Bit Error Rate Performance of Linear Companding Transforms for PAPR Reduction in OFDM Systems

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***Abstract*—This paper provides an analytical framework to study the performance of the linear companding techniques proposed in the OFDM literature, thus settling the numerous controversial claims that are based solely on simulation results. Linear companding transforms are widely employed to reduce the peak-to-average-power ratio (PAPR) in orthogonal frequency division multiplexing (OFDM) systems. Two mail linear companding classes have been considered in the literature: the linear symmetrical transform (LST) and the linear asymmetrical transform (LAST). In the literature, the bit error rate (BER) performance superiority of the basic LAST (with one discontinuity point) over the LST is claimed based on computer simulations. Also, it has been claimed that a LAST with two discontinuity points outperforms the basic LAST with one discontinuity point. These claims are however not substantiated with analytical results. Our analysis shows that these claims are, in general, not always true.**

We derive a sufficient condition, in terms of the companding parameters, under which the BER performance of a general LAST with $M-1$ discontinuity points is superior to that of the LST. The derived condition explains the contradictions between different reported results in the literature and validates some other reported simulation results. It also serves as a guideline in the process of choosing proper values for the companding parameters to obtain a specific trade-off between the PAPR reduction capability and the BER performance. In particular, the derived sufficient condition shows that the BER performance for the LAST depends on the slope values of the LAST rather than the number of discontinuity points as has been indicated so far. Moreover, we derive conditions for the companding parameters in order to keep the average transmitted power unchanged after companding. Our theoretical derivations are supported by simulation results.

I. INTRODUCTION

Despite the great advantages of OFDM, it has a few inherited drawbacks, the most serious of which is the varying signal envelope with high peaks. This results in a high peak-to-average power ratio (PAPR) for the transmitted signal. The PAPR for the discrete-time OFDM signal $x[n]$ is defined as the ratio of the maximum instantaneous power to the average power and can be expressed as

$$PAPR(x[n]) = \max_{0 \le n \le N-1} \frac{|x[n]|^2}{E\left[|x[n]|^2\right]}, \tag{1}$$

where $E[.]$ denotes the expectation operator. The high peaks drive the transmitter's power amplifier into the nonlinear region of operation, causing nonlinear distortions. They also demand analog-to-digital converters with large dynamic range. To overcome this problem, either power amplifiers with wider linear regions are required or power amplifiers must be forced to work at a reduced efficiency. These solutions come with a heavy price and are unsuitable from the practical point of view. Therefore, many PAPR reduction techniques have been proposed in the literature, such as clipping and filtering, companding transformation, peak windowing, selective mapping, partial transmit sequence, tone injection, tone reservation and linear block coding. An overview of these methods is provided in [1] and [2].

Companding transformation is an attractive and widely used PAPR reduction method due to its low implementation complexity regardless of the number of subcarriers of the OFDM signal. However, the major drawback of current companding techniques is that they lead to increased BER. This degradation in BER performance is mainly due to two factors: first, companding transforms distort the modulating symbols at the transmitter from the original constellation; second, the channel noise is expanded at the receiver by the decompanding transform resulting in an increased number of errors in the recovered data symbols leading to increased BER. Companding transforms are applied to the OFDM signal at the transmitter prior to the power amplifier in order to attenuate the high peaks and amplify the low amplitudes of the OFDM signal, thus decreasing the PAPR [3]. At the receiver, the decompanding transforms are applied in order to recover the original uncompanded OFDM signal. Figure 1 shows the basic block diagram of an OFDM transceiver with a compander used at the transmitter and a decompander at the receiver.

Different types of linear companding transforms, the linear symmetrical transform (LST) and the linear asymmetrical transform (LAST) have been extensively used in the literature to reduce the PAPR of OFDM signals [4], [5]. These transforms have been used in the literature to reduce the PAPR of the OFDM signal. The authors in [4] present a general companding design criteria to facilitate an effective tradeoff between PAPR reduction and BER performance. Specifically, they claimed, based on computer simulations, that the basic LAST (with one discontinuity point) outperforms the LST in terms of BER for some optimized companding parameters. In

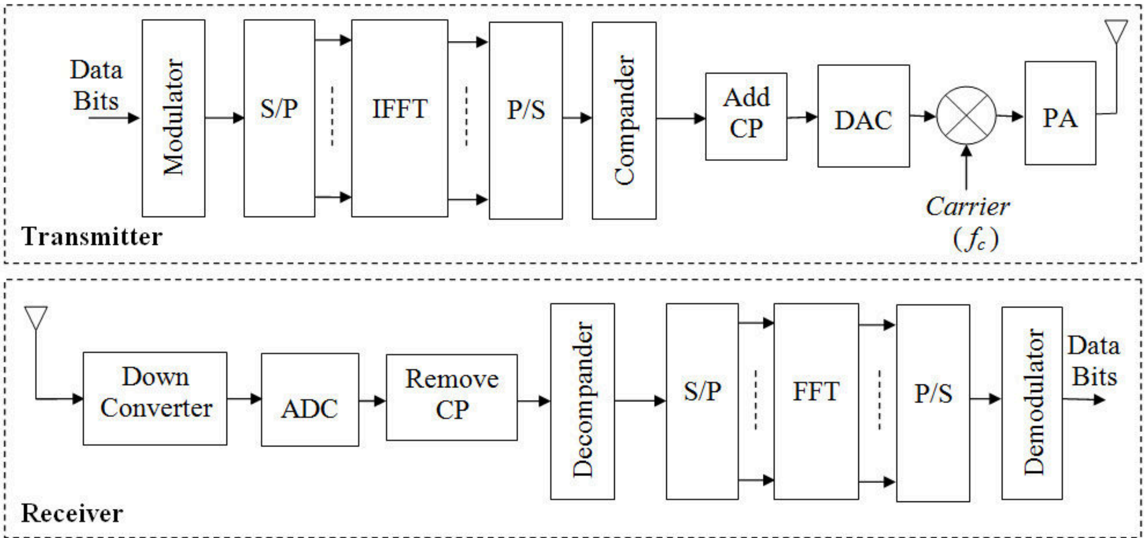

Fig. 1. OFDM transceiver system with companding transform.

[5], a LAST with two discontinuity points is proposed. Based on simulation results, the authors claim that the proposed LAST with two discontinuity points outperforms the basic LAST with one discontinuity point.

Two immediate questions arise from the previous work on linear companding transforms: First, for which parameter values does the LAST with one discontinuity point outperforms the LST? Second, for which parameter values does the LAST with two or more discontinuity points outperforms LST?

This paper tackles the above raised questions, by deriving a comprehensive analytical framework to study the BER performance of linear companding transforms. Specifically, we derive a sufficient condition that ensures the BER performance superiority of a general LAST with $M-1$ discontinuity points over LST. The derived condition explains the contradictions between different reported results in the literature that were unsubstantiated with analytical results and validates some other reported simulation results. Our analysis also serves as a guideline in the process of choosing proper values for the companding parameters to obtain a specific trade-off between the PAPR reduction capability and the BER performance. Furthermore, our analysis shows that the BER performance of the LAST depends on the slopes rather than the number of discontinuity points of the transform, as has been insinuated before. Hence, contrary to prior claims in the literature, LASTs with multiple discontinuities do not always perform better with respect to BER than those with single discontinuities. We also derive conditions, in terms of the companding parameters, to ensure that the average power of the companded OFDM signal remains statistically unchanged compared to the original OFDM signal.

The rest of this paper is organized as follows: Section II introduces the two classes of linear companding transforms, namely the LST and LAST, then derives a sufficient condition that ensures the BER performance superiority of a general LAST with $M-1$ discontinuity points over LST. The derived condition is found in terms of the slopes of the companding transforms. In Section III, we derive conditions for the companding parameters in order to keep the average transmitted power unchanged after companding. Section IV provides simulation results, which validate the theoretical analysis. Finally, Section V concludes the work presented in this paper.

II. Analysis Establishing the Superiority of LAST over LST

We consider two types of linear companding transforms that are widely used in the literature. These are the linear symmetrical transform (LST) and linear asymmetrical transform (LAST). Denoted as C_{LST} and C_{LAST}, respectively,

$$C_{LST}(x) = a\,x[n] + b\,, \tag{2}$$

with $0 < a < 1$ and $b > 0$, and

$$C_{LAST}(x) = \begin{cases} \frac{1}{u}\,x[n], & \text{if } |x| \le v \\ u\,x[n], & \text{if } |x| > v\ , \end{cases} \tag{3}$$

where $0 < u < 1$ and $0 < v < max(x[n])$ are, respectively, the slope and the discontinuity point of the LAST. Figure 2 shows typical profiles of the LST and LAST with one and two discontinuity points. At the receiver, the original OFDM signal can be recovered from the received signal $r[n]$ by applying the inverse LST and LAST, defined, respectively, as

$$\hat{x}_{LST}[n] = \frac{r[n]-b}{a}\,, \tag{4}$$

and

$$\hat{x}_{LAST}[n] = \begin{cases} u\,r[n], & \text{if } n \in \phi_1(v) \\ \frac{1}{u}\,r[n], & \text{if } n \in \phi_2(v)\ , \end{cases} \tag{5}$$

where $\phi_1(v) = \{n, \forall |x[n]| \le v\}$ and $\phi_2(v) = \{n, \forall |x[n]| > v\}$. Observe that it is assumed here that the index sets $\phi_1(v)$ and $\phi_2(v)$ of the OFDM samples are known at the receiver. In

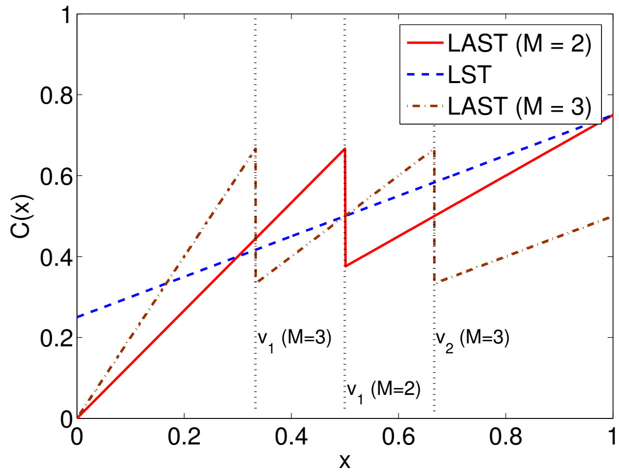

Fig. 2. Typical profiles of the LST and LAST with one and two discontinuity points.

[5], the LAST was extended to the case of two discontinuity points rather than one to offer greater design flexibility. Figure 2 shows typical profiles of the LST and LAST with one and two discontinuity points.

In the following proposition, we show that a sufficient condition for the superiority of the LAST over the LST, in terms of BER, is $\min\{u_k\} > a\,, k = 1, ..., M$, where u_k is the slope of the LAST line segment for $v_{k-1} < x[n] \leq v_k$, M is the number of segments in the LAST characteristics curve and a is the slope of the LST.

Proposition 1. *Consider the sampled OFDM signal $x[n]$. Denote by C_{LST} the linear symmetrical companding transform given by $C_{LST}\,(x[n]) = a\,x[n] + b$, with $0 < a < 1$ and $b > 0$, and let C_{LAST} denote a general linear asymmetrical companding transform with $M-1$ discontinuity points given by*

$$C_{LAST}(x[n]) = \begin{cases} u_1\, x[n], & \text{if } |x[n]| \leq v_1 \\ u_2\, x[n], & \text{if } v_1 < |x[n]| \leq v_2 \\ \vdots & \\ u_M\, x[n], & \text{if } |x[n]| > v_{M-1}\ , \end{cases} \tag{6}$$

where $0 < v_1 < v_2 < \ldots < v_{M-1} < \max(x[n])$. For a white Gaussian noise (WGN) channel, the LAST results in a smaller absolute error at the receiver compared to the LST if

$$\min_{k=1,\cdots,M} u_k > a\,. \tag{7}$$

Proof 1. *For the sake of simplicity, we write x to denote the discrete-time envelope of the OFDM signal $x[n]$. Let w be the channel's additive white Gaussian noise. The OFDM signal is companded before transmission through the WGN. The received signal is therefore given by $r = C(x)+w$. At the receiver, the decompanding function is applied to the received signal, then the proper demodulation is used to recover the data symbols correctly. Hence, the output of the decompander is given by $\hat{x} = C^{-1}[C(x) + w]$. We can then compute the absolute received error $|e| = |\hat{x} - x|$ as*

$$|e| = \left|C^{-1}\left[C\left(x\right) + w\right] - x\right|\ . \tag{8}$$

From Eq. (8), the absolute received error for the LST is

$$\begin{aligned} |e_{LST}| &= \left|\frac{[(ax+b+w)-b]}{a} - x\right| \\ &= \left|\frac{w}{a}\right|\ . \end{aligned} \tag{9}$$

For the LAST, the absolute error is given by

$$|e_{LAST}| = \begin{cases} |\frac{w}{u_1}|, & \text{if } n \in \phi_1 \\ |\frac{w}{u_2}|, & \text{if } n \in \phi_2 \\ \vdots & \\ |\frac{w}{u_M}|, & \text{if } n \in \phi_M\ , \end{cases} \tag{10}$$

where ϕ_k is the index set of the OFDM samples $x[n]$ with amplitudes that fall in the region $v_{k-1} < x[n] \leq v_k$ and M is the number of segments in the LAST characteristics curve. From Eqs. (9) and (10), it is clear that, a sufficient condition to have $|e_{LAST}| < |e_{LST}|$ is $\min_{k=1,\cdots,M}\ u_k > a$.

III. POWER CONSIDERATIONS

Although the condition given by Eq. (7) involves only the slope a of the LST (independently of the offset b) and the slopes $\{u_k\}$ of the LAST (independently of the discontinuity points $\{v_k\}$), it is necessary to relate a to b and $\{u_k\}$ to $\{v_k\}$ in order to achieve some design criteria. Similar to the work presented in [6], we follow the average power constraint design criterion, which ensures that the average power of the OFDM signal is kept unchanged after companding. The following proposition relates the companding parameters of the LST and LAST such that this criterion is met.

Proposition 2. *Consider the OFDM signal $x[n]$ with the linear symmetrical companding transform given by $C_{LST}\,(x[n]) = a\,x[n] + b$, with $0 < a < 1$ and $b > 0$, and the general linear asymmetrical companding transform with $M-1$ discontinuity points given by*

$$C_{LAST}(x[n]) = \begin{cases} u_1\, x[n], & \text{if } |x[n]| \leq v_1 \\ u_2\, x[n], & \text{if } v_1 < |x[n]| \leq v_2 \\ \vdots & \\ u_M\, x[n], & \text{if } |x[n]| > v_{M-1}\ , \end{cases} \tag{11}$$

where $0 < v_1 < v_2 < \ldots < v_{M-1} < max(x[n])$.

The average transmitted power of the companded OFDM signal by C_{LST} and C_{LAST}, respectively, can be kept unchanged compared to that of the original OFDM signal $x[n]$ if the companding parameters are chosen such that

$$a^2\left(2\,\sigma^2\right) + a\left(2\,\sigma\, b\,\sqrt{\frac{\pi}{2}}\,\right) + \left(b^2 - 2\,\sigma^2\right) = 0\,, \tag{12}$$

and

$$\frac{4-\pi}{2}\,\sigma^2 \sum_{k=1}^{M} u_k^2\, p(\phi_k) + \frac{\pi\sigma^2}{2}\left[\sum_{k=1}^{M} u_k\, p(\phi_k)\right]^2 = 2\,\sigma^2\,, \tag{13}$$

where σ is the strictly positive Rayleigh parameter of the probability density function of the OFDM signal's envelope, ϕ_k is the index set of the OFDM envelope samples $|x[n]|$ that fall in the region $v_{k-1} < |x[n]| \le v_k$, $p(\phi_k)$ is the probability that $v_{k-1} < |x[n]| \le v_k$ and M is the number of segments in the LAST characteristics curve.

Proof 2. *For the sake of simplicity, we write x to denote the discrete-time envelope of the OFDM signal $x[n]$. Using the central limit theorem (CLT), it is shown that for a large number of subcarriers, both the real and imaginary parts of the complex OFDM signal are asymptotically independent and identically distributed Gaussian random variables. Consequently, the envelope of the OFDM signal, x, follows the Rayleigh distribution [7] with the probability density function*

$$f_x(x) = \frac{x}{\sigma^2} \exp\left(\frac{-x^2}{2\,\sigma^2}\right), \tag{14}$$

where σ is the strictly positive adjustable Rayleigh parameter. The mean, variance and second moment of the Rayleigh distribution are given by $\sqrt{\pi/2}\,\sigma$, $\frac{4-\pi}{2}\,\sigma^2$ and $2\sigma^2$, respectively. To keep the average power of the OFDM signal unchanged after companding, the following condition must be satisfied

$$\begin{aligned} E\left[x^2\right] &= E\left[C^2(x)\right] \Leftrightarrow \\ 2\,\sigma^2 &= var\left[C(x)\right] + E^2\left[C(x)\right]. \end{aligned} \tag{15}$$

Using $C_{LST}(x)$ of the LST and substituting the values of $E[C_{LST}(x)]$ and $var[C_{LST}(x)]$ in Eq. (15) yields the condition given in (12).

Considering C_{LAST} of the LAST given by (11) and noting that

$$E[C_{LAST}(x)] = \sum_{k=1}^{M} u_k\, E[x]\, p(\phi_k), \tag{16}$$

and

$$var[C_{LAST}(x)] = \sum_{k=1}^{M} u_k^2\, var[x]\, p(\phi_k), \tag{17}$$

where $p(\phi_k)$ is given by

$$p(\phi_k) = CDF(v_k) - CDF(v_{k-1}), \tag{18}$$

we obtain the condition given in Eq. (13), where $CDF(v_k)$ is the cumulative distribution function of the Rayleigh distributed OFDM envelope at the point v_k, i.e., $CDF(v_k) = 1 - \exp(\frac{-v_k^2}{2\,\sigma^2})$.

The quadratic formula in Eq. (12) can be used to find a set of solutions for a as a function of b and σ. It is easy to show that, given the conditions $b > 0$ and $0 < a < 1$, the nontrivial solutions for a are obtained when b satisfies the following bound

$$0 < b < \sqrt{2}\,\sigma \tag{19}$$

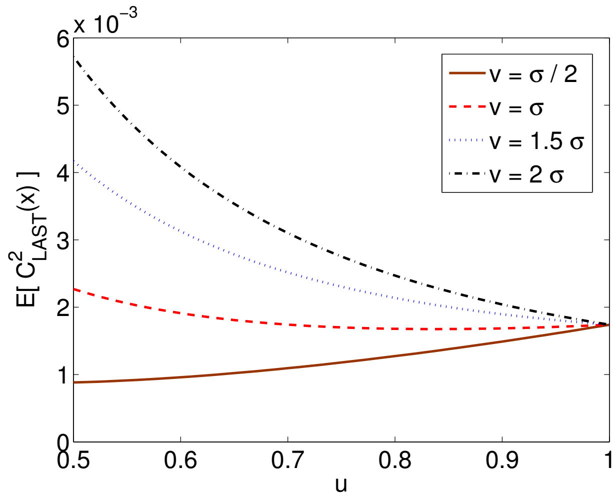

Fig. 3. The power of the companded OFDM signal using the LAST with one discontinuity point ($M = 2$) when different values of u and v are used.

IV. SIMULATION RESULTS

In order to validate the theoretical findings of the previous section, we conducted computer simulations for a baseband OFDM system model, which uses the LST and LAST compandings. Specifically, we consider the implementation of the WiMAX standard in the downlink partial use subcarrier (DL-PUSC) mode with 1024 subcarriers. The well known solid state power amplifier (SSPA) model is implemented to model the nonlinearity of the transmitter's power amplifier. This model amplifies the envelope of the signal without introducing phase distortions. The input-output relationship of the SSPA is given by

$$x_{out} = \frac{x_{in}}{\left[1 + \left(\frac{x_{in}}{A_{sat}}\right)^{2p}\right]^{1/2p}}, \tag{20}$$

where p is a positive parameter controlling the nonlinearity level of the power amplifier and A_{sat} is a normalization factor specifying the saturation level of the amplifier. In all simulations, we set $p = 2$ and $A_{sat} = 0.14$.

The transmission channel is modeled by a white Gaussian noise (WGN) channel and no channel coding or any other kind of channel diversity is used. For each signal-to-noise ratio (SNR) value, we transmit 100 OFDM frames, with 20 OFDM symbols of 1024 subcarriers in each frame, and compute the average of the BER.

We first consider a LAST companding scheme with one discontinuity point ($M = 2$), given by $v = \sigma \approx 0.0295$, with the slopes related according to Eq. (3). Figure (3) shows the power of the companded OFDM signal using the LAST ($E[C^2_{LAST}(x)]$) with one discontinuity point ($M = 2$) when different values of u and v are used. It is shown that choosing $v = \sigma$ keeps $E[C^2_{LAST}(x)] \approx 2\,\sigma^2 \approx 1.74 \times 10^{-3}$ for a much wider range of a compared to other choices of v.

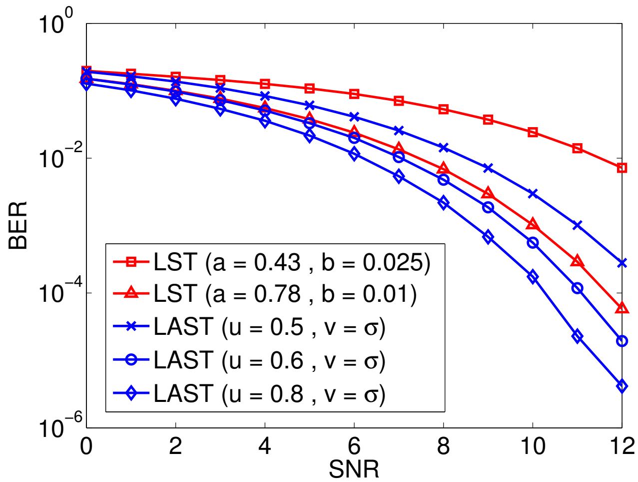

Fig. 4. BER for the LST and LAST with different companding parameters

Figure 4 shows the BER performance when the LST and LAST companding schemes are used for different companding parameters. Observe the two cases: LAST with $(u = 0.8, v = \sigma)$ versus LST with $(a = 0.78, b = 0.01)$ and LAST with $(u = 0.5, v = \sigma)$ versus LST with $(a = 0.43, b = 0.025)$. It is clear that both cases satisfy the sufficient condition derived in proposition 1. Hence LAST is superior to LST, in terms of BER, in both cases. It is important to emphasize, however, that the condition in Eq. (7) is only a sufficient and not necessary condition for LAST to outperform LST. In particular, violating this condition does not necessarily mean that LST will outperform LAST. Figure 4 shows two examples where the sufficient condition is violated, and, in each example, the performance of the LAST versus LST companding schemes is different. In the first example, the condition is violated with LAST $(u = 0.6, v = \sigma)$ versus LST $(a = 0.78, b = 0.01)$, and LAST outperforms LST in terms of BER. In the second example, the condition is violated with LAST $(u = 0.5, v = \sigma)$ versus LST $(a = 0.78, b = 0.01)$, and LST outperforms LAST. Although both of these LASTs fail to satisfy the sufficient condition of (7), we observe that at one hand the LAST with $(u = 0.6, v = \sigma)$ has a lower BER compared to LST with $(a = 0.78, b = 0.01)$. On the other hand, LAST with $(u = 0.5, v = \sigma)$ has a higher BER compared to the same LST.

In conclusion, the simulation results in Fig. (4) comply with the derived sufficient condition for the superiority of the LAST over LST and validate the fact that the condition is sufficient and not necessary. In other words, violating the sufficient condition does not necessarily imply that LST will outperform LAST.

V. CONCLUSIONS

In this paper, we derived a sufficient condition that ensures the BER performance superiority of a general linear asymmetrical transform (LAST) with any number of discontinuity points over the linear symmetrical transform (LST), in terms of the slope values of the companding transforms. The derived condition explains the controversy arising from some of the reported simulation results and claims in the literature, that were unsubstantiated with analytical results. Our results also serve as a guideline in the process of selecting proper companding parameters with desired PAPR reduction and BER performance. The sufficient condition shows that the BER performance of LAST depends mainly on the slopes rather than the number of discontinuity points of the transform. Simulation results validate our theoretical findings by comparing the BER of LAST with LST. Moreover, we derive the relationship between the companding parameters of LST and LAST in order to keep the average transmitted power unchanged after companding.

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