Theoretical Foundations of Spatially-Variant Mathematical Morphology Part I: Binary Images

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Abstract—We develop a general theory of spatially-variant (SV) mathematical morphology for binary images in the Euclidean space. The basic SV morphological operators (that is, SV erosion, SV dilation, SV opening, and SV closing) are defined. We demonstrate the ubiquity of SV morphological operators by providing an SV kernel representation of increasing operators. The latter representation is a generalization of Matheron's representation theorem of increasing and translation-invariant operators. The SV kernel representation is redundant, in the sense that a smaller subset of the SV kernel is sufficient for the representation of increasing operators. We provide sufficient conditions for the existence of the basis representation in terms of upper-semicontinuity in the hit-or-miss topology. The latter basis representation is a generalization of Maragos' basis representation for increasing and translation-invariant operators. Moreover, we investigate the upper-semicontinuity property of the basic SV morphological operators. Several examples are used to demonstrate that the theory of spatially-variant mathematical morphology provides a general framework for the unification of various morphological schemes based on spatially-variant geometrical structuring elements (for example, circular, affine, and motion morphology). Simulation results illustrate the theory of the proposed spatially-variant morphological framework and show its potential power in various image processing applications.

Index Terms—Mathematical morphology, spatially-variant morphology, adaptive morphology, circular morphology, affine morphology, median filter, kernel representation, basis representation, upper-semicontinuity, hit-or-miss transform.

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1 INTRODUCTION

SINCE it was first developed in the 1970s by Matheron [46] and Serra [53], mathematical morphology emerged as a powerful tool in signal and image processing applications [25], [41], [43], [44]. Mathematical morphology uses concepts from set theory, geometry, and topology to analyze geometrical structures in signals and images. It introduces the concept of *structuring element* (SE), which is a geometrical pattern used to probe the input image. The theory has been used in a wide range of applications including biomedical image processing [21], [1], shape analysis [31], coding and compression [34], [42], automated industrial inspection [19], texture analysis [22], [60], radar imagery [30], and multi-resolution techniques and scale-spaces [26], [27], [49]. Morphological operators have been efficiently implemented in numerous commercial software products and computer architectures for digital signal and image processing applications [23]. The ubiquity of morphological operators has been captured by Matheron's kernel representation theorem, which asserts that any increasing and translation-invariant operator can be exactly represented in terms of elementary morphological operators (that is, morphological erosions and dilations) using a collection of structuring elements in their kernel (that is, a set of structuring elements that characterizes the operator) [46]. Maragos, in his doctoral thesis [37], [39], has provided sufficient conditions under which the increasing and translation-invariant operators have basis representations.

Initially, the focus of mathematical morphology was devoted to translation-invariant operators (that is, the structuring element remains fixed in the entire space). However, the translation-invariance assumption is not appropriate in many applications. One of the earliest examples of adaptive (or spatially-variant) structuring elements is given by Beucher et al. [5] in the analysis of images from traffic control cameras. Because of the perspective effect, vehicles at the bottom of the image are closer and appear larger than those higher in the image. Hence, the structuring element (SE) should follow a law of perspective, for example, vary linearly with its vertical position in the image. In range imagery techniques, the gray-scale value of each pixel is proportional to its distance to the imaging device. Hence, the apparent length of a feature in such images is a function of its gray-value range. One can therefore process (for example, extract or eliminate) all the differently scaled instances of the object of interest by adapting the structuring element to the local range. Verly and Delanoy [58] developed an algorithm to design and apply adaptive structuring elements for object extraction in range images. The need for spatially-variant morphological image processing arises even in the most basic applications in image processing, namely, image smoothing and denoising. In morphological image denoising, there is a trade-off between noise removal and detail preservation in the image.

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Moreover, translation-invariant morphological filters are inherently incapable of restoring image structures that are smaller than the structuring element used [18], [43], [44], [52]. Thus, some geometric information of the image may be lost while reducing the noise. Many researchers proposed different adaptive smoothing algorithms, which consist of removing noise while preserving image features by adapting the structuring elements to local features of the image. We cite, in chronological order, some application-oriented algorithms for adaptive morphological denoising and smoothing. Morales [47] used an SE that changes adaptively according to the variance of the input signal in order to remove artifacts and noise from images. In [12], a family of SEs, which is closed under translation, is used to efficiently implement the maximum opening filter. The size of the SE is determined by the image characteristics and the noise patterns. This method allows for the elimination of more impulsive noise without blurring the edges compared to the conventional translation-invariant maximum opening filter. In [10], the SE changes with respect to the edge strength of the signal. That is, the line SE is shorter for a strong edge and longer for a weak one. So, the edges are not blurred by this method much as they are blurred by a conventional fixed SE. Chen et al. [11] developed the progressive umbra-filling (PUF) algorithm for adaptive signal smoothing. Their algorithm gradually fills the umbra of a signal with a set of overlapping SEs that vary from larger to smaller in scale. The PUF algorithm is shown to successfully reduce the bumping noise without oversmoothing the signal. In [45], a locally adaptive structuring element is used for contour extraction in ultrasound images corrupted by speckle noise. In [56], adaptive elliptical structuring elements were used in a morphological edge linking algorithm, where the size and orientation of the structuring element is adjusted according to the local properties of the image such as slope and curvature. Based on distance transformation, Cuisenaire [14] developed efficient algorithms for the implementation of the adaptive erosion, dilation, opening, and closing using ball structuring elements of varying sizes. Debayle and Pinoli [15], [16] used the concept of Adaptive Neighborhood (AN) that was proposed by Gordon and Rangayyan [20] to define AN-based structuring elements. The latter SEs depend on a morphological or geometrical criterion with a homogeneity tolerance so as to take into account the local features of the image. Lerallut et al. [36] proposed amoeba structuring elements, which adapt their size and shape to the content of the image.

Adaptive mathematical morphology for shape representation and image decomposition has been scarcely investigated. In [57], an adaptive decomposition of binary images into a number of simple shapes based on homothetics of a set of structuring elements was proposed in order to minimize the number of points in the representation of the image. In [61], eight structuring elements were used to generalize the morphological skeleton representation. In this representation, the number of points needed to represent a given shape is significantly lower than that in the standard morphological skeleton transform. In [7], we extended the morphological skeleton representation framework presented in [51] to the spatially-variant case. We also provided a practical algorithm to construct the optimal structuring elements, which minimize the cardinality of the spatially-variant morphological skeleton representation.

The examples cited above clearly illustrate the need to develop a unified spatially-variant mathematical morphology theory. The objective of this paper is, thus, to present a general theory of spatially-variant (SV) mathematical morphology in the Euclidean space, which will unify all the techniques proposed thus far into a comprehensive mathematical framework. The proposed theory preserves the concept of structuring element, which is crucial in the design of geometrical signal and image processing applications. This paper is the first in a sequence of two papers (Parts I and II). In this part, we will investigate the foundations of the theory of spatially-variant mathematical morphology in the Euclidean space for binary signals and images. The treatment of the gray-level case will be explored in Part II.

This paper is organized as follows: In Section 2, we review the previous work related to the extension of mathematical morphology to transformations that do not commute with the translation operator. In Section 3, we define the basic SV morphological operators (that is, SV erosion, SV dilation, SV opening, and SV closing). The properties of the basic SV morphological operators are enumerated in Appendix A. Subsequently, we provide several examples of nontranslation-invariant morphology such as affine morphology and adaptive neighborhood morphology, which establish our representation as a unified theory of spatially-variant mathematical morphology. Matheron's representation theorem is extended to the spatially-variant case in Section 4. The spatially-variant kernel representation is a powerful theoretical result since it demonstrates the ubiquity of the SV morphological operators by representing every increasing operator in terms of SV erosions or dilations. However, as was the case in translation-invariant mathematical morphology [37], the practical importance of the SV kernel representation is limited since it requires an infinite number of SV erosions or SV dilations to implement the operator. Following the development of Maragos' basis representation for translation-invariant operators presented in [2], [17], and [37], we provide, in Section 5, sufficient conditions under which the SV basis representation exists. These sufficient conditions are expressed in terms of upper-semicontinuity in the hit-or-miss topology. We subsequently investigate some topological properties of the SV erosion and SV dilation and provide, as an example, a basis representation of the adaptive median filter. In Section 6, we provide simulation results to show the power of the SV mathematical morphology theory in image reconstruction, pattern segmentation, and shape representation. A summary of the paper and concluding remarks are provided in Section 7.

2   RELATED WORK

An extension of the theory of mathematical morphology from the Euclidean space to complete lattices was initiated by Matheron [46] and Serra [54]. Heijmans and Ronse further pursued their work on lattice morphology in [28] and [29]. Lattice morphology is a powerful tool that provides an abstraction of mathematical morphology based on lattice theory, a topic devoted to the investigation of the algebraic properties of partially ordered sets [6], [25]. The general properties of lattice morphological operators depend only on the characteristics of the order relation and the supremum and infimum of the complete lattice. Although lattice morphology is an extremely powerful theory, it relies on

abstract mathematical concepts for the representation of the morphological operators and is, therefore, not accessible to engineers in the development of signal and image processing systems. In particular, the general theory of lattice morphology cannot be used to convey the notion of a structuring element, which is critical in the development of morphological image analysis applications. Thus, it is interesting to focus on special cases of the general lattice morphology theory, which convey the intuition provided by the translation-invariant theory of mathematical morphology in the Euclidean space.

Earlier efforts to extend mathematical morphology theory to nontranslation-invariant operators while preserving the notion of the structuring element have been presented in special cases. A generalization of Euclidean morphology to arbitrary abelian symmetry groups was investigated in [24] and [50]. This concept provides, for example, for the representation of morphological operators in terms of a family of circular structuring elements (that is, rotation and scaling of an elementary SE). Maragos, in [40], extended the notion of circular morphology by introducing the affine morphology framework. Heijmans and Ronse [28], [29] characterized morphological operators in complete lattices having a certain type of abelian group of automorphisms generalizing translations. Subsequently, they extended Matheron's theorem to increasing operators that are invariant under the group operator. Roedrink [48] extended Euclidean morphology by including invariance under more general groups of transformations (not necessarily abelian) such as the Euclidean motion group, the similarity group, the affine group, and the projective group. The most general representation, in the Euclidean space, was introduced by Serra in [54, Chapters 2, 3, and 4]. He defined the concept of a structuring function that associates to each point in space a local structuring element. This representation generalizes all prior efforts in the Euclidean space while preserving the concept of the structuring element. In his work, Serra uses the basic spatially-variant morphological operators to analyze the notion of connectivity and induced metrics. Chefchaouni and Schonfeld [8], [9] extended Serra's work on spatially-variant mathematical morphology and illustrated the concept of a spatially-variant kernel representation for binary increasing systems.

In this paper, we elaborate on Serra's and Chefchaouni and Schonfeld's work on spatially-variant mathematical morphology, in the Euclidean space, to provide a comprehensive theory of spatially-variant mathematical morphology, which captures the geometrical interpretation of the structuring element. The proposed theory is the most general framework of spatially-variant mathematical morphology that preserves the notion of the structuring element. For example, our approach unifies the work by Heijmans and Ronse on T-invariant operators, where T is an abelian group of automorphisms of a complete lattice [28] and the work of Maragos on affine morphology [40]. Through our work, we hope to provide a sound mathematical foundation to past and future research in spatially-variant morphological signal and image processing, which captures the geometrical intuition of practitioners in the engineering community.

Throughout the paper, we provide reference to known results and limit the presentation of proofs to new contributions.

3 SPATIALLY-VARIANT MATHEMATICAL MORPHOLOGY

3.1 Preliminaries

In this paper, we consider the continuous or discrete Euclidean space $\mathbf{E} = \mathbb{R}^n$ or $\mathbb{Z}^n$ for some $n > 0$.[1] The set $\mathcal{P}(\mathbf{E})$ denotes the set of all subsets of $\mathbf{E}$. Elements of the set $\mathbf{E}$ will be denoted by lower case letters, for example, a, b, and c. Elements of the set $\mathcal{P}(\mathbf{E})$ will be denoted by upper case letters, for example, A, B, and C. An order on $\mathcal{P}(\mathbf{E})$ is imposed by the inclusion $\subseteq$. We use $\cup$ and $\cap$ to denote the union and intersection in $\mathcal{P}(\mathbf{E})$, respectively. "$\Rightarrow$, $\Leftrightarrow$, $\forall$, and $\exists$" denote, respectively, "implies," "if and only if (iff)," "for all," and "there exist(s)." X^c denotes the complement of X. The translate of the set X by the element $a \in \mathbf{E}$ is defined by $X + a = \{x + a : x \in X\}$. The cardinality, $|X|$, of a set X is the total number of elements contained in the set. $X \ominus B$ and $X \oplus B$ denote the translation-invariant erosion and dilation, respectively, of the set X by the structuring element B. We use $\mathcal{O} = \mathcal{P}(\mathbf{E})^{\mathcal{P}(\mathbf{E})}$ to denote the set of all operators mapping $\mathcal{P}(\mathbf{E})$ into itself. The elements of the set $\mathcal{O}$ will be denoted by lower case Greek letters, for example, α, β, and γ. An order on $\mathcal{O}$ is imposed by the inclusion $\subseteq$, that is, $\alpha \subseteq \beta$ if and only if $\alpha(X) \subseteq \beta(X)$ for every $X \in \mathcal{P}(\mathbf{E})$. We shall restrict our attention to nondegenerate operators, that is, $\psi(\mathbf{E}) = \mathbf{E}$ and $\psi(\emptyset) = \emptyset$ for every $\psi \in \mathcal{O}$ (the set $\emptyset \in \mathcal{P}(\mathbf{E})$ is used to denote the empty set).

An operator $\psi \in \mathcal{O}$ is

- *increasing* if $X \subseteq Y \Longrightarrow \psi(X) \subseteq \psi(Y)$ $(X, Y \in \mathcal{P}(\mathbf{E}))$;
- *translation-invariant* if

$$\psi(X + a) = \psi(X) + a \qquad (X \in \mathcal{P}(\mathbf{E}), a \in \mathbf{E});$$

- *idempotent* if $\psi(\psi(X)) = \psi(X)$ $(X \in \mathcal{P}(\mathbf{E}))$;
- *extensive* (respectively, *anti-extensive*) if $X \subseteq \psi(X)$ (respectively, $\psi(X) \subseteq X$) $(X \in \mathcal{P}(\mathbf{E}))$.

The mapping ψ^* in $\mathcal{O}$ is the dual of the mapping ψ in $\mathcal{O}$ iff $\psi^*(X) = (\psi(X^c))^c$ $(X \in \mathcal{P}(\mathbf{E}))$.

3.2 Erosions and Dilations

Consider the spatially-variant structuring element θ given by a mapping from $\mathbf{E}$ into $\mathcal{P}(\mathbf{E})$. The class of all such mappings inherits the complete lattice structure of $\mathcal{P}(\mathbf{E})$ by setting $\theta_1 \leq \theta_2 \Longleftrightarrow \theta_1(z) \subseteq \theta_2(z)$ for every $z \in \mathbf{E}$. The transposed spatially-variant structuring element θ' is given by a mapping from $\mathbf{E}$ into $\mathcal{P}(\mathbf{E})$ such that

$$\theta'(y) = \{z \in \mathbf{E} : y \in \theta(z)\} \qquad (y \in \mathbf{E}). \qquad (1)$$

In the translation-invariant case, the mapping θ is the translation operator by a fixed set B, that is, $\theta(y) = B + y$, for every $y \in \mathbf{E}$. Therefore, $z \in \theta'(y) \Leftrightarrow y \in \theta(z) \Leftrightarrow y \in (B + z) \Leftrightarrow z \in (\check{B} + y)$, where $\check{B} = -B$ is the reflected set of B. Hence, the transposed mapping reduces, in the translation-invariant case, to the translation by the reflected set $\check{B}$. As in the translation-invariant mathematical morphology, the choice of the structuring element mapping is application oriented.

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1. The work presented in this paper is valid for complete atomic Boolean lattices except in Section 5, where specific conditions on the space $\mathbf{E}$ will be provided.
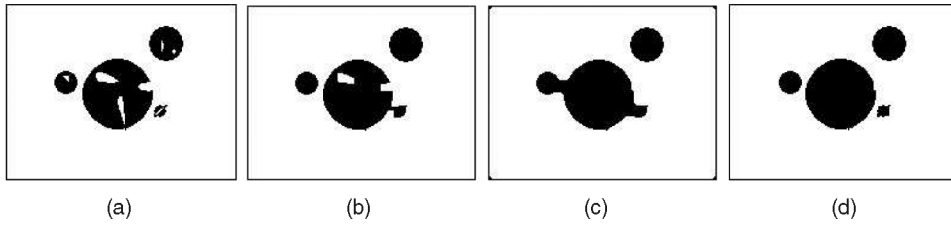
Fig. 1. Translation-invariant and spatially-variant closing of a binary image: (a) Original image, (b) translation-invariant closing with a circular SE of radius 3, (c) translation-invariant closing with a circular structuring element of radius 6, and (d) spatially-variant closing.

Definition 1. *The spatially-variant erosion $\mathcal{E}_\theta \in \mathcal{O}$ is defined as*

$$\mathcal{E}_\theta(X) = \{z \in \mathbf{E} : \theta(z) \subseteq X\} = \bigcap_{x \in X^c} \theta'^c(x) \qquad (X \in \mathcal{P}(\mathbf{E})).$$

(2)

Definition 2. *The spatially-variant dilation $\mathcal{D}_\theta \in \mathcal{O}$ is defined as*

$$\mathcal{D}_\theta(X) = \{z \in \mathbf{E} : \theta'(z) \cap X \neq \emptyset\} = \bigcup_{x \in X} \theta(x) \qquad (X \in \mathcal{P}(\mathbf{E})).$$

(3)

The right-hand side equalities in definitions 2 and 3 can be easily verified, and thus, their proof will be omitted. The SV erosion and SV dilation are increasing and form an adjunction. Hence, the basic properties of translation-invariant erosion and dilation can be transposed to the SV erosion and dilation. These properties are enumerated in Appendix A.

As in the translation-invariant mathematical morphology, a large class of SV binary operators can be built from the two basic SV operators. We give here only the spatially-variant version of the morphological opening and closing operators. The spatially-variant opening γ_θ is given by

$$\gamma_\theta(X) = \mathcal{D}_\theta(\mathcal{E}_\theta(X)) = \bigcup \{\theta(y) : \theta(y) \subseteq X; y \in \mathbf{E}\},$$

(4)

and the spatially-variant closing ϕ_θ is given by

$$\begin{aligned}\phi_\theta(X) &= \mathcal{E}_\theta(\mathcal{D}_\theta(X)) \\ &= \{z \in \mathbf{E} : \theta(y) \cap X \neq \emptyset, \ \forall \, \theta(y) : z \in \theta(y)\},\end{aligned}$$

(5)

for every $X \in \mathcal{P}(\mathbf{E})$.

The spatially-variant opening and closing are morphological filters, that is, they are increasing and idempotent. Moreover, it follows from (4) and (5) and the properties in Appendix A that the spatially-variant opening is antiextensive and the spatially-variant closing is extensive.

Fig. 1 illustrates a synthetic example of SV closing. Fig. 1a shows a binary image containing four circular objects corrupted by white pores. We want to process this image in order to fill in the pores without altering the topology of the image. Fig. 1b shows the output image of the translation-invariant closing using a circular structuring element of radius 3. Only the pores of dimension smaller than 3 were filled. The larger pores remain. Fig. 1c shows the output image of the translation-invariant closing using a circular structuring element of radius 6. All the pores were filled. However, the closing operation altered the topology of the original image by connecting originally disconnected objects in the image. The SV closing, shown in Fig. 1d, fills in all the

pores while preserving the connectivity properties of the original image.

Next, we present some examples of nontranslation-invariant mathematical morphology, which are special cases of the proposed framework of spatially-variant mathematical morphology.

3.3 Examples

3.3.1 Circular Morphology [28]

Consider $\mathbf{E} = \mathbb{R}^2 - 0$. Let (r_a, φ_a) be the polar coordinates of a given $a \in \mathbf{E}$. We define the operation $\odot$ by

$$a \odot b = (r_a r_b, \varphi_a + \varphi_b), \qquad (a, b \in \mathbf{E}).$$

(6)

Observe that the operation $\odot$ on $\mathbf{E}$ corresponds to the multiplication in the complex plane if we associate the complex number $r \exp^{j\varphi}$ to each $(r, \varphi) \in \mathbf{E}$. Consider a non empty set $A \in \mathcal{P}(\mathbf{E})$. Define the mapping θ as follows:

$$\theta(z) = A \odot z = \{a \odot z : a \in A\} \qquad (z \in \mathbf{E}).$$

(7)

That is, to each point $z \in \mathbf{E}$, the mapping θ associates the scaled and rotated version of the set A by the magnitude of the point z, r_z, and its angle φ_z. Then, $\theta'(z) = A^{-1} \odot z$, where $A^{-1} = \{\frac{1}{a} : a \in A\}$. The SV erosion and dilation defined in (2) and (3), respectively, become

$$X \ominus_c A = \{z \in \mathbf{E} : (A \odot z) \subseteq X\} \qquad (X \in \mathcal{P}(\mathbf{E})),$$

(8)

and

$$X \oplus_c A = \{z \in \mathbf{E} : (A^{-1} \odot z) \cap X \neq \emptyset\} \qquad (X \in \mathcal{P}(\mathbf{E})).$$

(9)

Equations (8) and (9) are the circular erosion and dilation, respectively, defined in [28]. Therefore, circular morphology is a special case of the proposed spatially-variant mathematical morphology.

3.3.2 Affine Morphology [40]

Let $\mathbf{E} = \mathbb{R}^2$, and let $\mathcal{G}$ be the set defined by

$$\mathcal{G} = \{(M, t) : M \in \mathbb{R}^{2 \times 2}, \det(M) \neq 0, t \in \mathbb{R}^2\}.$$

(10)

Consider a subset $\mathcal{S} \subseteq \mathcal{G}$. Define the structuring element mapping $\theta : \mathbf{E} \to \mathcal{P}(\mathbf{E})$ as

$$\theta(z) = \{Mz + t : (M, t) \in \mathcal{S}\}.$$

(11)

The transposed structuring element is then given by $\theta'(z) = \{M^{-1}(z - t) : (M, t) \in \mathcal{S}\}$. Hence, one can easily show that the spatially-variant erosion and dilation defined in (2) and (3), respectively, reduce to

$$X \ominus_a \mathcal{S} = \bigcap_{(M,t)\in\mathcal{S}} \{M^{-1}(x - t) : x \in X\}, \qquad (12)$$

and

$$X \oplus_a \mathcal{S} = \bigcup_{(M,t)\in\mathcal{S}} \{Mx + t : x \in X\}. \qquad (13)$$

Equations (12) and (13) are the affine erosion and dilation, respectively, defined in [40]. This establishes the affine morphology framework as a special case of the spatially-variant morphology theory. Observe that the affine group is not an abelian group, and therefore, the theory of Heijmans and Ronse on T-invariant operators presented in [28] does not apply.

3.3.3 Amoeba Morphology [36]

Consider $\mathbf{E} = \mathbb{Z}^2$. Denote by $I(x)$ the value of the image at position x. Let d be a distance defined between the values of the image. Let $\sigma = (x = x_0, x_1, \cdots, x_n = y)$ be a path between the points x and y. Let $\lambda > 0$ and $r > 0$. The length of the path σ is defined as

$$L(\sigma) = \sum_{i=1}^{n} [1 + \lambda d(I(x_i), I(x_{i+1}))]. \qquad (14)$$

The amoeba distance with parameter λ is defined by $d_\lambda(x, y) = \min_\sigma L(\sigma)$. Define the structuring element mapping θ as

$$\theta(x) = B_{\lambda,r}(x) = \{y : d_\lambda(x, y) \le r\}. \qquad (15)$$

Then, the SV erosion and dilation defined in (2) and (3), respectively, reduce to

$$\mathcal{E}_\theta(X) = \{z \in \mathbf{E} : B_{\lambda,r}(z) \subseteq X\}, \text{ and } \mathcal{D}_\theta(X) = \bigcup_{x \in X} B_{\lambda,r}(x). \qquad (16)$$

Equation (16) coincides with the definitions of the amoeba erosion and dilation introduced in [36]. Thus, amoeba morphology is another special case of the proposed spatially-variant mathematical morphology framework.

3.3.4 Adaptive Neighborhood Morphology [15], [16]

Consider $\mathbf{E} = \mathbb{R}^2$. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be a criterion mapping such as luminance or contrast. Let $m > 0$. For each $x \in \mathbf{E}$, define the connected set $V_m^h(x)$ by $V_m^h(x) = \{y : |h(y) - h(x)| \le m\}$. Choose the SE mapping θ as follows:

$$\theta(x) = \bigcup_{z \in \mathbf{E}} \{V_m^h(z) : x \in V_m^h(z)\}. \qquad (17)$$

The SV erosion and dilation defined in (2) and (3), respectively, reduce to

$$\mathcal{E}_\theta(X) = \{z \in \mathbf{E} : \exists y \in \mathbf{E} \text{ such that } z \in V_m^h(y) \text{ and } V_m^h(y) \subseteq X\}, \qquad (18)$$

and

$$\mathcal{D}_\theta(X) = \bigcup_{x \in X} \bigcup_{z \in \mathbf{E}} \{V_m^h(z) : x \in V_m^h(z)\}. \qquad (19)$$

Equations (18) and (19) are, respectively, the adaptive neighborhood erosion and dilation presented in [15]. Thus, adaptive neighborhood morphology is yet another special

case of the spatially-variant structuring element mathematical morphology theory.

The above examples are practical special cases of the proposed theory of spatially-variant mathematical morphology. Each example corresponds to a special choice of the structuring element mapping θ that is application oriented. For example, affine signal transformations are useful for modeling self-similarities in fractal images and shape deformations in visual motion [40]. Circular morphology is useful for circular-invariant material structure such as radar displays and echographic images [28]. Amoeba morphology is effective for denoising [36], and adaptive neighborhood morphology was illustrated for multiscale representation and segmentation [15].

In the next section, we demonstrate the ubiquity of the basic SV operators, that is, SV erosion and SV dilation, by proving that every increasing operator can be exactly represented in terms of SV erosions or SV dilations.

4 SPATIALLY-VARIANT KERNEL REPRESENTATION

4.1 Theoretical Analysis

We extend the concept of the kernel introduced by Matheron, for translation-invariant operators [46], to the spatially-variant case as follows:

Definition 3. *The kernel, $Ker(\psi)$, of an spatially-variant operator $\psi \in \mathcal{O}$ is given by*

$$Ker(\psi) = \{\theta : z \in \psi(\theta(z)), \text{ for every } z \in \mathbf{E}\}. \qquad (20)$$

The SV kernel of a nondegenerate operator is nontrivial as the following proposition shows:

Proposition 1. $Ker(\psi) \ne \emptyset, \forall \psi \in \mathcal{O}$.

An important property of the SV kernel of an increasing operator is that it is unique. Furthermore, the mapping that associates each operator $\psi \in \mathcal{O}$ to its kernel is an isomorphism.

Proposition 2. *Given two operators ψ_1 and $\psi_2 \in \mathcal{O}$, we have $\psi_1 \subseteq \psi_2$ if and only if $ker(\psi_1) \subseteq ker(\psi_2)$.*

We now provide the kernel representation of increasing operators based on SV erosions and SV dilations.

Theorem 1. *An operator $\psi \in \mathcal{O}$ is increasing if and only if ψ can be exactly represented as union of spatially-variant erosions by mappings in its kernel or equivalently as intersection of spatially-variant dilations by the transposed mappings in the kernel of its dual ψ^*, that is,*

$$\psi(X) = \bigcup_{\theta \in Ker(\psi)} \mathcal{E}_\theta(X) = \bigcap_{\theta \in Ker(\psi^*)} \mathcal{D}_{\theta'}(X), \qquad (X \in \mathcal{P}(\mathbf{E})). \qquad (21)$$

4.2 Examples

4.2.1 Circular Morphology [28]

We say that a mapping $\psi \in \mathcal{O}$ is circular invariant if for every $X \in \mathcal{P}(\mathbf{E})$ and for every $z \in \mathbf{E}$, $\psi(X \odot z) = \psi(X) \odot z$. It is straightforward to verify that the union and intersection of circular invariant operators are circular invariant. The following proposition shows that the circular erosion and dilation, defined in (8) and (9), are circular invariant.

Proposition 3. *Given a set $A \in \mathcal{P}(\mathbf{E})$, the circular erosion and the circular dilation, defined in (8) and (9), respectively, are circular invariant, that is, for every $z \in \mathbf{E}$, we have*

$$(X \odot z) \ominus_c A = (X \ominus_c A) \odot z, \qquad (X \in \mathcal{P}(\mathbf{E})), \qquad (22)$$

and

$$(X \odot z) \oplus_c A = (X \oplus_c A) \odot z, \qquad (X \in \mathcal{P}(\mathbf{E})). \qquad (23)$$

Therefore, from Theorem 1 and the kernel representation of group operators in [25, Theorem 5.35], every increasing and circular-invariant operator can be exactly represented as union of circular erosions or, equivalently, as intersection of circular dilations [28].

4.2.2 Affine Morphology [40]

Let $\mathbf{E} = \mathbb{R}^2$. Consider the set $\mathcal{G}$, defined in (10), and $\mathcal{S} \subset \mathcal{G}$. Define the affine transformation of a set $X \in \mathcal{P}(\mathbf{E})$ by the pair $(A, b) \in \mathcal{G}$ as the point by point affine transformation, that is, $AX + b = \{Ax + b : x \in X\}$. We say that an operator $\psi \in \mathcal{O}$ is affine invariant if and only if for every $(A, b) \in \mathcal{G}$, $\psi(AX + b) = A\psi(X) + b$. We show, using a counter example, that the affine erosion and dilation defined in (12) and (13), respectively, are not affine-invariant operators. Consider the set $\mathcal{S} = \{(I, t)\}$, where I is the identity matrix and $t \neq 0$. Then, from (12), we observe that, for every $A \neq I$, $(AX + b) \ominus_a \mathcal{S} = \{y - t : y \in (AX + b)\} = AX + b - t$, whereas $A(X \ominus_a \mathcal{S}) + b = A(X - t) + b = AX + b - At$. Hence, the affine erosion and affine dilation are not affine invariant. Nevertheless, Theorem 1 provides a sufficient condition for an operator to be represented as union of affine erosions, namely, if all the mappings in the kernel of an increasing operator ψ are of the form given by (11), then ψ can be exactly represented as union of affine erosions or, equivalently, as intersection of affine dilations.

5 BASIS REPRESENTATION

5.1 Motivation

The SV kernel representation, given in (20), is redundant, in the sense that a smaller subset of the kernel is sufficient for the representation of increasing operators. This can be seen, in the case of the representation by SV erosions, as follows: if θ_1 and $\theta_2 \in Ker(\psi)$ are such that $\theta_1 \leq \theta_2$, then $\mathcal{E}_{\theta_2} \subseteq \mathcal{E}_{\theta_1}$. Therefore, if the above θ_1 and θ_2 are contained in the kernel of an increasing operator ψ, its corresponding kernel representation will be redundant.

In the following proposition, we demonstrate that the kernel of an increasing operator is actually infinite.

Proposition 4. *Let $\psi \in \mathcal{O}$ be an increasing operator. Then, the kernel of ψ is infinite.*

In order to derive minimal representations for increasing operators, we need the notion of a basis of the kernel, which was first introduced by Maragos [37], [39] for translation-invariant operators.

Definition 4. *Let $\psi \in \mathcal{O}$ be an increasing operator. The basis $\mathcal{B}_\psi$ of $Ker(\psi)$ is the collection of minimal kernel mappings, formally defined as*

$$\mathcal{B}_\psi = \{\theta_M \in Ker(\psi) : \theta \in Ker(\psi) \text{ and } \theta \leq \theta_M \Longrightarrow \theta = \theta_M\}.$$

Observe that Definition 4 corresponds to the definition of a minimal basis. A more general definition of the basis as a subcollection of the kernel that is sufficient for representation can be found in [2]. If the basis of an increasing

operator exists, then the kernel representation of the operator reduces to a representation by the elements of the basis, which will allow in some cases a drastic reduction in the number of elements in the representation of the operator, as we will show in the examples.

Before proving that increasing and upper-semicontinuous operators have a basis representation, we briefly recall the definition of upper-semicontinuity in the hit-or-miss topology and study the topological properties of the SV basic morphological operators. For a comprehensive algebraic and topological background, we refer the reader to [4], [6], [37], [46], [53].

5.2 Upper-Semicontinuity in the Hit-or-Miss Topology

From now on, $\mathbf{E}$ is assumed to be a locally compact, Hausdorff and second countable topological space. We denote by $\mathcal{F}$ the set of all closed subsets of $\mathbf{E}$, by $\mathcal{G}$ the set of all open subsets of $\mathbf{E}$, and by $\mathcal{K}$ the set of all compact subsets of $\mathbf{E}$. Matheron defined a topology on $\mathcal{F}$ called the *hit-or-miss topology* [46]. We denote by $\mathcal{O}'$ the set of all operators mapping $\mathcal{F}$ into itself. From now on, we consider only mappings in $\mathcal{O}'$. In particular, the SV structuring element is now a mapping from $\mathbf{E}$ to $\mathcal{F}$.

A mapping ψ in $\mathcal{O}'$ is *upper-semicontinuous* if and only if for any $K \in \mathcal{K}$, the set $\psi^{-1}(\mathcal{F}^K)$ is open in $\mathcal{F}$ [46], where $\mathcal{F}^K$ is the class of the closed sets disjoint of K, that is, $\mathcal{F}^K = \{F : F \in \mathcal{F}, F \cap K = \emptyset\}$. A useful characterization of increasing upper-semicontinuous mappings in $\mathcal{F}$ is given by the following proposition due to Matheron:

Proposition 5 [46]. *Let ψ be an increasing mapping in $\mathcal{O}'$. ψ is upper-semicontinuous if and only if for every sequence $\{X_n\}_{n \in \mathbb{N}}$ of elements of $\mathcal{F}$ such that $X_n \downarrow X$ in $\mathcal{F}$ (that is, $X_1 \supseteq X_2 \supseteq \cdots \supseteq X_n \supseteq \cdots$ and $X = \bigcap_{n \geq 1} X_n$), we have $\psi(X_n) \downarrow \psi(X)$ in $\mathcal{F}$.*

Observe that continuity implies upper-semicontinuity but the converse is not true in general [46]. It is well known that the translation-invariant erosion of a closed set by a compact structuring element is upper-semicontinuous, and the translation-invariant dilation of a closed set by a compact structuring element is continuous [46], [54]. We generalize this result to the spatially-variant case. We say that the mapping θ is closed (respectively, compact) if $\theta(z)$ is a closed (respectively, compact) set, for every $z \in \mathbf{E}$. $\theta : \mathbf{E} \longrightarrow \mathcal{F}$ (respectively, $\mathcal{K}$) is continuous if and only if for every sequence $\{z_n\}_{n \in \mathbb{N}} \subset \mathbf{E}$ converging toward $z \in \mathbf{E}$, the sequence of sets $\{\theta(z_n)\}_{n \in \mathbb{N}}$ in $\mathcal{F}$ (respectively, $\mathcal{K}$) converges towards the set $\theta(z)$ in $\mathcal{F}$ (resp., $\mathcal{K}$) in the sense of that in [46, Theorem 1-2-2] (respectively, [46, Theorem 1-4-1]), and we write $\theta(x_n) \xrightarrow{\mathcal{F}} \theta(x)$ (respectively, $\theta(x_n) \xrightarrow{\mathcal{K}} \theta(x)$).

First, we prove that, under specific conditions on the SE mapping, the SV erosion and SV dilation are mappings from $\mathcal{F}$ into itself.

Proposition 6. *Consider the SE mapping θ.*

1. *If θ is continuous from $\mathbf{E}$ to $\mathcal{F}$, then $F \in \mathcal{F} \Rightarrow \mathcal{E}_\theta(F) \in \mathcal{F}$.*
2. *If θ' is continuous from $\mathbf{E}$ to $\mathcal{K}$, then $F \in \mathcal{F} \Rightarrow \mathcal{D}_\theta(F) \in \mathcal{F}$.*

Proposition 7.

1. *If the mapping θ is continuous from $\mathbf{E}$ to $\mathcal{F}$, then the spatially-variant erosion $\mathcal{E}_\theta$ is upper-semicontinuous from $\mathcal{F}$ to $\mathcal{F}$.*
2. *If the mappings θ' is continuous from $\mathbf{E}$ to $\mathcal{K}$, then the spatially-variant dilation is upper-semicontinuous from $\mathcal{F}$ to $\mathcal{F}$.*

For a general set mapping ψ, the property of upper-semicontinuity is not easily tractable. We provide the following easy test for upper-semicontinuity:

Proposition 8. *Let ψ_1 and ψ_2 be two increasing and upper-semicontinuous operators from $\mathcal{F}$ to $\mathcal{F}$. Then, their union $\psi = \psi_1 \cup \psi_2$ and their intersection $\psi' = \psi_1 \cap \psi_2$ are also increasing and upper-semicontinuous operators from $\mathcal{F}$ to $\mathcal{F}$.*

An obvious conclusion from Proposition 8 is that any finite union or intersection of increasing and upper-semicontinuous operators is also increasing and upper-semicontinuous. In particular, if a family of mappings $\{\theta_i\}_{i=1,\cdots,N}$ and its transpose $\{\theta'_i\}_{i=1,\cdots,N}$ are continuous from $\mathbf{E}$ to $\mathcal{K}$,[2] any finite union or intersection of SV erosions and SV dilations by mappings in $\{\theta_i\}_{i=1,\cdots,N}$ is an upper-semicontinuous increasing operator from $\mathcal{F}$ to $\mathcal{F}$.

5.3 Spatially-Variant Basis Representation
In order to prove that the SV kernel of an upper-semicontinuous increasing operator has a minimal element, we need the following lemma:

Lemma 1. *If $\mathcal{L}$ is a linearly ordered subset of $\mathcal{F}$, then there exists a sequence $\{X_n\}_{n\in\mathbb{N}}$ of elements of $\mathcal{L}$ such that $X_n \downarrow \bigcap \mathcal{L}$ for the hit-or-miss topology defined on $\mathcal{F}$.*

Theorem 2. *Let $\psi \in \mathcal{O}'$ be an increasing operator. If ψ is upper-semicontinuous, then the kernel of ψ has a minimal element.*

We now show that the minimal elements of the kernel are sufficient to represent the increasing and upper-semicontinuous operator ψ.

Theorem 3. *Let $\psi \in \mathcal{O}'$ be an upper-semicontinuous increasing operator. For every $\theta \in Ker(\psi)$, there exists a minimal element $\theta_M \in \mathcal{B}_\psi$ such that $\theta_M \leq \theta$.*

Finally, we provide the representation of an increasing upper-semicontinuous operator by its minimal elements.

Theorem 4. *Let $\psi \in \mathcal{O}'$ be an increasing upper-semicontinuous operator. Then, ψ is exactly represented as a union of spatially-variant erosions by mappings in its basis $\mathcal{B}_\psi$, that is,*

$$\psi(X) = \bigcup_{\theta\in\mathcal{B}_\psi} \mathcal{E}_\theta(X) \qquad (X \in \mathcal{F}). \qquad (24)$$

A minimal representation of an increasing upper-semicontinuous operator as an intersection of SV dilations is obtained by duality as follows:

Corollary 1. *If ψ is increasing from $\mathcal{G}$ to $\mathcal{G}$ and has an upper-semicontinuous dual ψ^* from $\mathcal{F}$ to $\mathcal{F}$, then ψ can be exactly represented as an intersection of spatially-variant dilations by the transposed mappings in the basis of its dual, that is,*

$$\psi(X) = \bigcap_{\theta\in\mathcal{B}_{\psi^*}} \mathcal{D}_{\theta'}(X) \qquad (X \in \mathcal{G}). \qquad (25)$$

In the discrete Euclidean space $\mathbb{Z}^n$, the set of open sets and closed sets are equivalent to the power set $\mathcal{P}(\mathbb{Z}^n)$. Therefore, every mapping ψ from $\mathcal{F}$ to $\mathcal{F}$ has a dual mapping ψ^* from $\mathcal{F}$ to $\mathcal{F}$. Hence, if ψ (respectively, ψ^*) is increasing and upper-semicontinuous, then the basis representation as union of spatially-variant erosions (respectively, intersection of spatially-variant dilations) exists.

5.4 Examples
SV erosion. Consider the SV erosion by the continuous SE mapping $\lambda : \mathbf{E} \rightarrow \mathcal{F}$. Then the smallest mapping in the kernel of the SV erosion is λ, that is, $\mathcal{B}_{\mathcal{E}_\lambda} = \{\lambda\}$.

SV dilation. Consider the SV dilation by the continuous SE mapping $\lambda : \mathbf{E} \rightarrow \mathcal{K}$. Then, the smallest mappings in the kernel of the SV dilation are the mappings that associate to each point $z \in \mathbf{E}$ a singleton $\{t_z\}$, where $t_z \in \lambda(z)$, that is,

$$\mathcal{B}_{\mathcal{D}_\lambda} = \{\theta : \theta(z) = \{t_z\}, \text{ for some } t_z \in \lambda(z), \forall z \in \mathbf{E}\}. \quad (26)$$

Thus, the SV erosion has only one basis set. If the cardinality of the mapping λ is finite, that is, $|\lambda(z)| \leq \vee_{z\in\mathbf{E}}|\lambda(z)| = n$, then, for each $z \in \mathbf{E}$, there are at most n mappings θ satisfying $\theta(z) = \{t_z\}$, for some $t_z \in \lambda(z)$. Define the support of the mapping λ as $\mathrm{Spt}(\lambda) = \{z \in \mathbf{E} : \lambda(z) \neq \emptyset\}$. If $\mathrm{Spt}(\lambda)$ is infinite, then there are an infinite number of mappings in the basis of the SV dilation even though λ has a finite cardinality. If, however, $\mathrm{Spt}(\lambda)$ is finite, then the basis of the SV dilation by the mapping λ is finite. In this case, let $N = |\mathrm{Spt}(\lambda)|$, then N^n is an upper bound for the number of elements in the basis of the SV dilation $\mathcal{D}_\lambda$.

Adaptive median filter. Consider $\mathbf{E} \subseteq \mathbb{Z}^2$. Let B be a mapping from $\mathbf{E}$ into $\mathcal{P}(\mathbf{E})$ such that $y \in B(y)$ and $|B(y)| = n = \text{is odd}, \forall y \in \mathbf{E}$. Let $r = \frac{n+1}{2}$. The adaptive (or spatially-variant) median, $med(X, B)$, of X with respect to the spatially-variant window B is given by

$$med(X, B) = \{y \in \mathbf{E} : |X \cap B(y)| \geq r\}. \qquad (27)$$

One can easily verify that the adaptive median is increasing and self dual. Therefore, from Theorem 1, the adaptive median has a kernel representation as union of spatially-variant erosions by mappings in its kernel or equivalently as intersection of spatially-variant dilations by the transposed mappings of its kernel. The kernel of the adaptive median filter is given by

$$Ker(med(., B)) = \{\theta : |\theta(z) \cap B(z)| \geq r, \forall z \in \mathbf{E}\}. \qquad (28)$$

Obviously, there are infinite number of mappings θ satisfying $|\theta(z) \cap B(z)| \geq r$, for all $z \in \mathbf{E}$. The following proposition shows that the adaptive median has a basis representation.

Proposition 9. *The adaptive median filter has $\binom{n}{r}$ mappings in its basis. They are given by*

$$\mathcal{B}_{med}(., B) = \{\theta : \theta \subseteq B \text{ and } |\theta| = r\}. \qquad (29)$$

Thus, from Theorem 4, the adaptive median filter has a basis representation given by

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2. If a mapping θ is continuous from $\mathbf{E}$ to $\mathcal{K}$, then θ is also continuous from $\mathbf{E}$ to $\mathcal{F}$ because every compact subset of a Hausdorff space is closed.

$$med(X, B) = \bigcup_{\theta \subseteq B, \; |\theta|=r} \; \bigcap_{x \in X^c} \theta'^c(x)$$
$$= \bigcap_{\theta \subseteq B, \; |\theta|=r} \; \bigcup_{x \in X} \theta(x). \tag{30}$$

We have thus a representation of the nonlinear adaptive median filter in terms of union and intersection of specified sets. In particular, no sorting is required. Observe that we did not prove the upper-semicontinuity of the adaptive median filter in order to find its basis. Instead, we found a finite basis that is not redundant. The upper-semicontinuity is only a sufficient condition for the existence of the basis. The question whether it is also a necessary condition or not remains still an open problem.

Spatially-variant hit-or-miss transform. The spatially-variant hit-or-miss transform. $\otimes(\theta_1, \theta_2)$ is given by

$$X \otimes (\theta_1, \theta_2) = \{z \in \xi : \theta_1(z) \subseteq X \subseteq \theta_2(z)\} \qquad (X \in \mathcal{P}(\xi)). \tag{31}$$

Consider mappings θ_1 and θ_2 from $\mathbf{E}$ into $\mathcal{P}(\mathbf{E})$. Let us use $[\theta_1, \theta_2]$ to denote the mapping segment given by $[\theta_1, \theta_2] = \{\theta : \theta_1 \subseteq \theta \subseteq \theta_2\}$. In the following theorem, we provide the spatially-variant kernel representation of operators in $\mathcal{O}$ (not necessarily increasing) based on spatially-variant hit-or-miss transforms.

Theorem 5. *Given a mapping $\psi \in \mathcal{O}$, we have*

$$\psi(X) = \bigcup_{[\theta_1, \theta_2] \subseteq Ker(\psi)} (X \otimes (\theta_1, \theta_2)) \qquad (X \in \mathcal{P}(\xi)). \tag{32}$$

The representation in Theorem 5 is redundant. To see this, let $[\theta_1, \theta_2]$ and $[\theta_3, \theta_4]$ be two segments such that $[\theta_1, \theta_2] \subseteq [\theta_3, \theta_4]$, that is, $\theta_3 \leq \theta_1 \leq \theta_2 \leq \theta_4$. Then, $X \otimes [\theta_1, \theta_2] \subseteq X \otimes [\theta_3, \theta_4]$, for every $X \in \mathcal{P}(\mathbf{E})$. Therefore, in the representation of an operator by SV hit-or-miss transforms, if the above segments $[\theta_1, \theta_2]$ and $[\theta_3, \theta_4]$ are contained in $Ker(\psi)$, the mapping $\otimes(\theta_1, \theta_2)$ will be redundant.

The basis of ψ, in this representation, is defined as the set of all the maximal intervals contained in $Ker(\psi)$. An interval is maximal if no other interval contained in $Ker(\psi)$ properly contains it. Bannon and Barrera [2] have a similar definition of the basis in the special case of translation-invariant operators. The extension of the derivation of the existence of the minimal basis to the SV case can be carried out based on the development in [2] and is out of the scope of this paper. One can verify that under the same sufficient condition of upper-semicontinuity, an operator, the domain of which is the collection of closed subsets of an Euclidean space, has a minimal kernel representation in terms of SV hit-or-miss transforms.

6   SIMULATIONS

The simulation results presented in this section are intended to illustrate the general theory of SV mathematical morphology and show its power in image analysis applications. For more practical examples of specific application-oriented spatially-variant structuring elements, we refer the reader to the references cited in Section 1.
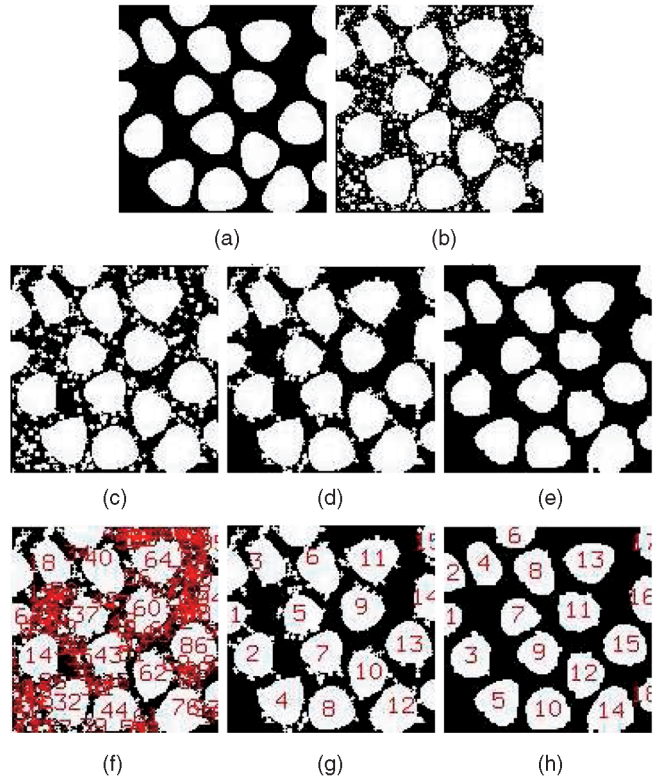

Fig. 2. Translation-invariant and spatially-variant opening by reconstruction and segmentation. (a) Original image. (b) Corrupted image by a germ grain noise model. (c) Opening by reconstruction using the rhombus SE (four connectivity) for the erosion and the reconstruction. (d) Opening by reconstruction using the rhombus SE dilated three times for the erosion and the rhombus SE for the reconstruction. (e) Spatially-variant opening by reconstruction. (f), (g), and (h) Segmentation of the reconstructed images (c), (d), and (e), respectively.

6.1   Spatially-Variant Opening by Reconstruction and Segmentation

Morphological filters by reconstruction [13], [55], [59] use an image, called a "marker" (usually a filtered version of the original image) to reconstruct objects or features of interest in the original image called the "mask." Thus, filters by reconstruction can be used for segmentation. However, a deceptive problem arises when dealing with noisy images: The reconstruction process reconnects portions of the noise along with the useful information in the image. This problem is intolerable in some applications such as contour extraction from noisy 2D and 3D medical images. We illustrate the morphological opening by a reconstruction scheme in noisy environments using the binary image of blobs in Fig. 2a and its corrupted version by a germ-grain noise model [53] in Fig. 2b. In morphological opening by reconstruction, the eroded image is used as the marker image. Then, the reconstruction process restores, through iterative dilations, the blobs that have not been totally removed by the erosion. Fig. 2c shows the output of the translation-invariant opening by reconstruction using the rhombus SE. Since the erosion by the rhombus SE is unable to eliminate the structures of noise bigger than the rhombus, most of the noise is reconstructed along with the original blobs. So, in a second experiment, we eroded the noisy image by the rhombus SE dilated three times to eliminate all the noise in the marker image. Nevertheless, the reconstructed image is still noisy, as shown in Fig. 2d. The
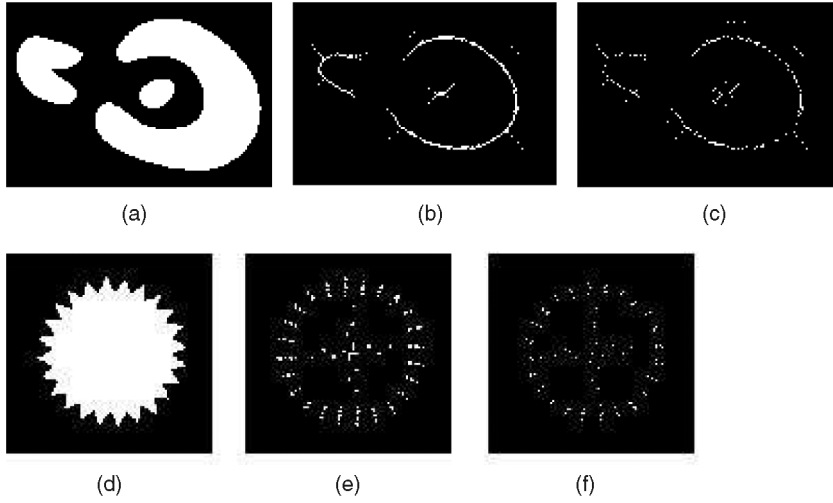
Fig. 3. Translation-invariant and spatially-variant morphological skeleton representation: (a) Original blob image. (b) Translation-invariant morphological skeleton representation using the rhombus SE. (c) Spatially-variant morphological skeleton representation. (d) Original gear image. (e) Translation-invariant morphological skeleton representation using the 3×3 square SE. (f) Spatially-variant morphological skeleton representation.

noise persists in the reconstructed image because it is connected to the original blobs and, hence, is viewed by the reconstruction algorithm as part of the blobs. A simple solution to avoid the noise reconstruction problem is to spatially vary the structuring element while eroding to form the marker image. The strategy adopted here is quite intuitive and can be summarized as follows:

1. At each pixel z of the image, decide by exploring its neighborhood whether it belongs to a noise grain or not (the germ-grain noise model is assumed to be known, a priori). The detection of the presence of a noise grain $C(z)$ centered at the pixel z is determined by selecting the largest possible grain C, which is present or absent in the degraded image Y (that is, $C + \{z\} \subseteq Y$ or $C + \{z\} \subseteq Y^c$). The SE mapping of the SV erosion is then selected as follows:

$$\theta(z) = \begin{cases} C(z) \oplus S, & \text{if } z \text{ is detected as a noisy pixel;} \\ S, & \text{otherwise,} \end{cases}$$

(33)

where S denotes the rhombus structuring element. This choice of the SE mapping ensures that all noise grains are removed completely (since the local SE is larger than the size of the noise grain) while preserving the small main blobs in the image (which have sizes bigger than the rhombus). The marker image is then obtained by SV erosion of the noisy image or the mask.

2. Label the pixels in the mask image as follows: If a pixel was detected as noisy in Step 1), label it differently from the main blobs (even if it is connected to a main blob). Each main blob is assigned a unique number.
3. Determine the labels that contain at least a pixel of the marker image.
4. The reconstructed image is obtained by removing all the pixels whose label is not one of the previous ones.

The result of SV opening by reconstruction is displayed in Fig. 2e. The noise is not reconstructed with the original blobs. This is important not only for denoising but also for

segmentation. A persisting noise in the reconstructed image has deleterious consequences for segmentation as it is either classified as main blobs (see Fig. 2f) or merges originally disconnected blobs (see Fig. 2g) and, in both cases, results in erroneous segmentation and blob detection. The third row in Fig. 2 displays the segmentation results. For visual display, we labeled each segmented blob by a number. The opening by reconstruction using the rhombus SE (respectively, the rhombus SE dilated three times) detected 104 different blobs (respectively, 15 blobs), whereas the SV opening by reconstruction resulted in the correct segmentation of the original 18 blobs.

6.2 Spatially-Variant Morphological Skeleton Representation

The translation-invariant morphological skeleton representation [35], [53] is known to be redundant, in the sense that a smaller subset of the morphological skeleton is sufficient for perfect reconstruction of the original image [7], [33], [38]. It has been shown that efficient encoding of the skeleton representation using run-length type codes can be used to provide an efficient compression routine for binary images [34], [42]. For practical purposes, we can assume that the coding efficiency of the skeleton depends only on the number of its points [42]. Therefore, minimizing the cardinality of the morphological skeleton representation, under the constraint of exact reconstruction, is crucial for efficient compression of binary images. In [7], we extended the morphological skeleton representation framework presented in [51] to the spatially-variant case. The theoretical properties of the SV morphological skeleton representation and the conditions for its invertibility were investigated in [7]. Fig. 3 shows the results of SV morphological skeleton representation compared to its translation-invariant counterpart for two binary images. The SV morphological skeleton representation has a cardinality that is less than 60 percent of the cardinality of its translation-invariant counterpart. Given an initial SE, B, the algorithm iteratively selects the center of the dilated structuring element, $nB = B \oplus B \cdots \oplus B$ (n times), which maximally intersects the image, for some integer n. The union

of these center points forms the SV morphological skeleton representation. The exact reconstruction of the original image is guaranteed given the SE B, the set of center points and their corresponding integer n.

7 CONCLUSION

In this paper, we presented a general theory of spatially-variant morphology for binary images in the Euclidean space. This theory provides a unified mathematical framework of numerous morphological schemes that have been employed by researchers in various image processing applications, for example, circular morphology [28] and affine morphology [40], as well as basic notions that have been introduced in several practical applications such as traffic measurements [5] and adaptive filtering [15], [36]. Our presentation has been confined to the Euclidean space but can be trivially extended to complete atomic Boolean lattices in the context of lattice morphology. A further generalization of the framework provided in this presentation to general lattices appears in [3]. However, such a generalization is very abstract and fails to capture the geometrical interpretation conveyed by the structuring element, which is crucial in signal and image processing applications.

In particular, in this paper, a kernel representation of increasing operators has been presented in terms of spatially-variant erosions and dilations. This result extends the theory of translation-invariant mathematical morphology introduced by Matheron. We have also extended our approach to the kernel representation of arbitrary (not necessarily increasing) SV operators based on a decomposition into SV hit-or-miss transforms. The SV kernel representation, like its translation-invariant counterpart, is redundant and consequently has limited practical importance. Following Maragos' development of the basis representation for increasing and translation-invariant operators, we proved that every increasing and upper-semicontinuous operator admits a basis representation. The domain of such an operator is restricted to a collection of closed subsets of the space. The basis representation allows, in some cases, a drastic reduction of the number of elements in the kernel as was shown in the adaptive median filter example. Finally, simulation results show the potential power of the general theory of spatially-variant mathematical morphology in several image processing and computer vision applications.

APPENDIX A

PROPERTIES OF SV EROSION AND SV DILATION

Adjunction. For every structuring element mapping θ, the pair $(\mathcal{E}_\theta, \mathcal{D}_\theta)$ forms an adjunction, that is,

$$\mathcal{D}_\theta(X) \subseteq Y \Longleftrightarrow X \subseteq \mathcal{E}_\theta(Y) \qquad (X, Y \in \mathcal{P}(\mathbf{E})).$$

Duality. The SV erosion $\mathcal{E}_\theta$ and SV dilation $\mathcal{D}_\theta$ are dual operators, that is,

$$\mathcal{E}_\theta^*(X) = \mathcal{D}_{\theta'}(X) \qquad (X \in \mathcal{P}(\mathbf{E})).$$

Increasingness. For every structuring element mapping θ, the SV erosion $\mathcal{E}_\theta$ and SV dilation $\mathcal{D}_\theta$ are increasing operators from $\mathcal{P}(\mathbf{E})$ to $\mathcal{P}(\mathbf{E})$.

Extensivity and anti-extensivity. If $z \in \theta(z)$, for all $z \in \mathbf{E}$, then the SV erosion $\mathcal{E}_\theta$ is anti-extensive, and the SV dilation $\mathcal{D}_\theta$ is extensive, that is,

$$\mathcal{E}_\theta(X) \subseteq X, \quad \text{and} \quad X \subseteq \mathcal{D}_\theta(X) \qquad (X \in \mathcal{P}(\mathbf{E})).$$

Observe that in the translation-invariant case with fixed SE B, that is, $\theta(z) = B + z$, the condition $z \in \theta(z)$ reduces to $0 \in B$.

Scaling with respect to the SE mapping. If $\theta_1 \subseteq \theta_2$, then

$$\mathcal{E}_{\theta_2}(X) \subseteq \mathcal{E}_{\theta_1}(X), \quad \text{and} \quad \mathcal{D}_{\theta_1}(X) \subseteq \mathcal{D}_{\theta_2}(X) \quad (X \in \mathcal{P}(\mathbf{E})).$$

Serial composition. Consider the mappings θ_1 and θ_2 from $\mathbf{E}$ into $\mathcal{P}(\mathbf{E})$. Let us use $\mathcal{E}_{\theta_1}(\theta_2)$ and $\mathcal{D}_{\theta_1}(\theta_2)$ to denote the mappings from $\mathbf{E}$ into $\mathcal{P}(\mathbf{E})$ given by $(\mathcal{E}_{\theta_1}(\theta_2))(z) = \mathcal{E}_{\theta_1}(\theta_2(z))$ and $(\mathcal{D}_{\theta_1}(\theta_2))(z) = \mathcal{D}_{\theta_1}(\theta_2(z))$, for every $z \in \mathbf{E}$. Then, we have

$$\mathcal{E}_{\theta_2}(\mathcal{E}_{\theta_1}) = \mathcal{E}_{\mathcal{D}_{\theta_1}(\theta_2)}, \text{ and } \mathcal{D}_{\theta_2}(\mathcal{D}_{\theta_1}) = \mathcal{D}_{\mathcal{D}_{\theta_2}(\theta_1)},$$

for every $X \in \mathcal{P}(\mathbf{E})$.

APPENDIX B

PROOF OF LEMMAS AND COROLLARIES

Proof of Lemma 1. Let $\mathcal{L}$ be a linearly ordered subset of $\mathcal{F}$. From [2, Lemma 4.1], $\bigcap \mathcal{L}$ is adherent to $\mathcal{L}$. Since $\mathcal{F}$ is second countable, there exists a sequence $\{X_n\}_{n \in \mathbb{N}}$ of the elements of $\mathcal{L}$ converging to $\bigcap \mathcal{L}$. The sequence $\{X_n\}_{n \in \mathbb{N}}$ is itself a linearly ordered subset of $\mathcal{L}$. Therefore, in [2, Lemma 4.1], $\bigcap_{n \in \mathbb{N}} X_n$ is adherent to the sequence $\{X_n\}_{n \in \mathbb{N}}$. The converging sequence $\{X_n\}_{n \in \mathbb{N}}$ has a unique adherent point and, therefore, $\bigcap \mathcal{L} = \bigcap_{n \in \mathbb{N}} X_n$. From the fact that the sequence $\{X_n\}_{n \in \mathbb{N}}$ is itself a linearly ordered subset of $\mathcal{L}$, we conclude that $X_n \downarrow \bigcap \mathcal{L}$.   □

Proof of Corollary 1. Consider an increasing operator ψ from $\mathcal{G}$ to $\mathcal{G}$. Then, the dual mapping ψ^* is also increasing and maps $\mathcal{F}$ to $\mathcal{F}$. Since ψ^* is upper-semicontinuous, it admits a basis representation as union of SV erosions (Theorem 4). The result follows then easily by duality. □

APPENDIX C

PROOF OF PROPOSITIONS

Proof of Proposition 1. Consider the mapping θ_X from $\mathbf{E}$ into $\mathcal{P}(\mathbf{E})$ given by

$$\theta_X(z) = \begin{cases} X, & z \in \psi(X); \\ \mathbf{E}, & z \notin \psi(X), \end{cases} \tag{34}$$

for some $X \in \mathcal{P}(\mathbf{E})$. Then, we have for any nondegenerate $\psi \in \mathcal{O}$

$$\psi(\theta_X(z)) = \begin{cases} \psi(X), & z \in \psi(X); \\ \mathbf{E}, & z \notin \psi(X). \end{cases} \tag{35}$$

Therefore, we observe that $z \in \psi(\theta_X(z))$, for every $z \in \mathbf{E}$. Therefore, from Definition 3, $\theta_X \in Ker(\psi)$.   □

Proof of Proposition 2. Assume that $\psi_1(X) \subseteq \psi_2(X)$ $(X \in \mathcal{P}(\mathbf{E}))$. From Definition 3, we observe that

$$\theta \in ker\ (\psi_1) \Longrightarrow z \in \psi_1(\theta(z)), \forall\ z \in \mathbf{E}$$
$$\Longrightarrow z \in \psi_2(\theta(z)), \forall\ z \in \mathbf{E}$$
$$\Longrightarrow \theta \in ker\ (\psi_2).$$

Therefore, we notice that $ker(\psi_1) \subseteq ker(\psi_2)$.

Assume now that $ker(\psi_1) \subseteq ker(\psi_2)$. Let us consider the mapping θ_X from $\mathbf{E}$ into $\mathcal{P}(\mathbf{E})$ given by (34) (with $\psi \longrightarrow \psi_1$). From (35), we observe that $z \in \psi_1(\theta_X(z))$, for every $z \in \mathbf{E}$. From Definition 3, we have

$$z \in \psi_1(\theta_X(z)), \forall\ z \in \mathbf{E} \Longrightarrow \theta_X \in ker\ (\psi_1)$$
$$\Longrightarrow \theta_X \in ker\ (\psi_2)$$
$$\Longrightarrow z \in \psi_2(\theta_X(z)), \forall\ z \in \mathbf{E}.$$

Let us consider $z \in \psi_1(X)$. From (34), we observe that $\theta_X(z) = X$. Therefore, $z \in \psi_2(\theta_X(z)) = \psi_2(X)$. Therefore, $\psi_1(X) \subseteq \psi_2(X)$, for every $X \in \mathbf{E}$. This completes the proof. $\square$

Proof of Proposition 3.

$$(X \ominus_c A) \odot z = \{y : (A \odot y) \subseteq X\} \odot z$$
$$= \{y \odot z : (A \odot y) \subseteq X\}$$
$$= \left\{y : \left[A \odot \left(y \odot \frac{1}{z}\right)\right] \subseteq X\right\}$$
$$= \left\{y : \left[(A \odot y) \odot \frac{1}{z}\right] \subseteq X\right\}$$
$$= \{y : (A \odot y) \subseteq (X \odot z)\}$$
$$= (X \odot z) \ominus_c A.$$

Therefore, we proved (22). A similar argument can be used to prove (23). $\square$

Proof of Proposition 4.
Let $\psi \in \mathcal{O}$ be an increasing operator. Consider the mapping θ_X defined in (34). $\theta_X \in Ker(\psi)$. Since ψ is increasing, we observe that every mapping θ from $\mathbf{E}$ into $\mathcal{P}(\mathbf{E})$ such that $\theta_X \leq \theta$ is also in the kernel of ψ. This proves that $Ker(\psi)$ is infinite. $\square$

Proof of Proposition 6.

1. Consider $F \in \mathcal{F}$ and let $\{x_n\}_{n \in \mathbb{N}} \subset \mathcal{E}_\theta(F)$ be a sequence converging toward $x \in \mathbf{E}$. By definition of the SV erosion, we have $\theta(x_n) \subseteq F$, $\forall n \in \mathbb{N}$. Since the mapping θ is continuous, $\theta(x_n) \xrightarrow{\mathcal{F}} \theta(x)$. In [46, Corollary 3(e)], we obtain $\theta(x) \subseteq F$, which is equivalent to $x \in \mathcal{E}_\theta(F)$. Therefore, every convergent sequence in $\mathcal{E}_\theta(F)$ has its limit point in $\mathcal{E}_\theta(F)$. Hence, $\mathcal{E}_\theta(F)$ is closed.

2. Consider $F \in \mathcal{F}$ and let $\{x_n\}_{n \in \mathbb{N}} \subset \mathcal{D}_\theta(F)$ be a sequence converging toward $x \in \mathbf{E}$. By definition of the SV dilation, we have $\theta'(x_n) \cap F \neq \emptyset$, $\forall n \in \mathbb{N}$. So, for each n, there exists $f_n \in F$ such that $f_n \in \theta'(x_n)$. Since θ' is continuous and compact, we have $\theta'(x_n) \xrightarrow{\mathcal{K}} \theta'(x)$. In [46, Theorem1-4-1], there exists $K_0 \in \mathcal{K}$ such that $\theta'(x_n) \subseteq K_0$, $\forall n$, and thus, $f_n \in K_0$. By compactness of K_0, there exists a convergent subsequence $f_{n_k} \to f \in F$ (since F is closed). By Condition 2 in [46, Theorem1-2-2], $f_{n_k} \in F$. Also, since f_{n_k} is a convergent subsequence in $\theta'(x_{n_k})$, $f \in \theta'(x)$. Hence, $f \in \theta'(x) \cap F$, which is

equivalent to $x \in \mathcal{D}_\theta(F)$. Therefore, every convergent sequence in $\mathcal{D}_\theta(F)$ has its limit point in $\mathcal{D}_\theta(F)$. Hence, $\mathcal{D}_\theta(F)$ is closed. $\square$

Proof of Proposition 7.

1. Let $\{F_n\}_{n \in \mathbb{N}}$ be a sequence in $\mathcal{F}$ converging toward $F \in \mathcal{F}$. Let $x \in \overline{\lim}\mathcal{E}_\theta(F_n)$. Then, by that in [46, Proposition 1-2-3 (b)], there exists a subsequence $x_{n_p} \in \mathcal{E}_\theta(F_{n_p})$ converging toward x. By definition of the SV erosion, $\theta(x_{n_p}) \subseteq F_{n_p}$. By continuity of the mapping θ, we have $\theta(x_{n_p}) \xrightarrow{\mathcal{F}} \theta(x)$. Using that in [46, Corollary 3(e)], we have $\theta(x) \subseteq F$, which is equivalent to $x \in \mathcal{E}_\theta(F)$. Therefore, every point $x \in \overline{\lim}\mathcal{E}_\theta(F_n)$ belongs also to $\mathcal{E}_\theta(F)$. Hence, $\overline{\lim}(\mathcal{E}_\theta(F_n)) \subseteq \mathcal{E}_\theta(F)$. By that in [46, Proposition 1-2-4 (a)], we conclude that $\mathcal{E}_\theta$ is upper-semicontinuous.

2. Let $\{F_n\}_{n \in \mathbb{N}}$ be a sequence in $\mathcal{F}$ converging toward $F \in \mathcal{F}$. Let $x \in \overline{\lim}\mathcal{D}_\theta(F_n)$. Then, by that in [46, Proposition 1-2-3(b)], there exists a subsequence $x_{n_p} \in \mathcal{D}_\theta(F_{n_p})$ converging toward x. Therefore, there exists $f_{n_p} \in F_{n_p}$ such that $f_{n_p} \in \theta'(x_{n_p})$. Since θ' is continuous and compact, we have $\theta'(x_{n_p}) \xrightarrow{\mathcal{K}} \theta'(x)$ and by that in [46, Theorem 1-4-1], there exists $K_0 \in \mathcal{K}$ such that $K_0 \supset \theta'(x_{n_p})$, $\forall n_p$. In particular, $f_{n_p} \in K_0$. By the compactness of K_0, the sequence $\{f_{n_p}\}$ admits a convergent subsequence toward a point $f \in F$. Without loss of generality, we can assume that the latter subsequence is f_{n_p}. By Criterion 2 in [46, Theorem1-2-2], $f \in \theta'(x)$. Hence, we have $f \in \theta'(x) \cap F$, which is equivalent to $x \in \mathcal{D}_\theta(F)$. Therefore, every point $x \in \overline{\lim}\mathcal{D}_\theta(F_n)$ belongs also to $\mathcal{D}_\theta(F)$. Hence, $\overline{\lim}(\mathcal{D}_\theta(F_n)) \subseteq \mathcal{D}_\theta(F)$. By that in [46, Proposition 1-2-4(a)], we conclude that $\mathcal{D}_\theta$ is upper-semicontinuous. $\square$

Proof of Proposition 8.

1. Consider first the union of two increasing and upper-semicontinuous operators, $\psi = \psi_1 \cup \psi_2$. We have, for X, $Y \in \mathcal{F}$ such that $X \subseteq Y$, $\psi(X) = \psi_1(X) \cup \psi_2(X) \subseteq \psi_1(Y) \cup \psi_2(Y) = \psi(Y)$. Thus, ψ is increasing. To prove the upper-semicontinuity of ψ, we use Proposition 5. Consider, then, a sequence $\{X_n\}_{n \in \mathbb{N}} \in \mathcal{F}$ such that $X_n \downarrow X$. We have, for every n, $X_n \supseteq X_{n+1} \supseteq X$. Therefore, since ψ is increasing, $\psi(X_n) \supseteq \psi(X_{n+1}) \supseteq \psi(X)$. Thus, the sequence $\{\psi(X_n)\}$ is decreasing. In particular, we have $\psi(X_n) \supseteq \psi(X)$, for every $n \in \mathbb{N}$. Thus,

$$\bigcap_{n \in \mathbb{N}} \psi(X_n) \supseteq \psi(X). \tag{36}$$

Consider now $x \in \bigcap_n \psi(X_n)$. That is, $x \in \bigcap_n (\psi_1(X_n) \cup \psi_2(X_n))$. Therefore, $x \in \psi_1(X_n) \cup \psi_2(X_n)$, for every n. We distinguish two cases:

Case 1. $x \in \psi_1(X_n)$ (or $\psi_2(X_n)$) only for a finite number of integers n. Since the problem is symmetric in ψ_1 and ψ_2, we can assume, without loss of generality, that $x \in \psi_1(X_n)$ only for a finite number of integers n. That is, there exists an integer N_0 such that for every $n \geq N_0$, $x \notin \psi_1(X_n)$. Thus, we have for every $n \geq N_0$, $x \in \psi_2(X_n)$.

Therefore, $x \in \bigcap_{n \geq N_0} \psi_2(X_n) = \bigcap_{n \in \mathbb{N}} \psi_2(X_n) = \psi_2(X)$, where the first equality follows from the fact that ψ_2 is increasing and the second equality follows from the upper-semicontinuity of ψ_2. Thus, $x \in \psi_2(X) \cup \psi_1(X) = \psi(X)$.

Case 2. $x \in \psi_1(X_n)$ and $\psi_2(X_n)$ for an infinite (and countable) number of integers n. That is, for every N_0, we have $x \in \psi_1(X_n) \cap \psi_2(X_n)$, for some $n \geq N_0$. Due to the monotonicity property of ψ_1 and ψ_2, we also have that $x \in \psi_1(X_n) \cap \psi_2(X_n)$, for every $n \leq N_0$. Hence, $x \in \psi_1(X_n) \cap \psi_2(X_n)$, for every $n \in \mathbb{N}$. Thus, $x \in \bigcap_n (\psi_1(X_n) \cap \psi_2(X_n)) = (\bigcap_n \psi_1(X_n)) \bigcap (\bigcap_n \psi_2(X_n)) = \psi_1(X) \cap \psi_2(X)$. In particular, $x \in \psi_1(X) \cup \psi_2(X) = \psi(X)$. Finally, we have

$$\psi(X) \supseteq \bigcap_{n \in \mathbb{N}} \psi(X_n). \tag{37}$$

Equations (36) and (37) and the increasing property of ψ are equivalent to $\psi(X_n) \downarrow \psi(X)$. Therefore, ψ is upper-semicontinuous.

2. Consider now the intersection of two increasing and upper-semicontinuous operators, $\psi' = \psi_1 \cap \psi_2$. We have, for $X, Y \in \mathcal{F}$ such that $X \subseteq Y$, $\psi'(X) = \psi_1(X) \cap \psi_2(X) \subseteq \psi_1(Y) \cap \psi_2(Y) = \psi'(Y)$. Thus, ψ' is increasing. Consider now a sequence $\{X_n\}_{n \in \mathbb{N}} \in \mathcal{F}$ such that $X_n \downarrow X$. Since ψ' is increasing, the sequence $\{\psi'(X_n)\}_{n \in \mathbb{N}}$ is decreasing. We have

$$x \in \bigcap_n \psi'(X_n) \Longleftrightarrow x \in \bigcap_n (\psi_1(X_n) \cap \psi_2(X_n))$$
$$\Longleftrightarrow x \in \left(\bigcap_n \psi_1(X_n)\right) \cap \left(\bigcap_n \psi_2(X_n)\right)$$
$$\Longleftrightarrow x \in \psi_1(X) \cap \psi_2(X) \Longleftrightarrow x \in \psi'(X).$$

Therefore, $\psi'(X_n) \downarrow \psi'(X)$. Therefore, ψ' is upper-semicontinuous $\square$

Proof of Proposition 9. Let $\mathcal{A}$ be the set of the $\binom{n}{r}$ subsets of B containing exactly r points. We denote by ker the kernel of the adaptive median filter. For every $\theta \in Ker$, there exists $\lambda \in \mathcal{A}$ such that $\lambda \leq \theta$. Therefore, by property $k)$ in Appendix A, $\mathcal{E}_\theta \subseteq \mathcal{E}_\lambda$. Thus, $\bigcup_{\theta \in Ker} \mathcal{E}_\theta \subseteq \bigcup_{\lambda \in \mathcal{A}} \mathcal{E}_\lambda$. Conversely, we have $\mathcal{A} \subseteq Ker$. Therefore, $\bigcup_{\lambda \in \mathcal{A}} \mathcal{E}_\lambda \subseteq \bigcup_{\theta \in Ker} \mathcal{E}_\theta$. Therefore, $\bigcup_{\lambda \in \mathcal{A}} \mathcal{E}_\lambda = \bigcup_{\theta \in Ker} \mathcal{E}_\theta$. We claim that the basis of the kernel of the adaptive median filter is $\mathcal{A}$. Otherwise, there exists $\theta \in Ker$ such that $\theta < \lambda$, for every $\lambda \in \mathcal{A}$. In particular, $|\theta| < r$. This contradicts the fact that $\theta \in ker$ and establishes $\mathcal{A}$ as the basis of the adaptive median operator. $\square$

APPENDIX D

PROOF OF THEOREMS

Proof of Theorem 1. Assume that $\psi(X) = \bigcup_{\theta \in Ker(\psi)} \mathcal{E}_\theta(X)$, for every $X \in \mathcal{P}(\mathbf{E})$. ψ is then increasing as union of increasing operators. Assume now that ψ is increasing and consider $X \in \mathcal{P}(\mathbf{E})$. Let $z \in \psi(X)$. Let us also consider the mapping θ_X from $\mathbf{E}$ into $\mathcal{P}(\mathbf{E})$ given by (34). $\theta_X \in ker(\psi)$. Therefore, $\theta_X(z) = X$. Hence, by

definition of the SV erosion, $z \in \mathcal{E}_{\theta_X}(X) \subseteq \bigcup_{\theta \in Ker(\psi)} \mathcal{E}_\theta(X)$. Therefore,

$$\psi(X) \subseteq \bigcup_{\theta \in Ker(\psi)} \mathcal{E}_\theta(X). \tag{38}$$

Let us now consider $z \in \bigcup_{\theta \in Ker(\psi)} \mathcal{E}_\theta(X)$. Therefore, there exists $\theta \in Ker(\psi)$ such that $z \in \mathcal{E}_\theta(X)$. From the definition of the SV erosion, this is equivalent to the existence of $\theta \in Ker(\psi)$ such that $\theta(z) \subseteq X$. Since, $\theta \in Ker(\psi)$ and ψ is increasing, we observe that $z \in \psi(\theta(z)) \subseteq \psi(X)$. Finally, we have shown that

$$\bigcup_{\theta \in Ker(\psi)} \mathcal{E}_\theta(X) \subseteq \psi(X). \tag{39}$$

From (38) and (39), we obtain the desired kernel representation of the increasing operator ψ. Since the class of increasing operators is closed under duality, the representation of an increasing operator as an intersection of SV dilations by the transposed mappings in the kernel of its dual is easily obtained by duality from its representation as a union of SV erosions by mappings in the kernel. $\square$

Proof of Theorem 2. Let $\psi \in \mathcal{O}'$ be an increasing and upper-semicontinuous operator. From Proposition 4, the kernel of ψ is a nonempty partially ordered set. Consider a linearly ordered subset $\mathcal{L}$ of $Ker(\psi)$. For every $z \in \mathbf{E}$, $\mathcal{L}_z = \{\theta(z) : \theta \in \mathcal{L}\}$ is a linearly ordered subset of $\mathcal{F}$. From Lemma 1, there exists a sequence $\{\theta_n(z) : n \in \mathbb{N}, \theta_n \in \mathcal{L}\}$ such that $\theta_n(z) \downarrow \bigcap_{n \in \mathbb{N}} \theta_n(z) = \bigcap \mathcal{L}_z$. From the fact that ψ is an increasing upper-semicontinuous operator, we have

$$\psi(\theta_n(z)) \downarrow \psi(\bigcap \mathcal{L}_z) \qquad (z \in \mathbf{E}). \tag{40}$$

By the uniqueness of limit points in Hausdorff spaces, we obtain

$$\psi(\bigcap \mathcal{L}_z) = \bigcap_{n \in \mathbb{N}} \psi(\theta_n(z)) \qquad (z \in \mathbf{E}). \tag{41}$$

From the above equation and using the fact that $\theta_n \in Ker(\psi)$, $\forall n \in \mathbb{N}$, we observe that the operator from $\mathbf{E}$ into $\mathcal{F}$ defined by

$$(\wedge \mathcal{L})(z) = \bigcap \mathcal{L}_z \qquad (z \in \mathbf{E}) \tag{42}$$

is an element of $Ker(\psi)$, where $\wedge \mathcal{L}$ is the infimum of the linearly ordered subset $\mathcal{L}$. Finally, we have showed that every linearly ordered subset of $Ker(\psi)$ has an infimum in $Ker(\psi)$. Using Zorn's Lemma [32], we conclude that the kernel of ψ has a minimal element. $\square$

Proof of Theorem 3. Let $\psi \in \mathcal{O}'$ be an increasing upper-semicontinuous operator, and let $\theta_A \in Ker(\psi)$. Then, there exists $\theta_B \in Ker(\psi)$ such that $\theta_B \leq \theta_A$; otherwise, θ_A is a minimal element. Therefore, for every $\theta_A \in Ker(\psi)$, we can construct a decreasing family $\mathcal{L}$ of $Ker(\psi)$ containing θ_A. From the fact that $\mathcal{L}$ is a linearly ordered subset of $Ker(\psi)$ and from Hausdorff's Maximality Principle [37], there exists a maximal linearly ordered subset $\mathcal{M}$ of $Ker(\psi)$ containing $\mathcal{L}$. Let $\theta_M(z) = (\wedge \mathcal{M})(z) = \bigcap_{\theta \in \mathcal{M}} \theta(z)$, for every $z \in \mathbf{E}$. From the proof of Theorem 2, $\theta_M \in Ker(\psi)$. We have $\theta_M \leq \bigcap \mathcal{L} \leq \theta_A$. Therefore, θ_M is a

minimal element of $Ker(\psi)$; otherwise, there exists $\theta_Y \in Ker(\psi)$ such that $\theta_Y \leq \theta_M$; the subset $\mathcal{M} \cup \{\theta_Y\}$ is then a linearly ordered subset of $Ker(\psi)$ containing $\mathcal{M}$; this contradicts the maximality of $\mathcal{M}$. Finally, we have shown that θ_M is a minimal element and $\theta_M \leq \theta_A$. This completes the proof. □

Proof of Theorem 4. Let $\psi \in \mathcal{O}'$ be an increasing and upper-semicontinuous operator. From the fact that $\mathcal{B}_\psi \subseteq Ker(\psi)$, we have $\bigcup_{\theta_M \in \mathcal{B}_\psi} \mathcal{E}_{\theta_M}(X) \subseteq \bigcup_{\theta \in Ker(\psi)} \mathcal{E}_\theta(X)$, for every $X \in \mathcal{F}$. From Theorem 3, every $\theta \in Ker(\psi)$ contains a minimal element $\theta_M \in \mathcal{B}_\psi$. Therefore, $\mathcal{E}_\theta \subseteq \mathcal{E}_{\theta_M}$. So, $\bigcup_{\theta \in Ker(\psi)} \mathcal{E}_\theta(X) \subseteq \bigcup_{\theta_M \in \mathcal{B}_\psi} \mathcal{E}_{\theta_M}(X)$, for every $X \in \mathcal{F}$. The result then follows by antisymmetry of the partial order $\subseteq$. □

Proof of Theorem 5. Let $\psi \in \mathcal{O}$. We know that any subset S of a partially ordered set P can be properly covered by the union of the closed intervals of P contained in S (see [2, Lemma 2.1] for a proof). Since $Ker(\psi)$ is a partially ordered set, and the mapping segment $[\theta_1, \theta_2]$ is a closed interval, we have

$$Ker(\psi) = \bigcup \{[\theta_1, \theta_2] : [\theta_1, \theta_2] \subseteq Ker(\psi)\}. \tag{43}$$

On the other hand, we observe that

$$Ker(. \otimes (\theta_1, \theta_2)) = [\theta_1, \theta_2]. \tag{44}$$

Combining (43) and (44), we obtain

$$Ker(\psi) = \bigcup \{Ker(. \otimes (\theta_1, \theta_2)) : [\theta_1, \theta_2] \subseteq Ker(\psi)\}. \tag{45}$$

From Proposition 2, the mapping $\psi \mapsto Ker(\psi)$ is an isomorphism. So, by applying the corresponding inverse mapping to both sides of (45), we obtain the desired result. □

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