ON THE PERFORMANCE OF LINEAR AND NONLINEAR COMPANDING TRANSFORMS IN OFDM SYSTEMS

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***Abstract*—Companding schemes are widely employed to reduce the peak-to-average-power ratio (PAPR) in orthogonal frequency division multiplexing (OFDM) systems. This paper considers the classes of linear and nonlinear companding schemes and derives a sufficient condition under which the bit error rate (BER) performance of one is superior to that of the other. The high PAPR of the OFDM signal drives the transmitter's power amplifier into its nonlinear region, thus causing nonlinear distortions. In the literature, signal companding is one of the widely used techniques for PAPR reduction. Furthermore, it has been claimed, based on simulation results, that linear companding performs better, in terms of BER, than nonlinear companding for an optimized set of companding parameters. In this paper, we derive sufficient conditions under which these claims are valid. The conditions derived also show that, in practice, nonlinear companding transforms perform better than linear companding transform. Our theoretical analysis is supported by simulation results.**

I. INTRODUCTION

OFDM is a popular multi-carrier modulation technique that can support high transmission rates over wireless channels and utilizes the available spectrum efficiently. It is the backbone of the physical layer for many commercial systems and wireless standards such as the Wireless Local Area Network (WLAN), Worldwide Interoperability for Microwave Access (WiMAX), Digital Audio Broadcasting (DAB), Digital Video Broadcasting (DVB) and Digital Subscriber Line (DSL). The OFDM scheme modulates multiple carriers using multiple data symbols simultaneously. It increases the data symbol duration, hence reducing its spectrum range. This process breaks the wide transmission band of the single-carrier modulation schemes into narrower, multiple subbands and allows the OFDM to effectively combat the frequency-selective fading encountered in multiple path propagation wireless channels.

Despite the great advantages OFDM offers, it has a few drawbacks, the most serious of which is the non-constant signal envelope with high peaks. These high peaks drive the power amplifier at the transmitter into nonlinear region of operation, thereby causing nonlinear distortion. To overcome this problem, power amplifiers with wider linear region are required. Otherwise, the power amplifier must be forced to work at a reduced efficiency. Since both solutions are not practical, many PAPR reduction techniques have been proposed in the literature [1]. Some examples of which include techniques such as clipping and filtering, signal companding, peak windowing, selective mapping, partial transmit sequence, tone injection, tone reservation and linear block coding. An overview of these methods is provided in [1].

The rest of this paper is organized as follows: Section II describes the basic OFDM system with companding transform. Section III defines the PAPR and discusses some of the aspects related to its computation. In Section IV, the concept of companding transforms and some of the widely used linear and nonlinear companding functions in the literature are introduced. In Section V, we derive a sufficient condition that ensures the BER performance superiority of linear companding over nonlinear companding transforms. Section VI presents three examples of nonlinear companding transforms and finds the conditions that relate their BER in terms of the companding parameters. In Section VII, we derive a relationship between the parameters of linear companding transform based on the power constraint criterion. This relationship leads to an upper bound for the linear companding offset. Finally, Sections VIII and IX show some simulation results to validate the derived conditions and conclude the work presented in this paper.

II. OFDM SYSTEM MODEL

For a binary data stream with rate R bits per seconds (bps), N modulated symbols, a_k, $0 \leq k \leq N-1$, data symbols are stored for an interval of $T_s = N/R$ using a serial-to-parallel (S/P) converter. Subsequently, N data symbols modulate N subcarrier and transmit them simultaneously. The OFDM signal $x(t)$ can be expressed as

$$\begin{aligned} x(t) &= \sum_{k=0}^{N-1} a_k \exp\left(j2\pi\left(f_c + k\,\Delta f\right)t\right) \\ &= \exp\left(j2\pi f_c\, t\right) \sum_{k=0}^{N-1} a_k \exp\left(j2\pi k\,\Delta f\, t\right) \\ &= \exp\left(j2\pi f_c\, t\right)\, a(t), \end{aligned} \tag{1}$$

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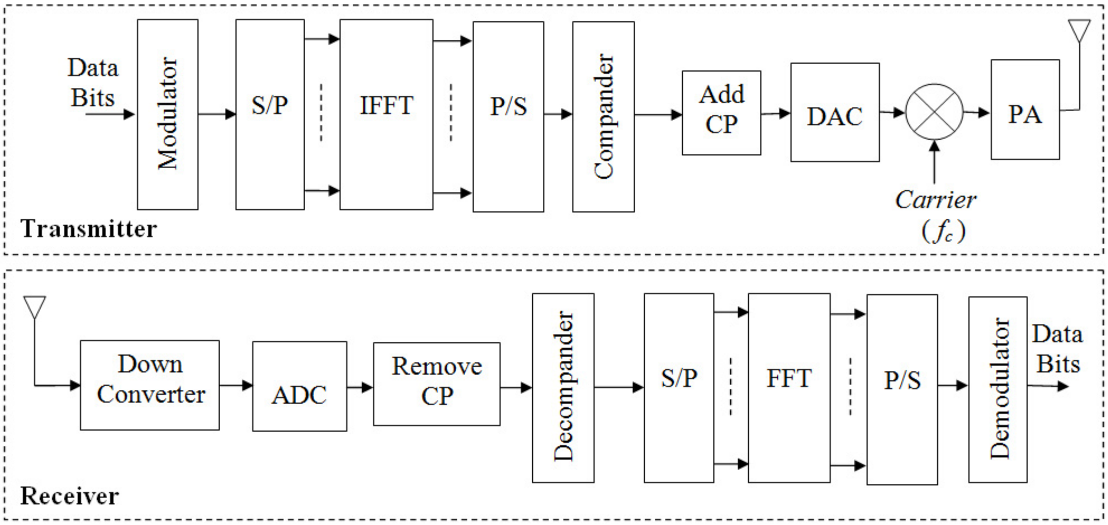


Fig. 1. OFDM transceiver system with companding.

where $f_k = f_c + k\,\Delta f, 0 \leq k \leq N-1$, is the k^{th} subcarrier, with f_c being the lowest subcarrier frequency and Δf is the frequency spacing between adjacent subcarriers, chosen to be $\frac{1}{T_s}$ to ensure that the subcarriers are orthogonal [2]. If $a(t)$ is sampled at the rate of R samples per second, then its sampled version $a[n]$ is represented by

$$a[n] = \sum_{k=0}^{N-1} a_k \exp\left(j2\pi kn/N\right), \tag{2}$$

which takes exactly the same form of the Inverse Discrete Fourier Transform (IDFT). Equations (1) and (2) demonstrate that the OFDM signal can be generated by modulating the Inverse Fast Fourier Transform (IFFT) of the sequence $\{a[n],\ 0 \leq n \leq N-1\}$ by a single carrier of frequency f_c instead of by modulating N data symbols by N subcarriers. Figure 1 shows the basic block diagram of an OFDM system with a companding (compressing) transform used at the transmitter to reduce the PAPR before amplifying the signal by the power amplifier. The decompanding (expanding) transform is used at the receiver to recover the data symbols correctly.

III. PEAK-TO-AVERAGE POWER RATIO

The PAPR for the discrete-time signal $x[n]$ is defined as the ratio of its maximum instantaneous power to its average power and can be expressed as

$$PAPR\left(x[n]\right) = \max_{0\leq n\leq N-1} \frac{\left|x\left[n\right]\right|^2}{E\left[\left|x[n]\right|^2\right]}, \tag{3}$$

where $E[.]$ denotes the expectation operator. The time-domain OFDM samples are obtained using an IFFT block at the Nyquist rate. The peak values computed using these samples may not accurately coincide with the peak values of the continuous-time OFDM signal [3]. As a result, the computed PAPR of the discrete-time OFDM samples may not be an accurate measure of the real PAPR. It is found that the PAPR of the oversampled discrete-time OFDM signal offers an accurate approximation of the continuous-time PAPR if the oversampling factor exceeds 4 [4].

The PAPR reduction capability is measured by the empirical complementary cumulative distributive function (CCDF), which indicates the probability that the PAPR is above a certain threshold.

IV. COMPANDING TRANSFORMS

The companding transformation is applied at the transmitter in order to attenuate the high peaks and amplify the low amplitudes of the OFDM signal, thus decreasing the PAPR [5], [6]. At the receiver, the decompanding process is applied by using the inverse companding function in order to recover the original OFDM signal. Companding is an attractive technique to reduce the PAPR of OFDM signals due to its low complexity regardless of the number of subcarriers in the OFDM signal. Moreover, the average transmitted power can be kept unchanged after companding by choosing proper companding parameters.

A major drawback of current companding techniques, however, is that they achieve PAPR reduction at the cost of increasing the BER. This inherent tradeoff between the PAPR reduction capability and the BER performance in companding systems is mainly due to two factors: First, companding distorts the modulating data symbols at the transmitter from their original constellation. Second, the channel noise is expanded at the receiver by the decompanding process resulting in an increased number of errors in the recovered data symbols and hence increased BER.

Many companding functions have been proposed and studied in the literature. The use of the μ-law companding technique to reduce the PAPR of the OFDM signal has been studied extensively in [7] and [8]. In [9], the symbol error rate of the companded OFDM is analyzed and compared

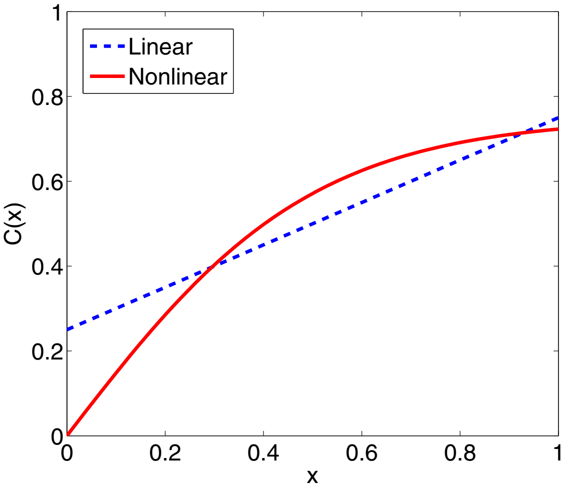


Fig. 2. Linear and nonlinear companding transform curves.

to that of the original OFDM. In [10], a nonlinear companding transform with parameters computed based on the statistics of the transmitted OFDM signal is proposed. A similar companding transform is proposed in [11] to transform the Rayleigh distribution of the envelope or the exponential distribution of the power of the original OFDM signal into a quasi-uniform distribution. Besides the error function [10], [11], other nonlinear companding functions like exponential [12], logarithmic [13], and hyperbolic tangent [14], [15], are used in the literature to reduce the PAPR of OFDM signals. The authors in [16] presented a general companding design criteria to facilitate an effective tradeoff between PAPR reduction and BER performance. Specifically, the performance of four typical companding transforms is investigated, and it is shown that significant PAPR reduction is possible by a proper selection of the companding forms and parameters.

The focus of this paper is on the BER performance of linear and nonlinear companding transforms. Figure 2 shows typical profiles of the linear and nonlinear companding transforms.

V. PERFORMANCE COMPARISON BETWEEN LINEAR AND NONLINEAR COMPANDING TRANSFORMS

In the following proposition, we show that a sufficient condition for the superiority of the linear companding over the nonlinear companding transforms, in terms of BER, is that the derivative curve of the nonlinear companding function remains below the horizontal line defined by the slope of the linear companding transform.

Proposition 1. *Consider a sampled OFDM signal with envelope x. Denote by C_L the linear companding transform given by $C_L(x) = (ax+b)$, with $0 < a < 1$ and $b > 0$, and let C_{NL} denote any nonlinear companding transform, for which the inverse exists. For a white Gaussian noise (WGN) channel, the linear companding transform results in a smaller absolute error at the receiver compared to the nonlinear companding transform if*

$$|C'_{NL}(x)| < a\,, \tag{4}$$

where $C'_{NL}(x)$ denotes the derivative of the function $C_{NL}(x)$ with respect to x.

Proof 1. *For the sake of simplicity, we write x to denote the discrete-time envelope of the OFDM signal $x[n]$. Let w be the channel's additive white Gaussian noise. At the receiver of an OFDM system, which employs companding, the absolute received error $|e| = |y - x|$, where y is the received signal, is given by*

$$|e| = \left|C^{-1}\left[C(x) + w\right] - x\right| . \tag{5}$$

At the receiver, the decompanding function is applied to the received signal prior to demodulation in order to recover the data symbols. From Eq. (5), the absolute received error for the linear companding transform is

$$\begin{aligned} |e_L| &= \left|\frac{[(ax+b+w)-b]}{a} - x\right| \\ &= \left|\frac{w}{a}\right| . \end{aligned} \tag{6}$$

In order to obtain the error at the receiver of the nonlinear companding system, we use the first order Taylor series expansion as an approximation of the function $C_{NL}^{-1}[C_{NL}(x)+w]$ around the point $C_{NL}(x)$. By noting that $\left[C_{NL}^{-1}(C_{NL}(x))\right]' = \frac{1}{C'_{NL}(x)}$, Eq. (5) becomes

$$\begin{aligned} |e_{NL}| &\approx \Big|C_{NL}^{-1}\left[C_{NL}(x)\right] - x \\ &\quad + w\left[C_{NL}^{-1}\left[C_{NL}(x)\right]\right]'\Big| \\ &= \left|\frac{w}{C'_{NL}(x)}\right| . \end{aligned} \tag{7}$$

Therefore, combining Eqs. (6) and (7), a sufficient condition to have $|e_L| < |e_{NL}|$ is $|C'_{NL}(x)| < a$.

VI. EXAMPLES

In this section, we apply the condition in Eq. (4) to three typical nonlinear companding functions widely used in the literature and find specific conditions in terms of the companding parameters.

A. Hyperbolic tangent (tanh) companding

The hyperbolic tangent ($tanh$) companding function is defined by [15]

$$C(x) = k_1 \tanh(k_2\, x)\ , \tag{8}$$

where k_1 and k_2 are positive numbers controlling the companding level applied to the envelope x. The derivative of $C(x)$ is given by

$$C'(x) = k_1 k_2\left[1 - \tanh^2(k_2\, x)\right] . \tag{9}$$

Given that $\left[1 - \tanh^2(k_2\, x)\right] \leq 1$, $\forall\, k_2, x$, a sufficient condition to have $|C'(x)| < a$ is $a > k_1 k_2$.

B. Error Function (erf) Companding

The error function (erf) is defined by [11], [10]

$$C\left(x\right) = k_1 \, \mathrm{erf}\left(k_2 \, x\right) \,, \tag{10}$$

where k_1 and k_2 are positive numbers controlling the level of companding. Taking the derivative, we obtain

$$C'\left(x\right) = \frac{2k_1k_2}{\sqrt{\pi}} \, \exp\left[-\left(k_2 \, x\right)^2\right] . \tag{11}$$

Since $\exp\left[-\left(k_2 \, x\right)^2\right] \leq 1, \, \forall \, k_2, x$. A sufficient condition to have $|C'\left(x\right)| < a$ is $a > \frac{2\,k_1k_2}{\sqrt{\pi}}$.

C. Logarithm Function (log) Companding

The logarithm (log_e) companding function is defined by [13]

$$C\left(x\right) = k_1 \, \log_e\left(1 + k_2 \, x\right) \,, \tag{12}$$

where k_1 and k_2 are two positive numbers controlling the amount of companding. The derivative of this function is given by

$$C'\left(x\right) = \frac{k_1k_2}{1 + k_2 \, x} \, . \tag{13}$$

Since both k_2 and x are positive, setting $k_1k_2 < a$ is a sufficient condition to ensure that $|C'(x)| < a$. In [13], the parameters of the logarithm function have been chosen as $k_1k_2 = 1$.

VII. COMPANDING PARAMETERS DESIGN CRITERIA

Although the condition given by Eq. (4) involves only the slope a of the linear companding transform (independently of the offset b), it is desirable to find a relationship to relate a and b based on some design criteria. We follow an average power constraint design criterion similar to the one presented in [17], which showed that the average transmitted power of the companded OFDM signal by a nonlinear companding transform, like the hyperbolic tangent or the error functions, is kept unchanged if $k_1 \approx 1/k_2$.

Using the central limit theorem (CLT), it is shown that for a large number of subcarriers, both the real and imaginary parts of the complex OFDM signal are asymptotically independent and identically distributed Gaussian random variables. Consequently, the envelope of the OFDM signal, x, follows the Rayleigh distribution with the probability density function

$$f_x\left(x\right) = \frac{x}{\sigma^2} \, \exp\left(\frac{-x^2}{2\,\sigma^2}\right) , \tag{14}$$

where σ is a strictly positive adjustable Rayleigh parameter. The mean and variance of the Rayleigh distribution are given by $\sigma\sqrt{\pi/2}$ and $\frac{4-\pi}{2} \, \sigma^2$, respectively. To keep the average power of the OFDM signal unchanged after companding, the following condition must be satisfied

$$\begin{aligned} E\left[x^2\right] &= E\left[C_L^2\left(x\right)\right] \Leftrightarrow \\ 2\,\sigma^2 &= E^2\left[C_L\left(x\right)\right] + var\left[C_L\left(x\right)\right] . \end{aligned} \tag{15}$$

Using $C_L(x) = ax + b$ and substituting the values of $E[x]$ and $var[x]$ in Eq. (15) yields

$$a^2\Big(2\,\sigma^2\Big) + a\Big(2\,\sigma\,b\sqrt{\frac{\pi}{2}}\Big) + \Big(b^2 - 2\,\sigma^2\Big) = 0 \, . \tag{16}$$

The quadratic formula in Eq. (16) can be used to find a set of solutions for a. Given the conditions $b > 0$, $0 < a < 1$ the following bound is found for the offset b in order to keep the average power unchanged after linear companding

$$0 < b < \sqrt{2}\,\sigma \tag{17}$$

VIII. SIMULATION RESULTS

In order to validate the theoretical findings, we conducted computer simulations for a baseband OFDM system model. Specifically, we consider the implementation of the WiMAX standard in the downlink partial use subcarrier (DL-PUSC) mode with 1024 subcarriers. At the transmitter, the solid state power amplifier (SSPA) model is implemented to model the nonlinearity of the power amplifier. This model amplifies the envelope of the signal without introducing phase distortions. The input-output relationship of the SSPA is given by

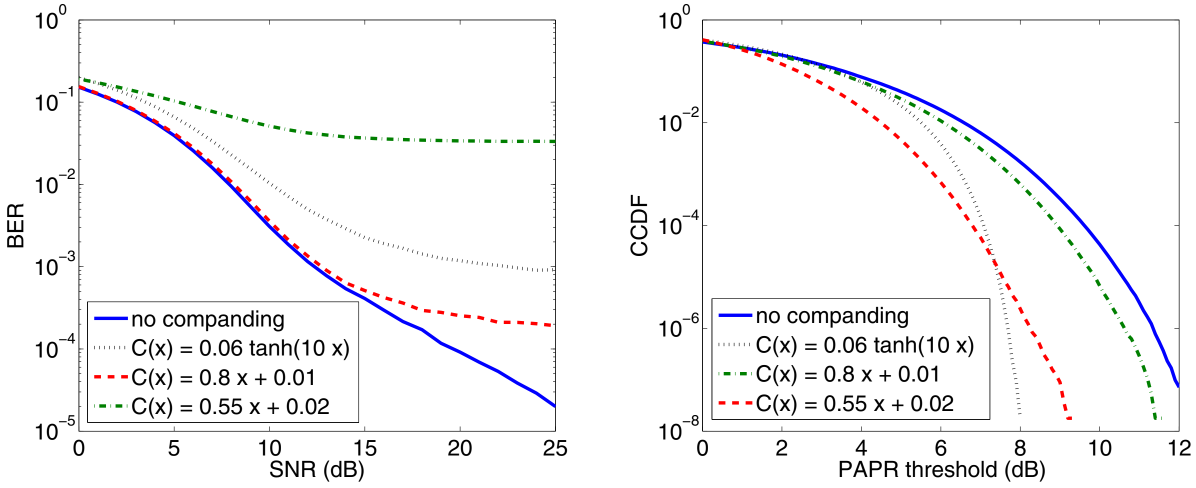
$$x_{out} = \frac{x_{in}}{\left[1 + \left(\frac{x_{in}}{A_{max}}\right)^{2p}\right]^{1/2p}} \,, \tag{18}$$

where p is a positive parameter controlling the nonlinearity of the amplifier and A_{max} is a normalization factor specifying the saturation level of the amplifier. We set $p = 2$ and $A_{max} = 0.16$ in all the simulations.

For each signal-to-noise ratio (SNR), we transmit 100 OFDM frames, with 20 OFDM symbols in each frame, and compute the average of the BER. It can be shown that Proposition 1 can be extended to the fading channel environment. Simulations are conducted for a fading channel using the Stanford University Interim-1 (SUI-1) channel model [18] for a transmission bandwidth of 10MHz. The fading channel is assumed to be quasi-static, so the channel characteristics are kept unchanged during the transmission of each OFDM frame. At the receiver, perfect channel estimation and symbol timing are assumed. Moreover, no channel coding or any other form of diversity is used.

Figure 3(a) shows the BER performance when the linear and nonlinear companding transforms are used. For the nonlinear companding transform, the hyperbolic tangent function presented in Section VI is used with $k_1 = 0.06$ and $k_2 = 10$. The linear companding function is used with parameters a and b satisfying Eq. (16). Simulation results confirm that for $a = 0.8 > k_1k_2 = 0.6$, $|e_L| < |e_{NL}|$. On the other hand, when $a = 0.55 < k_1k_2 = 0.6$, we obtain $|e_L| > |e_{NL}|$. Therefore, the simulation results validate the sufficient condition derived in Proposition 1.

Figure 3(b) shows the empirical complementary cumulative distributive function (CCDF) curves for the same sets of companding parameters used in Fig. 3 (a). The PAPR reduction capability is measured by the CCDF, which indicates the probability that PAPR is above a certain threshold. In



(a)

(b)

Fig. 3. Performance comparison between the linear and nonlinear companding transforms for different values of companding parameters: (a) BER performance; (b) CCDF performance.

particular, the BER is inversely proportional to the PAPR reduction capability in most cases.

IX. CONCLUSIONS

In this paper, we derived a sufficient condition for the superiority of either of the linear and the nonlinear companding transforms over the other, in terms of BER performance. It is noticeable that the derived condition depends only on the slope of the linear companding transform, and is independent of the offset. Both transforms are used in the literature to reduce the high PAPR of OFDM signals. Our simulation results validate the theoretical findings.

We also derived a relationship between the parameters a and b of the linear companding transform in order to keep the average transmitted power unchanged after companding. The average power constraint leads to an upper bound for the companding parameter b.

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