Paradoxical Euler: Integrating by Differentiating

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I. Introduction

Every student of calculus learns that one typically solves a differential equation by integrating it. However, as Euler shows in his 1758 paper (E236), *Exposition de quelques paradoxes dans le calcul intégral* (Explanation of certain paradoxes in integral calculus), there are differential equations that can be solved by actually differentiating them again. This seems paradoxical or as Euler describes it in the introduction of his paper ([1]):

> *Here I intend to explain a paradox in integral calculus that will seem rather strange: this is that we sometimes encounter differential equations in which it would seem very difficult to find the integrals by the rules of integral calculus yet are still easily found, not by the method of integration, but rather in differentiating the proposed equation again; so in these cases, a repeated differentiation leads us to the sought integral. This is undoubtedly a very surprising accident, that differentiation can lead us to the same goal, to which we are accustomed to find by integration, which is an entirely opposite operation.*

In this paper we explain Euler's paradoxical method and the geometrical problems that he poses (and solves) as applications of his new method. Moreover, we establish a theorem in Section II to mathematically characterize his method. In Section III we discuss a generalization of one of his problems to three dimensions and demonstrate how its solution, consisting of surfaces called *tangentially equidistant surfaces*, contains an interesting family of developable ruled surfaces.

II. Integrating by Differentiating

In his paper Euler presents four geometrical problems (I-IV) to demonstrate his paradoxical method of differentiation. We shall only treat Problems I and II. Problems III and IV are generalizations of Problems I and II and can be solved analogously. Here is Euler's statement of the first problem:

<u>PROBLEM I</u>

> *Given point A, find the curve EM such that the perpendicular AV, derived from point A onto some tangent of the curve MV, is the same size everywhere (Figure 1).*

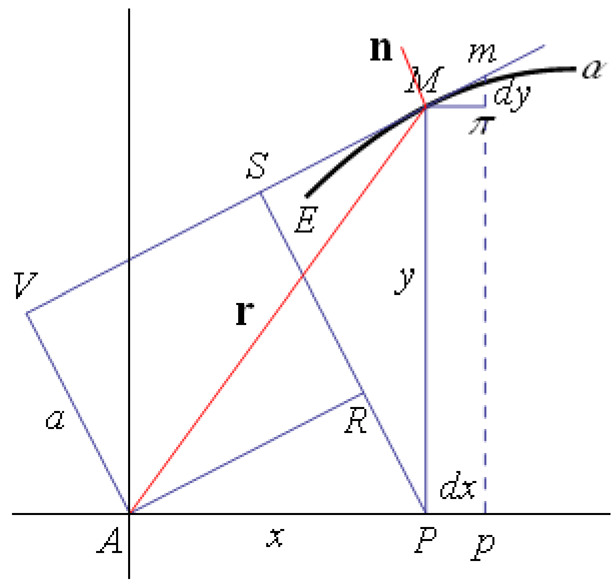

Figure 1

In modern terms, Problem I asks for a curve *EM* where every one of its tangent lines has fixed distance from point *A* (origin).

To solve this problem, Euler begins by introducing notation. In Figure 1, let α denote the curve *EM* and set $x = AP$, $y = PM$, $dx = Pp = Mm$, $dy = m\pi$, and $ds = Mm = \sqrt{dx^2 + dy^2}$. It follows from the similarity of the three triangles $\triangle APR$, $\triangle PMS$, and $\triangle Mm\pi$ that

$$\frac{PS}{PM} = \frac{M\pi}{Mm}, \quad \frac{PR}{AP} = \frac{m\pi}{Mm},$$

which implies

$$PS = \frac{PM \cdot M\pi}{Mm} = \frac{ydx}{ds}, \quad PR = \frac{AP \cdot m\pi}{Mm} = \frac{xdy}{ds}.$$

Thus,

$$a = AV = PS - PR = \frac{ydx - xdy}{ds},$$

or in differential form,

$$ydx - xdy = a\sqrt{dx^2 + dy^2}. \tag{1.1}$$

To solve (1.1), Euler applies the "ordinary" method of integrating a differential equation. Towards this end, he squares (1.1) to obtain

$$y^2dx^2 - 2xydxdy + x^2dy^2 = a^2dx^2 + a^2dy^2$$

and then solves for dy by extracting square roots:

$$dy = \frac{-xydx + a\,dx\sqrt{x^2 + y^2 - a^2}}{a^2 - x^2}.$$

This is equivalent to

$$a^2dy - x^2dy + xydx = a\,dx\sqrt{x^2 + y^2 - a^2}. \tag{1.2}$$

Next, Euler applies the substitution $y = u\sqrt{a^2 - x^2}$ to (1.2), and by assuming $x^2 \neq a^2$ (otherwise $y = 0$), he obtains

$$(a^2 - x^2)du = adx\sqrt{u^2 - 1}. \tag{1.3}$$

Now, it is straightforward to check that $u^2 = 1$ is a solution to (1.3) since $du = 0$. Therefore,

$$y = \pm\sqrt{a^2 - x^2},$$

or upon squaring both sides, yields the circle of radius *a* centered at the origin as the solution (see Figure 2):

$$x^2 + y^2 = a^2. \tag{1.4}$$

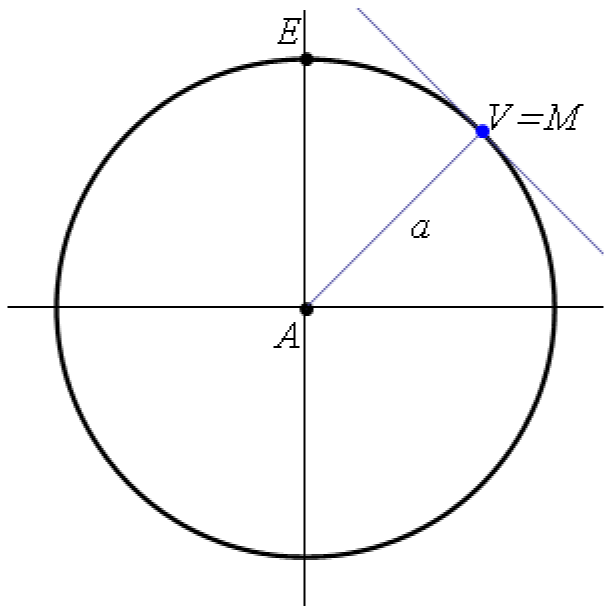

Figure 2

On the other hand, if $u^2 \neq 1$, then (1.3) can be separated as

$$\frac{du}{\sqrt{u^2 - 1}} = \frac{adx}{a^2 - x^2},$$

and upon integration Euler reveals a family of lines as the other solution to Problem I:

$$y = \frac{(n^2 - 1)}{2n}x + \frac{(n^2 + 1)}{2n}a. \tag{1.5}$$

Here, *n* is a constant of integration. Observe that the lines described by (1.5) are all tangent to the circle in (1.4) (see Figure 3) and reveals the circle as their envelope (see Figure 4).

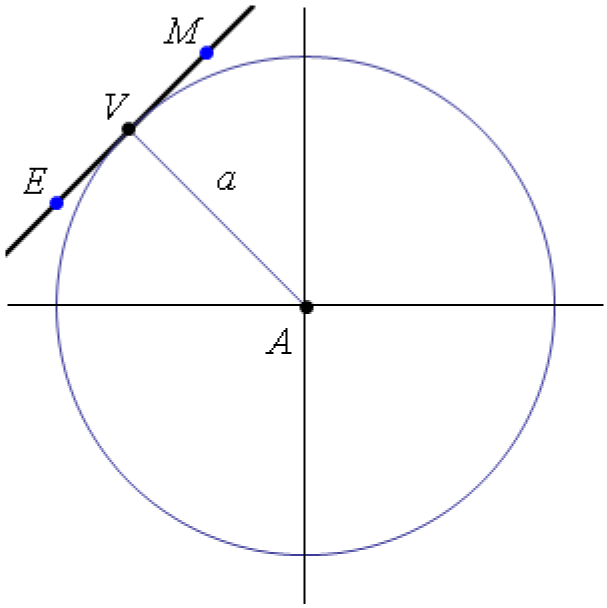

Figure 3

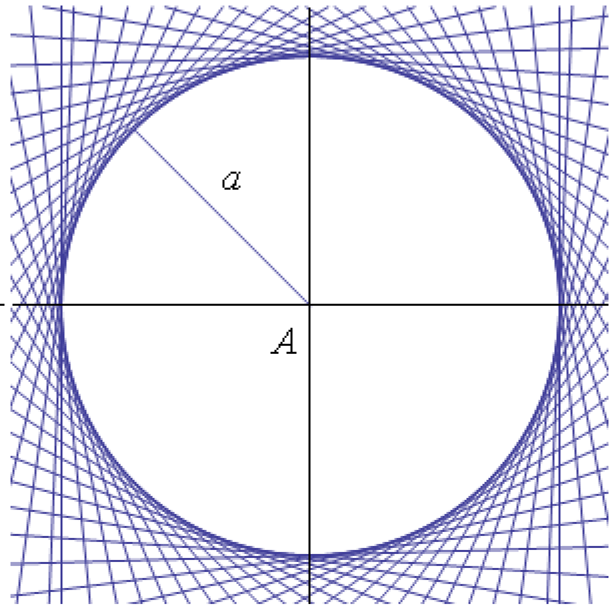

Figure 4

Having now solved Problem I by the traditional method of integration, Euler then points out that this technique in many cases is quite inefficient and impractical. For example, to separate variables in the third-order equation

$$ydx - xdy = a\sqrt[3]{dx^3 + dy^3}\ , \tag{1.6}$$

one would need to extract cube roots – not an easy task. Moreover, this would certainly not be possible in the general case

$$y\,dx - x\,dy = a\sqrt[n]{\sum_{k=0}^{n} c_k dx^{n-k} dy^k}\ , \tag{1.7}$$

where the constants c_k are arbitrary.

Euler proceeds to solve (1.1) again, but this time using his novel method of differentiation. Towards this end, he rewrites (1.1) in the form

$$y = px + a\sqrt{1+p^2} \tag{1.8}$$

where $p = dy/dx$. Then differentiating (1.8) yields

$$dy = xdp + pdx + \frac{ap}{\sqrt{1+p^2}}dp\,,$$

which simplifies to

$$0 = x\,dp + \frac{ap\,dp}{\sqrt{(1+p^2)}}\ .$$

because $dy = pdx$. Assuming $dp \neq 0$, Euler concludes that

$$x = -\frac{ap}{\sqrt{(1+p^2)}}$$

and

$$y = px + a\sqrt{1+p^2} = -\frac{ap^2}{\sqrt{(1+p^2)}} + a\sqrt{(1+p^2)} = \frac{a}{\sqrt{(1+p^2)}}.$$

To eliminate the parameter p in the solution above for x and y, he sums their squares to obtain the same circle found in (1.4):

$$x^2 + y^2 = \frac{a^2 p^2 + a^2}{1+p^2} = a^2.$$

On the other hand, if $dp = 0$, then $p = dy/dx = m$, a constant. This yields the linear solution

$$y = mx + a\sqrt{(1+m^2)}, \tag{1.9}$$

which agrees with the solution previously found in (1.5) upon setting $m = (n^2-1)/(2n)$.

To emphasize the usefulness of his new method, Euler then demonstrates how (1.6) can also be solved with ease by rewriting it in the form

$$y - px = a\sqrt[3]{(1+p^3)} \tag{1.10}$$

Differentiating (1.10) now yields

$$dy = p\,dx + x\,dp + \frac{ap^2\,dp}{\sqrt[3]{(1+p^3)^2}},$$

which reduces it to

$$0 = x\,dp + \frac{ap^2\,dp}{\sqrt[3]{(1+p^3)^2}}.$$

As before, by assuming $dp \neq 0$, Euler is able to solve for x and y:

$$\begin{aligned} x &= \frac{-ap^2}{\sqrt[3]{(1+p^3)^2}}, \\ y &= \frac{a}{\sqrt[3]{(1+p^3)^2}}. \end{aligned} \tag{1.11}$$

To eliminate p here, Euler sums the cube powers of x and y to obtain

$$y^3 + x^3 = \frac{a^3(1-p^6)}{(1+p^3)^2} = \frac{a^3(1-p^3)}{1+p^3} = -a^3 + \frac{2a^3}{1+p^3},$$

which allows him to solve for

$$\frac{1}{1+p^3} = \frac{a^3+x^3+y^3}{2a^3}.$$

Thus,

$$y = \frac{a}{\sqrt[3]{(1+p^3)^2}} = \frac{(a^3+x^3+y^3)^{2/3}}{a\sqrt[3]{4}}, \tag{1.12}$$

or equivalently,

$$4a^3y^3 = (a^3+x^3+y^3)^2. \tag{1.13}$$

On the other hand, if we require $dp = 0$, then by the same argument $p = dy/dx = m$, a constant. This produces the other solution:

$$y = mx + a\sqrt[3]{1+m^3}\,. \qquad (1.14)$$

Of course Euler does not stop here but proceeds to demonstrate the solution for the general case given by (1.7). We on the other hand shall not following him in this regard, but instead establish the even more general result:

Theorem: If

$$ydx - xdy = F(p)dx\,, \qquad (1.15)$$

where $p = dy/dx$ and $F(p)$ is a differentiable function of *p* with $dp \neq 0$, then

$$\begin{aligned} x &= -F'(p) \\ y &= F(p) - pF'(p) \end{aligned} \qquad (1.16)$$

Conversely, if $x = f(p)$ and $y = g(p)$ where $p = dy/dx$, $dp \neq 0$, and *f* (*p*) and *g*(*p*) are differentiable functions of *p*, then (1.15) and (1.16) hold with

$$F(p) = -\int f(p)dp \qquad (1.17)$$

To prove this theorem, we rewrite (1.15) in the form

$$y = xp + F(p) \qquad (1.18)$$

and differentiate it to get

$$dy = pdx + xdp + F'(p)dp\,. \qquad (1.19)$$

Then recognizing that $dy = pdx$, (1.19) simplifies to

$$0 = x\frac{dp}{dx} + F'(p)\frac{dp}{dx}\,. \qquad (1.20)$$

Assuming $dp \neq 0$, we obtain the parametric solution

$$\begin{aligned} x &= -F'(p) \\ y &= F(p) - pF'(p) \end{aligned}$$

as desired. On the other hand, if $dp = 0$, then $p = dy/dx = m$, a constant. Thus,

$$y = mx + F(m)\,. \qquad (1.21)$$

Conversely, suppose $x = f(p)$ and $y = g(p)$ where $p = dy/dx$, $dp \neq 0$, and *f* (*p*) and *g*(*p*) are differentiable functions. It is then easy to see that $dx = f'(p)dp$ and $dy = pdx$, therefore $dy = pf'(p)dp$. Using integration by parts, we further see that

$$y = pf(p) - \int f(p)dp\,.$$

By making the substitution $F(p) = -\int f(p)dp$, we have

$$\begin{aligned} x &= f(p) = -F'(p), \\ y &= g(p) = -pF'(p) + F(p). \end{aligned} \qquad (1.22)$$

Hence $g(p) - pf(p) = F(p)$, or equivalently,

$$ydx - xdy = F(p)dx\,.$$

As an application of this Theorem, suppose we modify Problem I to require that the distance *a* be proportional to *ds* (infinitesimal arclength), i.e.

$$\frac{ydx - xdy}{ds} = a = kds,$$

where k is the proportionality constant. The corresponding differential equation in this case takes the form

$$ydx - xdy = k(dx^2 + dy^2), \tag{1.23}$$

or equivalently,

$$ydx - xdy = k(1 + p^2)dx \tag{1.24}$$

where $p = dy/dx$. It follows from the Theorem with $F(p) = k(1+p^2)$ that

$$\begin{aligned} x &= -F'(p) = -2kp, \\ y &= -pF'(p) + F(p) = k(1-p^2). \end{aligned} \tag{1.25}$$

The solution is thus a parabola:

$$y = k\left(1 - \frac{x^2}{4k^2}\right). \tag{1.26}$$

We now move on to Euler's second problem:

PROBLEM II

On the axis AB, find the curve AMB such that having derived from one of its points M the tangent TMV, it intersects the two straight lines AE and BF, derived perpendicularly to the axis AB at the two given points A and B, so that the rectangle formed by the lines AT and BV is the same size everywhere. (Fig. 5)

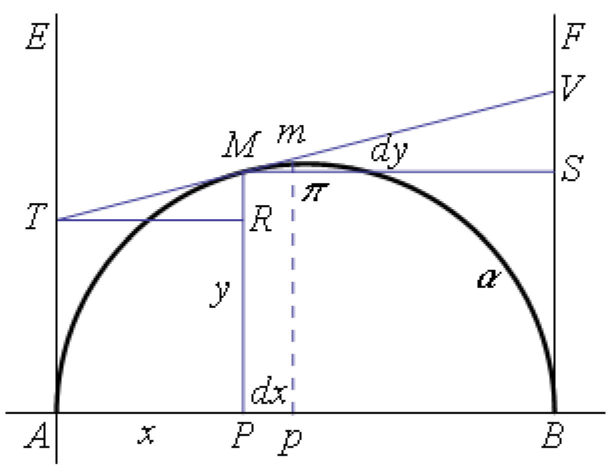

Figure 5

As in Problem I, Euler defines $AP = x$, $PM = y$, $Pp = M\pi = dx$, and $\pi m = dy$ (see Figure 5). Moreover, he defines the interval $AB = 2a$, which is fixed. Since the triangles $\Delta M\pi m$, ΔTRM, and ΔMSV are all similar to each other, it follows that

$$RM = \frac{x\,dy}{dx} \text{ and } SV = \frac{(2a-x)\,dy}{dx},$$

where Euler has used the fact that $MS = AB - AP = 2a - x$. Moreover,

$$AT = y - \frac{x\,dy}{dx} \text{ and } BV = y + \frac{(2a-x)\,dy}{dx}.$$

As the product $AT \cdot BV$ is required to be constant, the corresponding differential equation becomes

$$(y - \frac{x\,dy}{dx})(y - \frac{x\,dy}{dx} + \frac{2a\,dy}{dx}) = c^2.$$

Again, Euler mentions that the differential equation above is quite difficult to solve by the ordinary method of integration and instead applies his method of differentiation to further demonstrate its usefulness. As before, he first rewrites the equation in terms of $p = dy/dx$, resulting in

$$(y - px)(y - px + 2ap) = c^2.$$

Then completing the square in y yields the positive root

$$y = -(a-x)p + \sqrt{c^2 + a^2p^2} \tag{1.27}$$

This form now allows Euler to apply his method of differentiation, which yields

$$dy = -(a-x)\,dp + p\,dx + \frac{a^2p\,dp}{\sqrt{(c^2+a^2p^2)}}.$$

Again, cancellation of the terms dy and pdx allows Euler to solve for x (assuming $dp \neq 0$) and thus y from (1.27):

$$x = a - \frac{a^2p}{\sqrt{(c^2+a^2p^2)}},$$

$$y = \frac{-a^2p^2}{\sqrt{(c^2+a^2p^2)}} + \sqrt{(c^2+a^2p^2)} = \frac{c^2}{\sqrt{(c^2+a^2p^2)}}.$$

To eliminate the parameter p, Euler sums the squares of x and y, producing

$$\frac{(a-x)^2}{a^2} + \frac{y^2}{c^2} = \frac{a^2p^2+c^2}{c^2+a^2p^2} = 1,$$

which represents an ellipse with semi-axes of length a and c. This is illustrated in Figure 6.

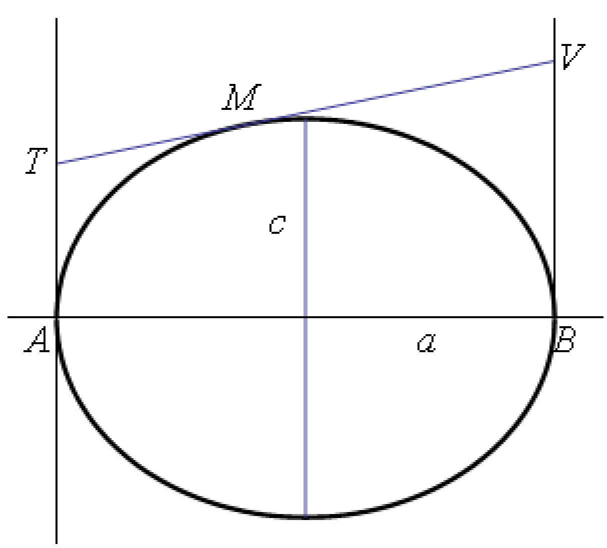

Figure 6

On the other hand, if $dp = 0$, then $p = n$ (constant) and thus the corresponding solution is the family of tangent lines (see Figures 7 and 8)

$$y = -n(a - x) + \sqrt{c^2 + n^2 a^2}\,.$$

Moreover, setting $x = 0$ and $x = 2a$ yields

$$AT = -na + \sqrt{(c^2 + n^2 a^2)} \text{ and } BV = na + \sqrt{(c^2 + n^2 a^2)},$$

respectively, which confirms that $AT \cdot BV = c^2$ is a fixed value. Observe that when $n = 0$, then TV is horizontal and thus $AT = BV = c$.

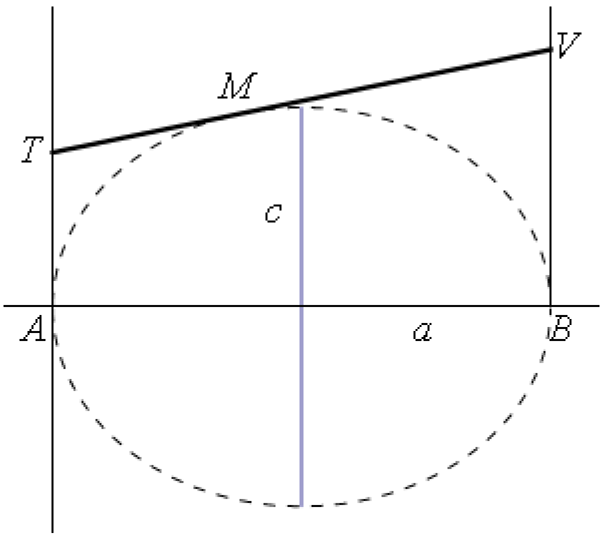

Figure 7

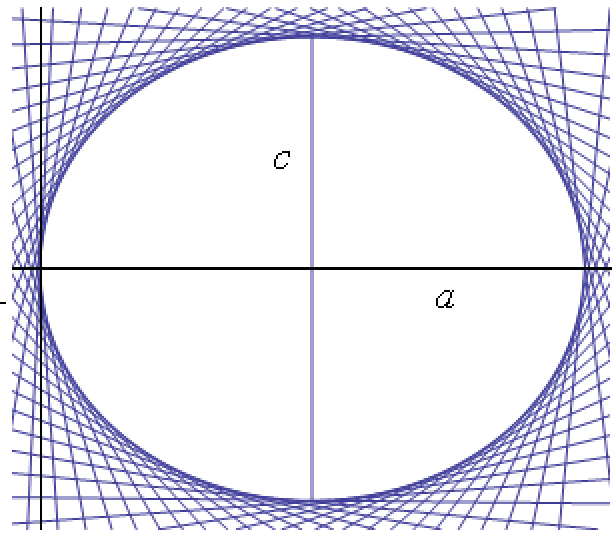

Figure 8

III. Generalization to Three Dimensions

The modern approach to deriving the differential equation for Problem I is to employ vectors. In particular, the value $a = AV$ can be viewed as the projection of the position vector $\mathbf{r} = \overrightarrow{AM} = (x, y)$ onto the normal vector $\mathbf{n} = (-dy, dx)$ for the tangent line (see Figure 9), i.e.

$$\frac{\mathbf{r} \bullet \mathbf{n}}{|\mathbf{n}|} = a\,.$$

It follows that

$$\frac{-xdy + ydx}{\sqrt{dx^2 + dy^2}} = a\,,$$

which is equivalent to (1.1).

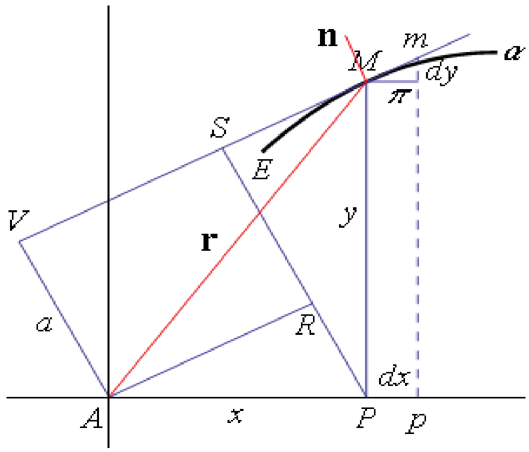

Figure 9

Problem I can now be generalized to three dimensions as follows:

PROBLEM I-3D

Determine a surface M whose tangent plane at every point P has constant distance k from the origin (Figure 10).

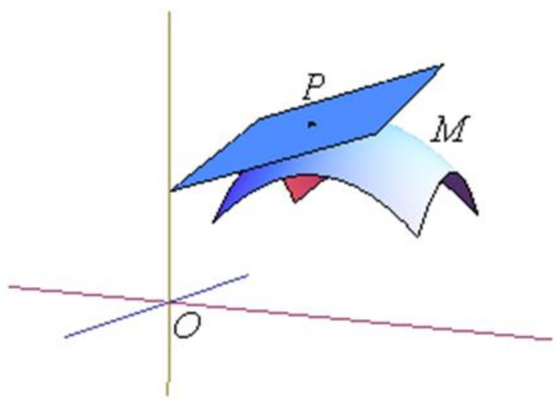

Figure 10

Let us call M a *tangentially equidistant (TED) surface of distance k*. To derive the corresponding differential equation for TED surfaces, we again view the distance k as the projection of the position vector $\mathbf{r} = \overrightarrow{OP} = (x, y, z)$ onto the normal vector $\mathbf{n} = (-\partial z / \partial x, -\partial z / \partial y, 1)$ for the tangent plane at P:

$$\frac{\mathbf{r} \bullet \mathbf{n}}{|\mathbf{n}|} = k \ .$$

It follows that S is modeled by the following nonlinear partial differentiation equation:

$$z - x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = k\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \ . \tag{1.28}$$

Using our intuition from Problem I, it is clear that (1.28) should have two types of solutions: the sphere $S^2(k)$ of radius k centered at the origin, i.e. $x^2+y^2+z^2=k^2$, and every one of its tangent planes. However, there is a third family of solutions that is quite interesting and consists of developable ruled surfaces generated from spherical curves lying on $S^2(k)$.

To derive these three families of solutions, denote by $p=\partial z/\partial x$ and $q=\partial z/\partial y$ so that (1.28) becomes

$$z-xp-yq=k\sqrt{1+p^2+q^2}\,. \tag{1.29}$$

Then following Euler we differentiate (1.29) with respect to x yields

$$\frac{\partial z}{\partial x}-p-x\frac{\partial p}{\partial x}-y\frac{\partial q}{\partial x}=\frac{k}{\sqrt{1+p^2+q^2}}\left(p\frac{\partial p}{\partial x}+q\frac{\partial q}{\partial x}\right). \tag{1.30}$$

Since $p=\partial z/\partial x$, (1.30) simplifies to

$$x\frac{\partial p}{\partial x}+y\frac{\partial q}{\partial x}=-\frac{k}{\sqrt{1+p^2+q^2}}\left(p\frac{\partial p}{\partial x}+q\frac{\partial q}{\partial x}\right). \tag{1.31}$$

Similarly, differentiating (1.29) with respect to y yields

$$x\frac{\partial p}{\partial y}+y\frac{\partial q}{\partial y}=-\frac{k}{\sqrt{1+p^2+q^2}}\left(p\frac{\partial p}{\partial y}+q\frac{\partial q}{\partial y}\right). \tag{1.32}$$

CASE I: Assume the partial derivatives for p and q to be non-zero:

$$\partial p/\partial x\neq 0,\quad \partial p/\partial y\neq 0,\quad \partial q/\partial x\neq 0,\quad \partial q/\partial y\neq 0\,.$$

Then equating coefficients for these partial derivatives on the left and right hand sides of (1.31) and (1.32) yields the following solution:

$$\begin{aligned} x&=-\frac{kp}{\sqrt{1+p^2+q^2}},\\ y&=-\frac{kq}{\sqrt{1+p^2+q^2}},\\ z&=xp+yq+k\sqrt{1+p^2+q^2}=\frac{k}{\sqrt{1+p^2+q^2}}, \end{aligned} \tag{1.33}$$

which represents a sphere of radius k (Figure 11):

$$x^2+y^2+z^2=k^2\,.$$

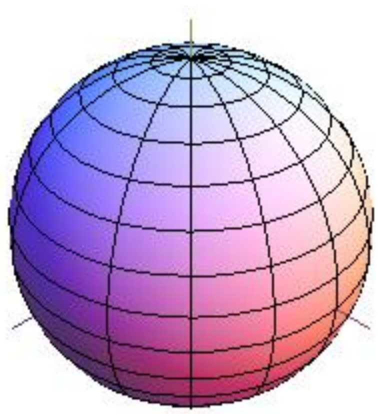

Figure 11

CASE II: Assume all four partial derivatives vanish identically:

$$\partial p/\partial x = 0, \quad \partial p/\partial y = 0, \quad \partial q/\partial x = 0, \quad \partial q/\partial y = 0\,.$$

It follows that p and q are both constant, say $p = m$ and $q = n$. Thus, we obtain a family of planes as our second solution set:

$$z = mx + ny + k\sqrt{1+m^2+n^2}\,. \tag{1.34}$$

Ruled TED Surfaces

In this section we present a third family of TED surfaces and demonstrate how they can be constructed as ruled surfaces generated from spherical curves. A surface M is called a *ruled surface* if it has a coordinate patch $x : D \subset \mathbb{R}^2 \to M \subset \mathbb{R}^3$ of the form (see [2])

$$\mathbf{x}(u,v) = \beta(u) + v\delta(u)\,. \tag{1.35}$$

Here, $\beta(u)$ and $\delta(u)$ are curves in $\mathbb{R}^3$ and the surface S can be viewed as consisting of lines emanating from $\beta(u)$ (directrix) and moving in the direction $\delta(u)$ (ruling). To obtain ruled TED surfaces, we restrict β to being a spherical curve lying on $S^2(k)$. Since $S^2(k)$ is an equidistant surface, it follows that $\mathbf{x}(u,v)$ describes an TED surface M if every tangent plane of M is also a tangent plane of $S^2(k)$. This holds if both parameter tangent vectors

$$\mathbf{x}_u(u,v) = \beta'(u) + v\delta'(u)$$
$$\mathbf{x}_v(u,v) = \delta(u)$$

lie on the tangent plane $T_{\beta(u)}S^2(k)$, or equivalently, if $\delta(u) \in T_{\beta(u)}S^2(k)$ and all three vectors $\beta'(u)$, $\delta(u)$, and $\delta'(u)$ are coplanar . In that case the unit normal

$$U = \frac{x_u \times x_v}{|x_u \times x_v|}$$

for M does not depend on v since T_pM is constant in the v-direction and so the normal curvature of M is zero in the same direction. Thus, M is a developable surface, i.e. a surface having zero Gaussian curvature. We summarize this formally in the following theorem.

Theorem: Let M be a ruled surface having a coordinate patch of the form

$$\mathbf{x}(u,v) = \beta(u) + v\delta(u)\,,$$

where β is a spherical curve on $S^2(k)$ and $\delta(u) \in T_{\beta(u)}S^2(k)$. If $\beta'(u)$, $\delta(u)$, and $\delta(u)$ are coplanar, then $T_{\mathbf{x}(u,v)}M = T_{\beta(u)}S^2(k)$ and thus *M* is a developable ruled TED surface of distance *k*.

To construct such surfaces based on our theorem, define

$$\delta(u) = \beta(u) \times \beta'(u)\,.$$

We claim that this choice of δ yields a developable TED surface *M* defined by (1.35). To prove this, first observe that $\mathbf{x}_v = \delta(u) \in T_{\beta(u)}S^2(k)$ since $\delta(u)$ is perpendicular to $\beta(u)$ and thus perpendicular to the unit normal $U = \beta(u)/|\beta(u)|$ for $S^2(k)$. To prove that $\beta'(u)$, $\delta(u)$, and $\delta'(u)$ are coplanar, we will show that their scalar triple product vanishes. Towards this end recall that $\beta(u)$ is perpendicular to $\beta'(u)$ since β has constant distance *k* from the origin and so β, β', δ are mutually orthogonal. It follows that

$$\beta'(u) \times \delta(u) = |\beta'(u)|^2\, \beta(u)\,.$$

Thus,

$$\delta' \bullet (\beta' \times \delta) = (\beta' \times \beta' + \beta \times \beta'') \bullet (|\beta|^2\, \beta) = (\beta \times \beta'') \bullet (|\beta|^2\, \beta) = 0\,.$$

This proves that the vectors $\beta'(u)$, $\delta(u)$, and $\delta'(u)$ are coplanar. Thus, by our theorem *M* is a developable ruled TED surface of distance *k*.

Let us now finish our discussion by considering a couple of examples of developable ruled TED surfaces generated from our construction.

Example 1: Assume β is a parallel (latitude) of $S^2(k)$ of the form

$$\beta(u) = (\cos u \cos v_0, \sin u \cos v_0, \sin v_0)\,,$$

where v_0 is its latitude. Then the corresponding developable ruled TED surface *M* is a cone circumscribing the sphere (Figure 12), unless β is an equator ($v_0 = 0$), in which case *M* is a cylinder (Figure 13).

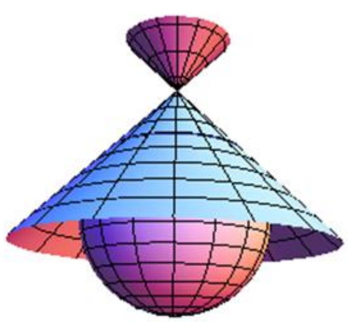

Figure 12

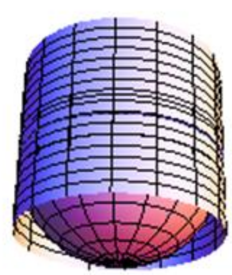

Figure 13

Example 2: Assume β is the spherical figure-8 curve (see Figure 14) given by

$$\beta(u) = (\sin u \cos u, \sin^2 u, \cos u)$$

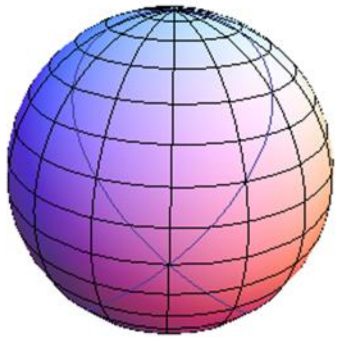

Figure 14

Then the corresponding developable ruled TED surface M is shown in Figures 15 and 16 (side views) circumscribing the sphere.

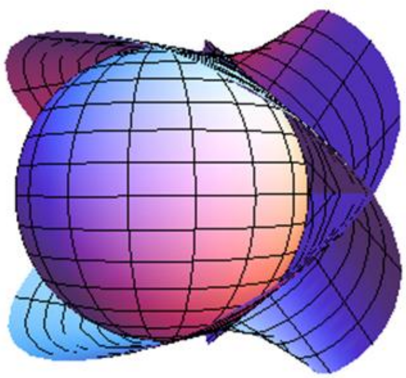

Figure 15

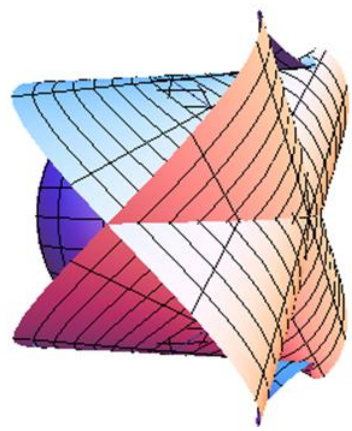

Figure 16

Observe that other TED surfaces can be constructed by taking any region ΔS of the sphere $S^2(k)$ and attaching to it the developable ruled TED surface corresponding to the boundary of ΔS (assumed to be a simple closed spherical curve). One such example is the silo surface obtained as the union of the upper hemisphere and the cylinder generated as a ruled surface from the circular boundary (equator) of the hemisphere (see Figure 17).

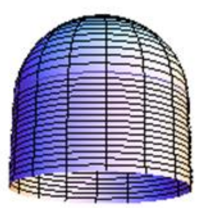

Figure 17

We conclude by asking whether the converse holds true, i.e. whether every TED surface must either be the sphere of radius k, a developable ruled surface, or unions of developable ruled TED surfaces with regions of the sphere. Our intuition says that it should be true but we have not been able to prove this. Of course, counterexamples are most welcome!

References

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