# Numerical Analysis

## Chapter 4 Interpolation and Approximation

### 4.1 Polynomial Interpolation

**Goal** Given $n+1$ data points

$$(x_0, y_0),\ (x_1, y_1),\ \cdots (x_n, y_n),$$

to find the polynomial of degree less than or equal to $n$ that passes through these points.

**Remark** There is a unique polynomial of degree less than or equal to $n$ passing through $n+1$ given points. (Give a proof for $n = 2$.)

**Linear Interpolation** Given two points $(x_0, y_0)$ and $(x_1, y_1)$, the linear polynomial passing through the two points is the equation of the line passing through the points. One way to write its formula is

$$P_1(x) = y_0\, \frac{x_1 - x}{x_1 - x_0} + y_1\, \frac{x_- x_0}{x_1 - x_0}.$$

**Example** For the data points $(2, 3)$ and $(5, 7)$ find $P_1(x)$.

**Solution:**

$$P_1(x) = 3\, \frac{5-x}{5-2} + 7\, \frac{x-2}{5-2} = (5-x) + \frac{5}{3}(x-2)$$

**Example** For the data points $(0.82, 2.270500)$ and $(0.83, 2.293319)$, find $P_1(x)$ and evaluate $P_1(0.826)$.

**Solution:**

$$P_1(x) = 2.270500\, \frac{.83 - x}{.83 - .82} \ +\ 2.293319\, \frac{x - .82}{.83 - .82} = 227.0500\ (.83 - x)\ +\ 229.3319(x - .82)$$

and hence

$$P_1(.826) = 2.2841914.$$

**Remark.** If $f(x) = e^x$, then $f(.82) \approx 2.270500$, $f(.83) \approx 2.293319$, and $f(.826) \approx 2.2841638$. Note then that $P_1(x)$ is an approximation of $f(x) = e^x$ for $x \in [.82, .83]$.

In general, if $y_0 = f(x_0)$ and $y_1 = f(x_1)$ for some function $f$, then $P_1(x)$ is a linear approximation of $f(x)$ for all $x \in [x_0, x_1]$.

**Quadratic Interpolation** If $(x_0, y_0),\ (x_1, y_1),\ (x_2, y_2)$, are given data points, then the quadratic polynomial passing through these points can be expressed as

$$P_2(x) = \ y_0\, L_0(x)\ +\ y_1\, L_1(x)\ +\ y_2\, L_2(x)$$

where

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}$$
$$L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$$
$$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

The polynomial $P(x)$ given by the above formula is called **Lagrange's interpolating polynomial** and the functions $L_0,\ L_1,\ L_2$ are called **Lagrange's interpolating basis functions**.

**Remark** Note that $\deg(P_2) \le 2$ and that

$$L_i(x_j) = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$\delta_{ij}$ is called the **Kronecker delta** function.

**Example** Construct $P_2$ from the data points $(0, -1), (1, -1), (2, 7)$.

**Solution:**

$$P_2(x) = (-1)\,\frac{(x-1)(x-2)}{2}\ +\ (-1)\,\frac{x(x-2)}{-1}\ +\ 7\,\frac{x(x-1)}{2} = \frac{-1}{2}(x-1)(x-2)\ +\ x(x-2)\ +\ \frac{7}{2}x(x-1)$$

**Example** See Example 4.1.4 on page 122 of the text.

**Higher-Degree Interpolation** Given $n+1$ data points

$$(x_0, y_0),\ (x_1, y_1),\ \cdots (x_n, y_n),$$

the $n$ Lagrange interpolating polynomial is given by

$$P_n(x) = \ y_0\, L_0(x)\ +\ y_1\, L_1(x)\ +\ y_2\, L_2(x)\ +\ y_n\, L_n(x)$$

where

$$L_0(x) = \frac{(x - x_1)(x - x_2)(x - x_3)\cdots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)\cdots(x_0 - x_n)}$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)(x - x_3)\cdots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)\cdots(x_1 - x_n)}$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)(x - x_3)\cdots(x - x_n)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)\cdots(x_2 - x_n)}$$

$$\vdots$$

$$L_n(x) = \frac{(x - x_0)(x - x_1)(x - x_2)\cdots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)(x_n - x_3)\cdots(x_n - x_{n-1}}$$

__**Newton's Divided Difference**__ Given distinct points $x_0$ and $x_1$ in the domain of a function $f$, we define

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

This is called the **first-order divided difference** of $f(x)$.

**Remark.** Note that if $f$ is differentiable on $[x_0, x_1]$, then by Mean Value Theorem, there exists a $c \in (x_0, x_1)$ such that $f[x_0, x_1] = f'(c)$. Furthermore, if $x)0$ and $x_1$ are close to each other, then we have

$$f[x_0, x_1] \approx f'(d) \qquad \text{with} \quad d = \frac{x_0 + x_1}{2}.$$

**Example** Consider $f(x) = \cos x$, $x_0 = 0.2$, and $x_1 = 0.3$. Compute $f[x_0, x_1]$.

**Solution:**

$$f[x_0, x_1] = \frac{\cos(0.3) - \cos(0.2)}{0.3 - 0.2} \approx -0.2473009$$

Note that

$$f'\left(\frac{x_0 + x_1}{2}\right) = -\sin(0.25) \approx -0.247404$$

**Definition** **Higher order divided differences** are defined recursively using the lower-order ones.

Suppose $x_0, x_1, x_2$ are distinct point in the domain of $f$. Then the **second-order divided difference** is given by

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

Suppose $x_0, x_1, x_2, x_3$ are distinct points in the domain of $f$. Then the **third-order divided difference** is given by

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$$

In general, if $x_0, x_1, x_2 \cdots x_n$ are distinct points in the domain of $f$, then the $n$**th-order divided difference** is given by

$$f[x_0, x_1, x_2, \cdots x_n] = \frac{f[x_1, x_2, \cdots x_n] - f[x_0, x_1, c \ldots, x_{n-1}]}{x_n - x_0}$$

**Theorem** Suppose $x_0, x_1, x_2, \ldots, x_n$ are distinct points in $[a, b]$ and suppose $f$ is $n$ times continuously differentiable. Then there exists a point $c$ between the smallest and largest of $x_0, x_1, \cdots, x_n$ such that

$$f[x_0, x_1, \cdots, x_n] = \frac{f^{(n)}(c)}{n!}.$$

**Example** Let $f(x) = \cos x,\ x_0 = 0.2,\ x_1 = 0.3,\ x_2 = 0.4.$ Compute $f[x_0, x_1, x_2]$.

**Solution:** From the previous example, we have $f[x_0, x_1] \approx -0.2473009$ and

$$f[x_1, x_2] = \frac{\cos(0.4) - \cos(0.3)}{0.4 - 0.3} \approx -0.3427550$$

Thus

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \approx \frac{-0.3427550 - (-0.2473009)}{0.4 - 0.2} \approx -0.4772705.$$

With $n = 2$ and $c = 0.3$ ( a point between 0.2 and 0.3) we have

$$\frac{f''(c)}{2} = -\frac{1}{2}\cos(0.3) \approx -0.4776682$$

which is very close to $f[x_0, x_1, x_2]$ as claimed in the theorem.

## Basic Properties of Divided differences

1) $f[x_0, x_1] = f[x_1, x_0]$ and $f[x_0, x_1, x_2] = f[x_1, x_0, x_2] = f[x_1, x_2, x_0] = \cdots$. In general if $\{i_0,\ i_2,\ \cdots,\ i_n\}$ is a permutation of $\{0, 1, 2, \cdots n\}$, then

$$f[x_0, x_1, \cdots, x_n] = f[x_{i_0},\ x_{i_1}, \cdots,\ x_{i_n}]$$

2)

$$f[x_0, x_1, x_2] = \frac{f(x_0)}{(x_0 - x_1)(x_1 - x_2)} + \frac{f(x_1)}{(x_1 - x_2)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)}$$

3) From the definition we have

$$\lim_{x_1 \to x_0} f[x_0, x_1] = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(x_0).$$

The we can define

$$f[x_0, x_0] = f'(x_0)$$

In general, if $x_0 = x_1 = x_2 = \cdots = x_n$, then

$$f[x_0, x_0, \cdots, x_0] = \frac{f^{(n)}}{n!}.$$

4) If $x_0 = x_2 \neq x_1$, then

$$f[x_0, x_1, x_0] = f[x_0, x_0, x_1] = \frac{f[x_0, x_1] - f[x_0, x_0]}{x_1 - x_0}$$

## Newton's Divided Difference Interpolating Polynomial Or Newton's Form

Define

$$
\begin{aligned}
P_1(x) &= f(x_0) + f[x_0, x_1](x - x_0) \\
\\
P_2(x) &= f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_1) \\
\\
&= P_1(x) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\
\\
P_3(x) &= f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_1) + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2) \\
\\
&= P_2(x) + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2) \\
\\
&\vdots \\
\\
P_n(x) &= P_{n-1}(x) + f[x_0, x_1, \cdots, x_{n-1}](x - x_0)(x - x_1)\cdots(x - x_{n-1})
\end{aligned}
$$

The polynomial $P_n$ is called **Newton's divided deference formula for the interpolating polynomial or Newton's form for the interpolating polynomial**. Note that $P_n(x_i) = f(x_i)$.

—**Example** Determine the Newton form for the interpolating polynomial for the data set $\{(-1, 5), (0, 1), (1, 1), ($
Then use this polynomial to approximate $y$ if $x = 1.5$.

**Solution**

| $i$ | $x_i$ | $f[x_i] = f(x_i)$ | $f[x_i, x_{i+1}]$ | $f[x_i, x_{i+1}, x_{i+2}]$ | $f[x_0, x_1, x_2, x_3]$ |
|---|---|---|---|---|---|
| 0 | -1 | 5 | | | |
| | | | -4 | | |
| 1 | 0 | 1 | | 2 | |
| | | | 0 | | 1 |
| 2 | 1 | 1 | | 5 | |
| | | | 10 | | |
| 3 | 2 | 11 | | | |

Therefore

$$
\begin{aligned}
P_3(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2) \\
&= 5 - 4(x - (-1)) + 2(x - (-1))(x - 0) + 1(x - (-1))(x - 0)(x - 2) \\
&= 5 - 4(x + 1) + 2x(x + 1) + x(x + 1)(x - 1)
\end{aligned}
$$

And so $P_3(1.5) = 4.375$.

## 4.2 Error in Polynomial Interpolation

**Theorem** Let $f$ be a given function on $[a,b]$ and $P_n$ be the polynomial of degree less than or equal to $n$ interpolating the $f$ at the $n+1$ data points $x_0,\ x_1,\ x_2,\ \cdots, x_n$ in $[a,b]$:

$$P_n(x) = f(x_0)\cdot L_0(x) \ + \ f(x_1)\cdot L_1(x) \ + f(x_2)\cdot L_2(x) + \ \cdots \ + \ f(x_n)\cdot L_n(x).$$

If $f$ has $n+1$ continuous derivatives and $x_j$ are distinct, then

$$f(x) - P_n(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_n)}{(n+1)!} f^{(n+1)}(c_x)$$

where $a \le x \le b$ and $c_x$ is between the maximum and minimum of $x,\ x_0,\ x_1,\ x_2,\ \cdots,\ x_n$.

**Example** Let $f(x) = e^x$ on $[0,1]$ and let $0 \le x_0 < x_1 \le 1$. Then by the theorem,

$$f(x) - P_1(x) = \frac{(x-x_0)(x-x_1)}{2} e^{c_x}$$

where $0 \le x \le 1$ and $c_x$ is between the maximum and minimum of $x$, $x_0$, and $x_1$. If we assume that $x_0 \le x \le x_1$, then $c_x$ is between $x_0$ and $x_1$, and we have

$$\frac{(x-x_0)(x_1-x)}{2} e^{x_0} \le |f(x) - P_1(x)| \le \frac{(x-x_0)(x_1-x)}{2} e^{x_1}$$

Note that if $h = x_1 - x_0$, then

$$\max_{x_0 \le x \le x_1} \frac{(x-x_0)(x_1-x)}{2} = \frac{h^2}{8}$$

and hence

$$|f(x) - P_1(x)| \le \frac{(x-x_0)(x_1-x)}{2} e^{x_1} \le \frac{h^2}{8} e.$$

In particular, if $x_0 = 0.82,\ x_1 = 0.83$ and if $x = 0.826$, then the above reduces to

$$|e^x - P_1(x)| \le 0.000340$$

Note that the actual error is $-0.0000276$.

**Example** Let $f(x) = e^x$ on $[0,1]$ and let $0 \le x_0 < x_1 < x_2 \le 1$. Then by the theorem,

$$f(x) - P_2(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{6} e^{c_x}$$

where $0 \le x \le 1$ and $c_x$ is between the maximum and minimum of $x$, $x_0$, $x_1$, and $x_2$. If we assume that $x_0 \le x \le x_2$ and that $h = x_1 - x_0 = x_2 - x_1$, then $c_x$ is between $x_0$ and $x_2$, and we have

$$|f(x) - P_2(x)| \le \left| \frac{(x-x_0)(x_1-x)(x-x_2)}{6} e^{x_2} \right| \le \left| \frac{(x-x_0)(x_1-x)(x-x_2)}{6} \right| e$$

Note that for $h = x_1 - x_0 = x_2 - x_1$, we have

$$\max_{x_0 \le x \le x_2} \left| \frac{(x-x_0)(x_1-x)}{2} \right| = \frac{h^3}{9\sqrt{3}}.$$

Thus,

$$|f(x) - P_2(x)| \le \frac{h^3}{9\sqrt{3}} e \approx 0.174 h^3.$$

In particular, if $h = 0.01$ then the above reduces to

$$|e^x - P_2(x)| \le 1.74 \cdot 10^{-7}$$

## 4.3 Interpolation Using Splines

**Remark** Consider the data points (0, 2.5), (1, 0.5), (2, 0.5), (2.5, 1.5), (3, 1.5), (3.5, 1.125), (4, 0). The iterating polynomial of Newton and Lagrange are of degree 6. Figure 4.8 on page 148 shows the graph of $P_6(x)$. Such polynomials are not always easy to evaluate and there may be loss of significant digits involved in their calculations .

For these reasons it is desirable to consider **piecewise polynomial interpolation**. This involves finding a continuous function $g$ on $[0, 4]$ and that is a polynomial of 'small' degree in each of the intervals $[0, 1]$, $[1, 2]$, $[2, 2.5]$, $[2.5, 3]$, $[3, 3.5$, and $[3.5, 4]$. Clearly we need $g$ to interpolate the data set.

Such a function $g$ is called a **piecewise linear interpolation** if each of the polynomials on the subintervals are of degree less than or equal to 1.

We say $g$ is a **piecewise quadratic interpolation** if each of the polynomials on the subintervals are of degree less than or equal to 2.

**Example** For a **piecewise linear interpolation** of the above data points, see Figure 4.7 on page 147 of your text. Figure 4.9 at the bottom of page 148 shows a **piecewise quadratic interpolation**.

**Remark** Both the linear and the quadratic interpolating functions are inadequate in that the function $g$ is not differentiable at the **node points.** Thus if smoothness at the node points is required we need the degree of the polynomials to be at least less than or equal to three. As the following theorem states this is all we need.

**Theorem** If $a = x_1 < x_2 < \cdots x_{n-1} < x_n = b$, then there is a unique interpolating function $s(x)$ of the data points $(x_1, y_1)$, $(x_2, y_2)$, $\cdots$, $(x_n, y_n)$ such that

**S1** $s(x)$ is a polynomial of degree $\leq$ 3 on each of the subintervals $[x_{i-1}, x_i]$ for $i = 2, 3, \cdots n$.

**S2** $s(x)$, $s'(x)$, and $s''(x)$ are all continuous on $[a, b]$

**S3** $s''(x_1) = s''(x_n) = 0$

The function $s$ satisfying the above theorem is called the **natural cubic spline**

**Construction of the natural cubic spline $s(x)$**

To simplify notation, we assume that

$$h = x_2 - x_1 = x_3 - x_2 = \cdots = x_n - x_{n-1} = \frac{b-a}{n}.$$

Note that $s''(x)$ is at most linear (why?). Define

$$M_j = s''(x_j).$$

Then, since $s''(x_{j-1}) = M_{j-1}$ and $s''(x_j) = M_j$ are points on the linear function $s''(x)$ and since $s''(x)$ is the equation of the line passing through the points $(x_{j-1}, M_{j-1})$ and $(x_j, M_j)$, we can write its equation as

$$s''(x) = \frac{M_{j-1}}{h}(x_j - x) \ + \ \frac{M_j}{h}(x - x_{j-1})$$

We now integrate $s''(x)$ twice and use the continuity of $s'$ and $s$, and the fact that $s(x_i) = y_i$ to obtain

$$s(x) = \frac{M_{j-1}}{6h}(x_j - x)^3 + \frac{M_j}{6h}(x - x_{j-1})^3 + \frac{y_{j-1}}{h}(x_j - x) + \frac{y_j}{h}(x - x_{j-1}) - \frac{hM_{j-1}}{6}(x_j - x) - \frac{hM_j}{6}(x - x_{j-1})$$

For a general formula we can replace $h$ by $x_j - x_{j-1}$. (See text on page 150.)

The $M_j$ are obtained from the following $n-2$ equation

$$\frac{h}{6}M_{j-1} + \frac{2h}{3}M_j + \frac{h}{6} = \frac{y_{j+1} - y_j}{h} - \frac{y_j - y_{j-1}}{h}$$

and the two conditions form **S3** of the theorem, which in this case translate to

$$M_1 = 0, \qquad M_n = 0$$

.

**Example** For the data points $(1, 1),\ (2, 1/2),\ (3, 1/3),\ (4, 1/4)$, find the natural cubic spline.

**Solution:** Here $n = 4$ and $h = 1$. The last system of equation is then

$$\begin{array}{ccccccccc} \frac{1}{6}M_1 & + & \frac{2}{3}M_2 & + & \frac{1}{6}M_3 & & & = & \frac{1}{3} \\ & & \frac{1}{6}M_2 & + & \frac{2}{6}M_3 & + & \frac{1}{6}M_4 & = & \frac{1}{12} \end{array}$$

Since $M_1 = M =_4 = 0$, the above reduces to

$$\begin{array}{ccccc} \frac{2}{3}M_2 & + & \frac{1}{6}M_3 & = & \frac{1}{3} \\ \frac{1}{6}M_2 & + & \frac{2}{6}M_3 & = & \frac{1}{12} \end{array}$$

Solving this we get $M_2 = 1/2$ and $M_3 = 0$

Substituting $M_1 = M_3 = M_4 = 0$, $M_2 = 1/2$ and $y_1 = 1, ; y_2 = 1/2,\ y_3 = 1/3,\ y_4 = 1/4$ in the formula for $s(x)$ and simplifying we get

$$s(x) = \begin{cases} \frac{1}{12}x^3 - \frac{1}{4}x^2 - \frac{1}{3}x + \frac{1}{3}, & 1 \le x \le 2 \\ -\frac{1}{12}x^3 - \frac{3}{4}x^2 - \frac{7}{3}x + \frac{17}{6}, & 2 \le x \le 3 \\ -\frac{1}{12}x + \frac{7}{12}, & 3 \le x \le 4. \end{cases}$$