M-Idempotent and Self-Dual Morphological Filters: Supplemental Material

A. Properties of Increasing Operators

The following propositions will be useful for the subsequent proofs.

Proposition 5 *[3] Let α_1 and α_2 be two increasing operators. We have $\alpha_1 \subseteq \alpha_2$ if and only if Ker$(\alpha_1) \subseteq$ Ker(α_2).*

Proposition 6 *[6] Let α_1 and α_2 be two increasing operators. We have*

$$Ker(\alpha_1) \cup Ker(\alpha_2) \subseteq Ker(\alpha_1 \cup \alpha_2), \quad (38)$$
$$Ker(\alpha_1 \cap \alpha_2) = Ker(\alpha_1) \cap Ker(\alpha_2). \quad (39)$$

B. Proof of Lemmas and Corollaries

Proof of Corollary 1: Let $\alpha_2^* \subseteq \rho \subseteq \alpha_1^*$. From Theorem 2, we have

$$\begin{aligned} \rho = \rho^* \quad &\iff \quad Id \cap \alpha_1^* \subseteq \rho \subseteq Id \cup \alpha_2^* \quad (40)\\ &\iff \quad Id \cap \alpha_1^* \subseteq (Id \cup \alpha_1) \cap \alpha_2 =\subseteq Id \cup \alpha_2^* \\ &\iff \quad Id \cap \alpha_1^* \subseteq \alpha_2 \text{ and } \alpha_1 \subseteq Id \cup \alpha_2^*. \end{aligned}$$

■

Proof of Corollary 2: The proof follows from Lemma 2 and Theorem 3, by letting $\theta_i(z) = (A_i)_z$ for every $i \in I$ and for every $z \in \mathbf{E}$, and then by letting $z = 0$. ■

Proof of Corollary 3: The proof follows immediately from Theorem 5 by letting $\theta_i(z) = (A_i)_z$ for every $z \in \mathbf{E}$ and for every $i \in I$ and then by letting $z = 0$. ■

Proof of Corollary 4: The proof follows immediately from Proposition 1 by letting $\theta_i(z) = (A_i)_z$ for every $z \in \mathbf{E}$ and for every $i \in I$ and then by letting $z = 0$. ■

Proof of Corollary 5: The proof follows immediately from Theorem 6 by letting $\theta_i(z) = (A_i)_z$, for every $z \in \mathbf{E}$ and for every $i \in I$ and then by letting $z = 0$. ■

Proof of Corollary 6: The proof is obtained from Theorem 7 by letting $\theta_i(z) = (A_i)_z$, for every $i \in I$ and for every $z \in \mathbf{E}$ and then by letting $z = 0$ in conditions (a) and (b) of Theorem 7. ■

Proof of Corollary 7: The proof can be obtained from Theorem 8 by letting $\theta_i(z) = (A_i)_z$ for every $i \in I$ and for every $z \in \mathbf{E}$, and then letting $z = 0$ in conditions (1) and (2) of Theorem 8. Condition (1) of Corollary 7 is similar to Condition (a) of Corollary 6. Condition (2) of Corollary 7 can be obtained from Condition (2) of Theorem 8 by noticing that the latter condition can be written in the increasing and translation-invariant case as: "There exists $(x_1, x_2, \cdots, x_{2m}) \in \mathbf{E}^{2m}$ such that $x_1, x_{2m} \in \bigcap_{i\in I} A_i$ and $x_{i+1} - x_i \in \bigcap_{i\in I} A_i$ for $i = 1, 2, \cdots, 2m-1$." This last condition is equivalent to Condition (2) of Corollary 7. ■

Proof of Corollary 8: The proof can be obtained from Theorem 9 by letting $\alpha_i(z) = (A_i)_z$ and $\beta_j(z) = (B_j)_z$, for every $i \in I, j \in J$ and for every $z \in \mathbf{E}$ and then by letting $z = 0$. ■

Proof of Corollary 9: The proof can be obtained from Theorem 10 by letting $\alpha_i(z) = (A_i)_z$ and $\beta_j(z) = (B_j)_z$, for every $i \in I, j \in J$ and for every $z \in \mathbf{E}$ and then by letting $z = 0$. ■

Proof of Lemma 1: $\alpha_2 = \alpha_1^* \Rightarrow \alpha_1 = \alpha_2^*$. Hence, we have

$$\rho \;=\; (Id \cup \alpha_2^*) \cap \alpha_1^* = [(Id \cap \alpha_2) \cup \alpha_1]^* = \rho^*. \quad (41)$$

■

Proof of Lemma 2: Let $\alpha_1 = \bigcup_{i\in I} \mathcal{E}_{\theta_i}$. We have $\alpha_1^* = \bigcap_{i\in I} \mathcal{D}_{\theta_i'}$. It follows that

$$\begin{aligned} \alpha_1 \subseteq \alpha_1^* \quad &\Leftrightarrow \quad \forall\, i, j,\ \mathcal{E}_{\theta_i} \subseteq \mathcal{D}_{\theta_j'} \quad (42)\\ &\Leftrightarrow \quad \forall i, j, z, X,\ (\theta_i(z) \subseteq X) \Rightarrow (\theta_j(z) \cap X \neq \emptyset). \end{aligned}$$

By taking $X = \theta_i(z)$, we obtain $\forall i, j, z, \theta_j(z) \cap \theta_i(z) \neq \emptyset$. ■

Proof of Lemma 3: By recalling the identity: $X \subseteq (Y \cup Z) \iff (X \cap Z^c) \subseteq Y$, we have

$$\begin{aligned} &\alpha = Id \cap \alpha_2 \cup \alpha_1,\ \alpha_1 \subseteq \alpha_2 \quad (43)\\ &\iff \alpha_1 \subseteq \alpha \subseteq \alpha_2 \text{ and } Id \cap \alpha_2 \subseteq \alpha \subseteq Id \cup \alpha_1 \\ &\iff \alpha_1 \subseteq \alpha \subseteq Id \cup \alpha_1 \text{ and } Id \cap \alpha_2 \subseteq \alpha \subseteq \alpha_2 \\ &\iff \alpha \cap Id^c \subseteq \alpha_1 \subseteq \alpha \text{ and } \alpha \subseteq \alpha_2 \subseteq \alpha \cup Id^c. \end{aligned}$$

■

Proof of Lemma 4: Assume that α is an overfilter. For every $z \in \mathbf{E}$ and for every $X \in \mathcal{P}(\mathbf{E})$ we have

$$\begin{aligned} &z \in \alpha(X) \Longrightarrow z \in \alpha(\alpha(X)) = \alpha^2(X) \quad (44)\\ \iff \quad &\exists i : z \in \mathcal{E}_{\theta_i}(X) \Longrightarrow \exists j \in I : z \in \mathcal{E}_{\theta_j}(\alpha(X)) \\ \iff \quad &\exists i : \theta_i(z) \subseteq X \Longrightarrow \exists j \in I : \theta_j(z) \subseteq \alpha(X). \end{aligned}$$

By letting $X = \theta_i(z)$, we obtain the result. ■

C. Proof of Propositions

Proof of Proposition 1: From the increasing property of α, we have $\alpha(Id \cap \alpha) \subseteq \alpha$. Thus,

$$\begin{aligned} &\alpha \text{ is an inf-overfilter} \\ &\iff \alpha \subseteq \alpha(Id \cap \alpha) \\ &\iff Ker(\alpha) \subseteq Ker(\alpha(Id \cap \alpha)) \\ &\iff \theta \in Ker(\alpha) \Longrightarrow \theta \in Ker(\alpha(Id \cap \alpha)) \\ &\iff \forall z \in \mathbf{E}, z \in \alpha(\theta(z)) \Longrightarrow z \in \alpha(\theta(z) \cap \alpha(\theta(z))) \\ &\iff \forall z \in \mathbf{E},\ \exists i \in I : \theta_i(z) \subseteq \theta(z) \Longrightarrow \exists j \in I : \\ &\theta_j(z) \subseteq (\theta(z) \cap \alpha(\theta(z))). \end{aligned}$$

By letting $\theta = \theta_i$, we obtain

$$\forall z \in \mathbf{E},\ \forall i \in I,\ \exists j \in I : \theta_j(z) \subseteq \theta_i(z) \text{ and } \theta_j(z) \subseteq \alpha(\theta_i(z)). \quad (45)$$

Since we assume that there is no inclusion between the elements of the kernel generating α, we have $\theta_j = \theta_i$, and thus $\forall z \in \mathbf{E},\ \forall i \in I,\ \theta_i(z) \subseteq \alpha(\theta_i(z))$. ■

Proof of Proposition 2: From the increasing property of α, we have $\alpha \subseteq \alpha(Id \cup \alpha)$. Thus, α is an sup-underfilter

$$\begin{aligned} \iff \quad &\alpha(Id \cup \alpha) \subseteq \alpha \\ \iff \quad &\forall\ X \in \mathcal{P}(\mathbf{E}), 0 \in \alpha(X \cup \alpha(X)) \Longrightarrow 0 \in \alpha(X) \\ \iff \quad &\forall\ X \in \mathcal{P}(\mathbf{E}), \exists A_i \subseteq X \cup \alpha(X) \Longrightarrow \exists A_j \subseteq X \\ \iff \quad &\forall i \in I, \forall B_i \subseteq A_i, \forall \text{ mapping } k : A_i \setminus B_i \mapsto I, \\ &\exists j \in I : B_i \cup \bigcup_{y \in A_i \setminus B_i} (A_{k(y)})_y \supseteq A_j. \end{aligned}$$

■

Proof of Proposition 3: First, notice that $\alpha^* = \bigcap_{i\in I} \mathcal{D}_{\theta'_i}$. We have

$$\begin{aligned}\theta \in \mathit{Ker}(\alpha^*) &\iff z \in \alpha^*(\theta(z)),\ \forall z \in \mathbf{E} \quad (46)\\ &\iff \theta(z) \cap \theta_i(z) \neq \emptyset,\ \forall i \in I, \forall z \in \mathbf{E}\\ &\iff \exists y_i \in \theta_i(z) \text{ and } y_i \in \theta(z),\ \forall i \in I, \forall z \in \mathbf{E}\\ &\iff \bigcup_{\substack{y_i\in\theta_i(z)\\ i\in I}} \{y_i\} \subseteq \theta(z),\ \forall z \in \mathbf{E}. \quad (47)\end{aligned}$$

Assume now that $\theta \in \mathit{Ker}(\rho)$. The kernel of ρ is generated by the kernel of $Id \cap \alpha^*$ and the kernel of α. From Eq. (47), $\mathit{Ker}(Id\cap\alpha^*)$ is generated by the mappings of the form $\theta(z) = \{z\} \cup \bigcup_{\substack{y_i\in\theta_i(z)\\ i\in I}}\{y_i\}$ for every $z \in \mathbf{E}$ and the kernel of α is generated by the mappings of the form $\theta(z) = \theta_i(z)$ for every $z \in \mathbf{E}$ and every $i \in I$.

The proof of the translation-invariant case is simply obtained from the proof above by letting $\theta_i(z) = (A_i)_z$ for every $z \in \mathbf{E}$ and every $i \in I$ and then by letting $z = 0$. ■

Proof of Proposition 4: The proof of this proposition follows exactly the same steps involved in the proof of Proposition 4. ■

D. PROOF OF THEOREMS

Proof of Theorem 2: We have

$$\begin{aligned}\rho = \rho^* &\iff \rho = Id \cap \alpha_1^* \cup \alpha_2^* = Id \cup \alpha_2^* \cap \alpha_1^* \quad (48)\\ &\iff Id \cap \alpha_1^* \subseteq \rho \subseteq Id \cup \alpha_2^* \text{ and } \alpha_2^* \subseteq \rho \subseteq \alpha_1^*.\end{aligned}$$

■

Proof of Theorem 3: The proof follows immediately from Lemmas 1 and 2. ■

Proof of Theorem 4: Assume that there exists $\alpha_1 \subseteq \alpha_1^*$ such that $\alpha = Id \cap \alpha_1^* \cup \alpha_1$. From Lemma 1, α is self-dual.

Assume that $\alpha \in \mathcal{O}$ is self-dual. Let $\alpha_1 = \alpha - Id$ and $\alpha_2 = \alpha \cup Id^c$. Notice that $\alpha_2 = \alpha_1^*$, and $\alpha_1 \subseteq \alpha_2$. From Lemma 3, we have $\alpha = Id \cap \alpha_1^* \cup \alpha_1$. ■

Proof of Theorem 5: From Lemma 4, we have

$$\begin{aligned}&\alpha \text{ is an overfilter} \quad (49)\\ &\iff \forall z \in \mathbf{E},\ \forall i \in I,\ \exists j \in I : \theta_j(z) \subseteq \alpha(\theta_i(z)) \quad (50)\\ &\iff \forall z \in \mathbf{E},\ \forall i \in I,\ \exists j \in I:\ \forall y \in \theta_j(z), y \in \alpha(\theta_i(z))\\ &\iff \forall z \in \mathbf{E},\ \forall i \in I,\ \exists j \in I:\ \forall y \in \theta_j(z), \exists k(y) \in I\\ &\text{such that } \theta_{k(y)}(y) \subseteq \theta_i(z).\end{aligned}$$

■

Proof of Theorem 6: From Proposition 5 and the increasing property of α, we have

$$\begin{aligned}&\alpha^2 \subseteq \alpha\\ &\iff z \in \alpha(\alpha(X)) \implies z \in \alpha(X), \forall z \in \mathbf{E}, \forall X \in \mathcal{P}(\mathbf{E})\\ &\iff \exists i \in I : \theta_i(z) \subseteq \alpha(X) \implies \exists l \in I : \theta_l(z) \subseteq X,\\ &\forall z \in \mathbf{E}, \forall X \in \mathcal{P}(\mathbf{E})\\ &\iff \exists i \in I : \forall y \in \theta_i(z), \exists k(y) \in I : \theta_{k(y)}(y) \subseteq X \implies\\ &\exists l \in I : \theta_l(z) \subseteq X, \forall z \in \mathbf{E}, \forall X \in \mathcal{P}(\mathbf{E})\\ &\iff \forall z \in \mathbf{E}, \forall i \in I, \forall \text{ mapping } k : \theta_i(z) \mapsto I, \exists l \in I :\\ &\theta_l(z) \subseteq \bigcup_{y\in\theta_i(z)} \theta_{k(y)}(y).\end{aligned}$$

■

Proof of Theorem 7: Since ρ is self-dual, it is sufficient to prove that it is an overfilter. From Proposition 3, recall that the kernel of ρ is generated by the mappings θ of the form either (i) $\theta(z) = \{z\} \cup \bigcup_{i\in I}\{y_i, y_i \in \theta_i(z)\}$ for every $z \in \mathbf{E}$ or (ii) $\theta(z) = \theta_i(z)$ for every $i \in I$ and for every $z \in \mathbf{E}$. We will cosnider both cases. From Theorem 5, we have

- Case (i) Assume that θ has the form (i). For $y = z$ we obviously have $\theta(z) \subseteq \theta(z)$. For $y = y_i \in \theta_i(z)$, by using the symmetry assumption we have $\theta(z) = \{z\} \cup \{y_i\} \cup \bigcup_{j\in I, j\neq i}\{y_j : y_i \neq \theta_j(z), y_j \in \theta_j(z)\}$ because if $y_i \in \theta_j(z)$ then $z \in \theta_j(y_i)$. From Condition (a) and the symmetry assumption, we have $\theta(z) = \{y_i\} \cup \{z\} \cup \bigcup_{j\in I, j\neq i}\{y_j : y_j \in \theta_j(y_i)\} = \{y_i\} \cup \bigcup_{j\in I}\{t_j : t_j \in \theta_j(y_i)\} = \tilde{\theta}(y_i) \subseteq \theta(z)$, where $\tilde{\theta}$ is a mapping of the form (i). Thus, we have proved that Theorem 5 holds for mappings θ of the form (i).
- Case (ii) Assume that θ has the form (ii). From Condition (b), there exists $x \in \bigcap_{i\in I}\theta_i(z)$ and therefore $\theta_1(z) = \{z, x\}$ is an element of the kernel of ρ. For $y = z$, we have obviously $\theta_i(z) \subseteq \theta(z) = \theta_i(z)$. For $y = x$, by using Condition (b) and the symmetry assumption and by letting $\theta_2(z) = \{x\} \cup \bigcup_{i\in I}\{y_i : y_i \in \theta_i(x)\}$, we have $\theta_2(x) = \{x\} \cup \bigcup_{i\in I}\{y_i : y_i \in \theta_i(z)\} \subseteq \theta(z)$. Thus, we have proved that Theorem 5 holds for $\theta(z)$ of the form (ii).

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Proof of Theorem 8: We will show that under conditions (1) and (2), we have $\rho^m \subseteq \rho^{m+1}$ and from the self-duality of ρ, it can be deduced that $\rho^{m+1} = \rho^m$. Let $z \in \rho^m(X)$ for some $X \in \mathcal{P}(\mathbf{E})$. From the SV kernel representation of ρ and proposition 3, there exists some $\theta(z)$ of the form either (i) $\theta(z) = \{z\} \cup \bigcup_{i\in I}\{y_i : y_i \in \theta_i(z)\}$ or of the form (ii) $\theta(z) = \theta_i(z)$ for some $i \in I$, such that $\theta(z) \subseteq \rho^{m-1}(X)$.

- Case (i) Let $\theta(z)$ be of the form (i). From Condition (1), from the symmetry assumption and by using a similar argument as in the proof of Theorem 7 for the case (i), we have $\theta(z) = \{z\} \cup \bigcup_{j\in I, j\neq i}\{y_j : y_j \in \theta_j(z)\} = \tilde{\theta}(y_i) \subseteq \rho^{m-1}(X)$. This shows that $y_i \in \rho^m(X)$ for every $y_i \in \theta(z)$. Therefore, $\theta(z) \subseteq \rho^m(X)$ and $z \in \rho^{m+1}(X)$.
- Case (ii) Let $\theta(z)$ be of the form (ii). From condition 2 and the symmetry assumption, there exists $x_1, x_{2m} \in \rho^{m-1}(X)$ such that $x_1, x_{2m} \in \bigcap_{j\in I}\theta_j(z)$ and there exist $x_i, x_{m+i} \in \rho^{m-i}(X)$ such that $x_i \in \bigcap_{j\in I}\theta_j(x_{i-1})$ for $i = 2, \cdots, m-1$. If $x_1 \in \rho^{m-2}(X)$, we have $\{x_2, x_1\} = \theta_2(x_2) \subseteq \rho^{m-2}(X)$. This proves that $x_2 \in \rho^{m-1}(X)$. Consequently, we have $x_1 \in \rho^m(X)$ and therefore $\{z, x_1\} = \theta_3(z) \subseteq \rho^m(X)$. The latter is equivalent to $z \in \rho^{m+1}(X)$. A similar argument can be made if $x_{2m} \in \rho^{m-2}(X)$. If x_1 and $x_{2m} \notin \rho^{m-2}(X)$, then x_2 and $x_{2m-2} \in \rho^{m-2}(X)$ and the above process can be repeated. In the worst case, we get x_m and $x_{m+1} \in \rho(X)$. In this case, by going backwards and using Condition (2) it is easy to see that $x_{m-1} \in \rho^2(X)$ and as $x_{m-2} \in \rho^2(X)$ we have $x_{m-3} \in \rho^3(X)$ and so on

until we obtain $z \in \rho^{m+1}(X)$.

■

Proof of Theorem 9: The proof follows exactly the proof of Theorem 7. Conditions (a) and (b) imply that ρ is an overfilter and conditions (c) and (d) imply that ρ^* is an overfilter. Notice again that conditions (c) and (d) are obtained from conditions (a) and (b) by simply exchanging the roles of α_i and β_j. ■

Proof of Theorem 10: The proof follows exactly the proof of Theorem 8. ■