\underline{1} \qquad G_n(\theta) = e^{\theta}$$

$$\theta_{k+1} = \theta_k + \left(N e^{2\theta_k} - \sum_{n=0}^{N-1} e^{\theta_k} \left(x[n] - e^{\theta_k} \right) \right)^{-1}$$

$$\cdot \; e^{\theta_k} \mathbf{1}^T \left(\mathbf{x} - e^{\theta_k} \mathbf{1} \right)$$

$$= \theta_k + \frac{e^{\theta_k} \left(N\bar{x} - N e^{\theta_k} \right)}{N e^{2\theta_k} - N\bar{x} e^{\theta_k} + N e^{2\theta_k}}$$

$$= \theta_k + \frac{\bar{x} - e^{\theta_k}}{2 e^{\theta_k} - \bar{x}}$$

To find analytically let $\alpha = e^{\theta} \Rightarrow \hat{\alpha} = \bar{x}$
and from Prob 8.26 $\hat{\theta} = \ln \bar{x}$.