Sur l'accord des deux dernieres eclipses du soleil et de la lune avec mes tables, pour trouver les vrais momens des pleni-lunes et novi-lunes

Synopsis of Leonard Euler's E141 Paper

Andrew Fabian and Hieu D. Nguyen

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ON THE AGREEMENT
OF THE LAST TWO ECLIPSES OF THE SUN
AND MOON WITH MY TABLES, FOR FINDING THE ACTUAL TIMES OF THE HALF-MOON AND NEW MOON

BY MR. EULER

In this 1750 paper Leonard Euler comments on his predictions from a previous Memoir ([E117]) regarding the times and durations of two eclipses that occurred in 1748: the solar eclipse on July 25 and the lunar eclipse between August 8 and 9. In the first section Euler discusses his accurate prediction for the solar eclipse. In particular his calculations for the beginning and end times of the solar eclipse are in close agreement with observation, but not for the duration of the annulus. He corrects his error for this duration by using a more accurate measurement for the latitude of Berlin recently obtained by Maupertuis. In the second section Euler discusses his somewhat inaccurate prediction for the lunar eclipse. He is surprised by this, especially since he had updated his astronomical tables for many other lunar eclipses, and devotes the remainder of the paper (section III-XV) towards reworking his calculations to obtain values that are in much closer agreement with observation.

I. In this section Euler discusses the accuracy of his prediction for the solar eclipse on July 25, 1748 based on observation. He points out that his time predictions for the beginning and end of the eclipse were accurate, but not the duration of the annulus, in comparison with the observed times, as seen in the table below:

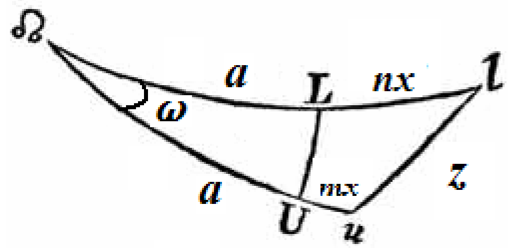
Solar Eclipse – July 25, 1748	Euler's Prediction	Observation	Difference
Beginning Time	10^h, 17′, 45″	< 10^h, 18′	-15″
Ending Time	1^h, 24′, 0″	1^h, 24′, 30″	-30″
Duration Annulus	5′, 10″	1′, 20″	+3′ 50″

Euler then explains the source of his error for the duration of the annulus as stemming from the latitude of Berlin, which he places too far north by 4′, 30″. He argues that if the more recent observed calculation for the latitude of Berlin were used instead (52°, 31′, 30″), then he would have obtained a much more accurate prediction.

II. In this section Euler mentions that his predictions for the lunar eclipse on December 8-9, 1748 are not in agreement with observation. His Astronomical Almanac had predicted that it begin at 11^h, 0′, 14″ (11:00 PM) on December 8 and end at 1^h, 14′, 4″ (1:14 AM) on December 9, yet it was observed to begin at 11^h, 5′ (11:05 PM) and ended at 1^h, 18′ (1:18 AM). He is surprised by this since he had earlier corrected his astronomical tables for many earlier lunar eclipses. Euler therefore devotes the rest of the paper in reworking his calculations to see if there were any errors in his original calculations. In the end he obtains a more accurate prediction (in hindsight) but does not point out any of these errors (if any) and so it is unclear where his calculations were improved upon.

III - VIII. In these sections Euler provides tables of astronomical values relating to the positions and movements of the Sun and Moon that will be used in his calculations in later sections.

IX. **Finding a relation for *UL*:**



To the right is a more complete figure of the problem. The known values are denoted by *a*, *n*, *m*, and ω. Euler proceeds to find relations for the distances *UL* and *z* and the elapsed time *x* in terms of these known values.

To obtain a relation for *UL* at the moment of opposition, Euler begins by applying the *Law of Cosines* for spherical trigonometry, which states that in a spherical triangle with sides *a*, *b*, *c* and an angle *C* opposite to side *c*, we have

$$\cos c = \cos a \cos b + \sin a \sin b \cos C\,.$$

This is applied to the spherical triangle $\triangle \Omega UL$ in the figure, which gives the relation

$$\cos UL = \cos\omega \sin^2 a + \cos^2 a\,. \tag{1.1}$$

The *Pythagorean Identity* states that $\cos^2\theta + \sin^2\theta = 1$, thus $\cos^2 a = 1 - \sin^2 a$. Hence, equation (1.1) can be rewritten as

$$\cos UL = 1 - \sin^2 a\,(1-\cos\omega)\,. \tag{1.2}$$

But the *Double Angle Formula* for cosine states that $\cos(2\theta) = 1 - 2\sin^2\theta$, thus $1-\cos\omega = 2\sin^2(\tfrac{1}{2}\omega)$. Hence, equation (1.2) becomes

$$2\sin^2(\tfrac{1}{2}UL) = 2\sin^2 a \sin^2(\tfrac{1}{2}\omega)\,,$$

which finally gives

$$\sin(\tfrac{1}{2}UL) = \sin a \sin(\tfrac{1}{2}\omega)\,. \tag{1.3}$$

X. **Finding a relation for *z* in terms of the variable *x*:**

Now that Euler has solved for the simple case at the moment of opposition, he now proceeds to solve for *z*, the distance between the centers *x* hours after the opposition. Euler begins by again applying the *Law of Cosines* to the spherical triangle $\triangle \Omega ul$, giving

$$\cos z = \cos\omega \sin(a+mx)\sin(a+nx) + \cos(a+mx)\cos(a+nx)\,. \tag{1.4}$$

Euler then states the *Product-to-Sum Formulas*:

$$\sin b \sin c = \tfrac{1}{2}\cos(b-c) - \tfrac{1}{2}\cos(b+c)\,,$$
$$\cos b \cos c = \tfrac{1}{2}\cos(b-c) + \tfrac{1}{2}\cos(b+c)\,.$$

Making these substitutions into (1.4) and simplifying gives

$$\begin{aligned}\cos z = {} & \tfrac{1}{2}\cos\omega\cos((n-m)x) - \tfrac{1}{2}\cos\omega\cos(2a+(n+m)x) \\ & + \tfrac{1}{2}\cos((n-m)x) + \tfrac{1}{2}\cos(2a+(n+m)x)\,.\end{aligned}$$

Next, Euler groups like terms to obtain

$$\cos z = \tfrac{1}{2}\cos((n-m)x)(1+\cos\omega) + \tfrac{1}{2}\cos(2a+(n+m)x)(1-\cos\omega)\,,$$

but because of the *Half Angle Formulas* this reduces to

$$\cos z = \cos((n-m)x)\cos^2(\tfrac{1}{2}\omega) + \cos(2a+(n+m)x)\sin^2(\tfrac{1}{2}\omega)\,. \tag{1.5}$$

From his previous reasoning $\cos z = 1 - 2\sin^2(\tfrac{1}{2}z)$ and $\cos^2(\tfrac{1}{2}\omega) = 1-\sin^2(\tfrac{1}{2}\omega)$, thus (1.5) becomes

$$\begin{aligned}1-2\sin^2(\tfrac{1}{2}z) = {} & \cos((n-m)x) - \sin^2(\tfrac{1}{2}\omega)\cos((n-m)x) \\ & + \sin^2(\tfrac{1}{2}\omega)\cos(2a+(n+m)x)\end{aligned} \tag{1.6}$$

Euler then states the *Sum-Difference Formula* for $\cos(2a+(n+m)x)$:

$$\cos(2a+(n+m)x) = \cos(2a)\cos((n+m)x) - \sin(2a)\sin((n+m)x)\,,$$

which applied to (1.6) gives

$$\begin{aligned}1-2\sin^2(\tfrac{1}{2}z) = {} & \cos((n-m)x) - \sin^2(\tfrac{1}{2}\omega)\cos((n-m)x) \\ & + \cos(2a)\sin^2(\tfrac{1}{2}\omega)\cos((n+m)x) \\ & - \sin(2a)\sin^2(\tfrac{1}{2}\omega)\sin((n+m)x)\end{aligned}\,. \tag{1.7}$$

Euler then uses the approximations

$$\cos((n-m)x) \approx 1 - \tfrac{1}{2}(n-m)^2x^2\,,$$
$$\cos((n+m)x) \approx 1 - \tfrac{1}{2}(n+m)^2x^2\,,$$

and

$$\sin((n+m)x) \approx (n+m)x\,,$$

since the angles are relatively small, to simplify (1.7):

$$\begin{aligned}4\sin^2(\tfrac{1}{2}z) = {} & (n-m)^2x^2 + 2\sin^2(\tfrac{1}{2}\omega) - \sin^2(\tfrac{1}{2}\omega)(n-m)^2x^2 \\ & - 2\cos(2a)\sin^2(\tfrac{1}{2}\omega) + (n+m)^2x^2\cos(2a)\sin^2(\tfrac{1}{2}\omega) \\ & + 2\sin(2a)(n+m)x\sin^2(\tfrac{1}{2}\omega),\end{aligned} \tag{1.8}$$

but the *Double Angle Formula* for sine states that $\sin(2\theta) = 2\sin\theta\cos\theta$, thus equations (1.8) becomes

$$\begin{aligned}4\sin^2(\tfrac{1}{2}z) = {} & (n-m)^2x^2 + 2\sin^2(\tfrac{1}{2}\omega) - \sin^2(\tfrac{1}{2}\omega)(n-m)^2x^2 \\ & - 2\cos(2a)\sin^2(\tfrac{1}{2}\omega) + (n+m)^2x^2\cos(2a)\sin^2(\tfrac{1}{2}\omega) \\ & + 4(n+m)x\sin a\cos a\sin^2(\tfrac{1}{2}\omega).\end{aligned}$$

Grouping like terms gives

$$\begin{aligned}4\sin^2(\tfrac{1}{2}z) = {} & (n-m)^2x^2(1-\sin^2(\tfrac{1}{2}\omega)) + 2\sin^2(\tfrac{1}{2}\omega)(1-\cos(2a)) \\ & + (n+m)^2x^2\cos(2a)\sin^2(\tfrac{1}{2}\omega) + 4(n+m)x\sin a\cos a\sin^2(\tfrac{1}{2}\omega)\,,\end{aligned}$$

but from previously $1-\sin^2(\tfrac{1}{2}\omega) = \cos^2(\tfrac{1}{2}\omega)$ and $1-\cos(2a) = 2\sin^2 a$, thus yielding the first result of this section:

$$\begin{aligned}4\sin^2(\tfrac{1}{2}z) &= 4\sin^2 a\sin^2(\tfrac{1}{2}\omega)+(n-m)^2x^2\cos^2(\tfrac{1}{2}\omega)\\ &\quad+(n+m)^2x^2\cos(2a)\sin^2(\tfrac{1}{2}\omega)+4(n+m)x\sin a\cos a\sin^2(\tfrac{1}{2}\omega)\end{aligned}. \quad (1.9)$$

Another *Double Angle Formula* for cosine states that $\cos(2\theta)=\cos^2\theta-\sin^2\theta$, thus (1.9) becomes

$$\begin{aligned}4\sin^2(\tfrac{1}{2}z) &= 4\sin^2 a\sin^2(\tfrac{1}{2}\omega)+(n-m)^2x^2\cos^2(\tfrac{1}{2}\omega)+(n+m)^2x^2\cos(2a)\sin^2(\tfrac{1}{2}\omega)\\ &\quad+(n+m)^2x^2\sin^2 a\sin^2(\tfrac{1}{2}\omega)+4(n+m)x\sin a\cos a\sin^2(\tfrac{1}{2}\omega).\end{aligned}$$

Euler then perceives that $(4\sin^2 a+4(n+m)x\sin a\cos a+(n+m)^2x^2\cos^2 a)\sin^2(\tfrac{1}{2}\omega)=$ $(2\sin a+(n+m)x\cos a)^2\sin^2(\tfrac{1}{2}\omega)$, hence finally yielding the second result of the section:

$$\begin{aligned}4\sin^2(\tfrac{1}{2}z) &= (2\sin a+(n+m)x\cos a)^2\sin^2(\tfrac{1}{2}\omega)+(n-m)^2x^2\cos^2(\tfrac{1}{2}\omega)\\ &\quad-(n+m)^2x^2\sin^2 a\sin^2(\tfrac{1}{2}\omega)\end{aligned} \quad (1.10)$$

XI. **Finding *x* and plugging it into the relation for *z*:**

Euler begins by minimizing the first result of part X by differentiating with respect to *x* and equating to zero, which gives

$$(n-m)^2x\cos^2(\tfrac{1}{2}\omega)+(n+m)^2x\cos(2a)\sin^2(\tfrac{1}{2}\omega)+2(n+m)\sin a\cos a\sin^2(\tfrac{1}{2}\omega)=0.$$

Then solving this equation for *x* yields

$$x=\frac{-(n+m)\sin(2a)\sin^2(\tfrac{1}{2}\omega)}{(n-m)^2\cos^2(\tfrac{1}{2}\omega)+(n+m)^2\cos(2a)\sin^2(\tfrac{1}{2}\omega)}. \quad (1.11)$$

Substituting solution (1.11) for *x* into (1.10), the second result of part X, gives

$$\sin(\tfrac{1}{2}z)=\sin a\sin(\tfrac{1}{2}\omega)\sqrt{\frac{(n-m)^2\cos^2(\tfrac{1}{2}\omega)-(n+m)^2\sin^2 a\sin^2(\tfrac{1}{2}\omega)}{(n-m)^2\cos^2(\tfrac{1}{2}\omega)+(n+m)^2\cos(2a)\sin^2(\tfrac{1}{2}\omega)}}. \quad (1.12)$$

Observe that this substitution is not as simple as Euler presents it. Appendix A is a step-by-step verification of this substitution.

Formula (1.12) is somewhat complicated. To obtain a more simple formula for $\sin(\tfrac{1}{2}z)$, he considers an approximation for *x*. Observe that the terms in (1.11) that are multiplied by $\sin^2(\tfrac{1}{2}\omega)$ are extremely small, thus *x* can be expanded as a geometric power series since it can be expressed in the form $\dfrac{\alpha}{1-r}$. The condition that $\sin^2(\tfrac{1}{2}\omega)$ is relatively small guarantees that the ratio *r* is less than 1.

To see this, first rewrite the numerator and denominator for *x* in (1.11) as such:

$$x=\frac{\dfrac{-(n+m)\sin(2a)\sin^2(\tfrac{1}{2}\omega)}{(n-m)^2\cos^2(\tfrac{1}{2}\omega)}}{\dfrac{(n-m)^2\cos^2(\tfrac{1}{2}\omega)+(n+m)^2\cos(2a)\sin^2(\tfrac{1}{2}\omega)}{(n-m)^2\cos^2(\tfrac{1}{2}\omega)}},$$

or equivalently,

$$x = \frac{-\dfrac{(n+m)}{(n-m)^2}\sin(2a)\tan^2(\tfrac{1}{2}\omega)}{1+\dfrac{(n+m)^2}{(n-m)^2}\cos(2a)\tan^2(\tfrac{1}{2}\omega)}.$$

Denote by $\alpha = -\dfrac{(n+m)}{(n-m)^2}\sin(2a)\tan^2(\tfrac{1}{2}\omega)$ and $r = -(\dfrac{n+m}{n-m})^2\cos(2a)\tan^2(\tfrac{1}{2}\omega)$ so that $x = \dfrac{\alpha}{1-r}$. Since $|r|<1$, the formula

$$\frac{\alpha}{1-r} \approx \alpha(1+r)$$

is a good approximation for $\dfrac{\alpha}{1-r}$. Thus,

$$x \approx \alpha(1+r) = -\frac{(n+m)}{(n-m)^2}\sin(2a)\tan^2(\tfrac{1}{2}\omega)(1-(\frac{n+m}{n-m})^2\cos(2a)\tan^2(\tfrac{1}{2}\omega)). \quad (1.13)$$

Euler substitutes this new approximate value for *x* into (1.9) (the first result from part X) to obtain

$$\sin(\tfrac{1}{2}z) = \sin a \sin(\tfrac{1}{2}\omega)(1-\frac{(n+m)^2\cos^2 a\tan^2(\tfrac{1}{2}\omega)}{2(n-m)^2}). \quad (1.14)$$

This reduction is even lengthier than the first! For those that are interested in the details of this substitution, see Appendix B.

XII. **Working Backwards:**

Now Euler supposes that *z* is known, so the only unknown value is now the time *x*. He begins by rewriting relation (1.3) in terms of $\sin(\tfrac{1}{2}z)$ and an angle φ defined implicitly as follows:

$$\sin(\tfrac{1}{2}z) = \frac{\sin a \sin(\tfrac{1}{2}\omega)}{\cos\phi}, \quad (1.15)$$

where

$$\cos\phi = \frac{\sin(\tfrac{1}{2}UL)}{\sin(\tfrac{1}{2}z)}.$$

Substituting (1.15) into (1.9) and simplifying gives

$$\begin{aligned}4\sin^2 a\sin^2(\tfrac{1}{2}\omega)\tan^2\phi &= (n-m)^2x^2\cos^2(\tfrac{1}{2}\omega)+(n+m)^2x^2\cos(2a)\sin^2(\tfrac{1}{2}\omega)\\ &\quad+2(n+m)x\sin(2a)\sin^2(\tfrac{1}{2}\omega)\end{aligned} \quad (1.16)$$

But the last two terms containing $\sin^2(\tfrac{1}{2}\omega)$ on the right hand side of (1.16) are negligible and so

$$4\sin^2 a\sin^2(\tfrac{1}{2}\omega)\tan^2\varphi = (n-m)^2x^2\cos^2(\tfrac{1}{2}\omega).$$

Thus, solving for *x* yields the first-order approximation

$$x = \frac{2\sin a\tan(\tfrac{1}{2}\omega)\tan\varphi}{n-m}.$$

This value of x is not exact because of the neglected terms; hence there is an error y defined by

$$x = \frac{2\sin a \tan(\tfrac{1}{2}\omega)\tan\phi}{n-m} - y\ . \tag{1.17}$$

To find a formula for y and thus obtain a second-order approximation for x, Euler substitutes (1.17) back into (1.16), rearranged as

$$\begin{aligned}4\sin^2 a\sin^2(\tfrac{1}{2}\omega)\tan^2\phi = {}& 2(n+m)x\sin(2a)\sin^2(\tfrac{1}{2}\omega) + x^2((n-m)^2\cos^2(\tfrac{1}{2}\omega)\\ &+(n+m)^2\cos(2a)\sin^2(\tfrac{1}{2}\omega))\end{aligned},$$

and simplifying gives

$$\begin{aligned}4\sin^2 a\sin^2(\tfrac{1}{2}\omega)\tan^2\phi = {}& 4\frac{n+m}{n-m}\sin a\sin(2a)\sin^2(\tfrac{1}{2}\omega)\tan(\tfrac{1}{2}\omega)\tan\phi\\ &-2(n+m)\sin(2a)\sin^2(\tfrac{1}{2}\omega)y\\ &+4\sin^2 a\cos^2(\tfrac{1}{2}\omega)\tan^2(\tfrac{1}{2}\omega)\tan^2\phi\\ &+4(\frac{n+m}{n-m})^2\sin^2 a\cos(2a)\sin^2(\tfrac{1}{2}\omega)\tan^2(\tfrac{1}{2}\omega)\tan^2\phi\\ &-4(n-m)\sin a\cos^2(\tfrac{1}{2}\omega)\tan(\tfrac{1}{2}\omega)\tan\phi\ y\\ &-4\frac{(n+m)^2}{n-m}\sin a\cos(2a)\sin^2(\tfrac{1}{2}\omega)\tan(\tfrac{1}{2}\omega)\tan\phi\ y\\ &+(n-m)^2\cos^2(\tfrac{1}{2}\omega)y^2+(n+m)^2\cos(2a)\sin^2(\tfrac{1}{2}\omega)y^2\end{aligned}$$

But the last two terms containing y^2 on the right-hand side above are negligible in comparison to y (assumed to be relatively small), and the remaining equation can be grouped as

$$\begin{aligned}&4\sin^2 a\sin^2(\tfrac{1}{2}\omega)\tan^2\phi - 4\sin^2 a\cos^2(\tfrac{1}{2}\omega)\tan^2(\tfrac{1}{2}\omega)\tan^2\phi =\\ &-4(n-m)\sin a\cos^2(\tfrac{1}{2}\omega)\tan(\tfrac{1}{2}\omega)\tan\phi\ y\\ &+4(\frac{n+m}{n-m})^2\sin^2 a\cos(2a)\sin^2(\tfrac{1}{2}\omega)\tan^2(\tfrac{1}{2}\omega)\tan^2\phi\\ &+4\frac{n+m}{n-m}\sin a\sin(2a)\sin^2(\tfrac{1}{2}\omega)\tan(\tfrac{1}{2}\omega)\tan\phi\\ &-2(n+m)\sin(2a)\sin^2(\tfrac{1}{2}\omega)y\\ &-4\frac{(n+m)^2}{n-m}\sin a\cos(2a)\sin^2(\tfrac{1}{2}\omega)\tan(\tfrac{1}{2}\omega)\tan\phi\ y\end{aligned} \tag{1.18}$$

Again, the last two terms containing y on the right-hand side above are relatively small in comparison to first term containing y, thus (1.18) reduces to

$$\begin{aligned}&4\sin^2 a\sin^2(\tfrac{1}{2}\omega)\tan^2\phi - 4\sin^2 a\cos^2(\tfrac{1}{2}\omega)\tan^2(\tfrac{1}{2}\omega)\tan^2\phi =\\ &-4(n-m)\sin a\cos^2(\tfrac{1}{2}\omega)\tan(\tfrac{1}{2}\omega)\tan\phi\ y\\ &+4(\frac{n+m}{n-m})^2\sin^2 a\cos(2a)\sin^2(\tfrac{1}{2}\omega)\tan^2(\tfrac{1}{2}\omega)\tan^2\phi\\ &+4\frac{n+m}{n-m}\sin a\sin(2a)\sin^2(\tfrac{1}{2}\omega)\tan(\tfrac{1}{2}\omega)\tan\phi\end{aligned} \tag{1.19}$$

Making some trigonometric substitutions, (1.19) in turn reduces to

$$0 = -4(n-m)y\sin a\sin(\tfrac{1}{2}\omega)\cos(\tfrac{1}{2}\omega)\tan\phi$$

$$+4\left(\frac{n+m}{n-m}\right)^2\sin^2 a\cos(2a)\sin^2(\tfrac{1}{2}\omega)\tan^2(\tfrac{1}{2}\omega)\tan^2\phi$$

$$+4\frac{n+m}{n-m}\sin a\sin(2a)\sin^2(\tfrac{1}{2}\omega)\tan(\tfrac{1}{2}\omega)\tan\phi$$

Now solve this equation for y to get

$$y = \frac{n+m}{(n-m)^2}\sin(2a)\tan^2(\tfrac{1}{2}\omega) + \frac{(n+m)^2}{(n-m)^3}\sin a\cos(2a)\tan^3(\tfrac{1}{2}\omega)\tan\phi.. \qquad (1.20)$$

Lastly, substitute this value for y into (1.17) and simplify to get the desired value of x:

$$x = \frac{\sin a\tan(\tfrac{1}{2}\omega)}{n-m}\left(2\tan\phi - \frac{2(n+m)}{n-m}\cos a\tan(\tfrac{1}{2}\omega) - \frac{(n+m)^2}{(n-m)^2}\cos(2a)\tan^2(\tfrac{1}{2}\omega)\tan\phi\right) \quad (1.21)$$

XIII. In this section Euler determines what state the umbra and Moon have to be in when the Eclipse is starting and ending. Using z, the distance between the centers of the umbra and the Moon, he reasons that z is the sum of the radii of the umbra and the Moon when the Eclipse is beginning and ending. This is because with this condition, the disks are just touching at the edges. The immersion and emersion would be the moments where z is the difference of the radii. Euler also notes that the condition $\sin(\tfrac{1}{2}z) > \sin a\sin(\tfrac{1}{2}\omega)$ is required to obtain a meaningful result. If this condition is not met, then much accuracy would be lost in the computations due to the relative insignificances of some of the terms in the formula.

XIV. Euler previously derived $\sin(\tfrac{1}{2}UL) = \sin a\sin(\tfrac{1}{2}\omega)$ where UL is the z value at the moment of opposition. Euler uses his Astronomical Tables to find the values of a and ω, which are 0^s, 8°, 48′, 46″ and 0^s, 5°, 16′, 33″ respectively. Euler then uses base-10 logarithms to simplify his work. Either using the log tables or computing directly, UL = 48′, 29″, thus at the moment of opposition z = 48′, 29″.

XV. Euler provides the values for m and n, the hourly movements of the Sun and the Moon respectively, to be m = 152″ and n = 2277″. Euler can now solve for x at the moment of closest proximity since $x \approx \dfrac{-(n+m)\sin(2a)\tan^2(\tfrac{1}{2}\omega)}{(n-m)^2}$ where this formula is the optimization of the position versus time formula, and where the error caused by approximation is negligible. Using the more accurate numbers from the *Opera Omnia* gives x = –4′, 17″, or the moment when the disks' centers are closest is the time of opposition + x = August 8^d, 12^h, 10′, 22″ in 1748 true time.

XVI. In this section the formula

$$\sin(\tfrac{1}{2}z) = \sin a\sin(\tfrac{1}{2}\omega)\left(1 - \frac{(n+m)^2\cos^2 a\tan^2(\tfrac{1}{2}\omega)}{2(n-m)^2}\right)$$

is used to find z at x. This formula can also be expressed as

$$\sin(\tfrac{1}{2}z)=\sin(\tfrac{1}{2}UL)-\frac{(n+m)^2}{2(n-m)^2}\sin a\sin(\tfrac{1}{2}\omega)\cos^2 a\tan^2(\tfrac{1}{2}\omega),$$

or approximately as

$$\tfrac{1}{2}z=\tfrac{1}{2}UL-\frac{(n+m)^2}{2(n-m)^2}\sin a\sin(\tfrac{1}{2}\omega)\cos^2 a\tan^2(\tfrac{1}{2}\omega)\,. \tag{1.22}$$

Thus, doubling (1.22) gives

$$z=UL-\frac{(n+m)^2}{(n-m)^2}\sin a\sin(\tfrac{1}{2}\omega)\cos^2 a\tan^2(\tfrac{1}{2}\omega) \tag{1.23}$$

where UL has already been found. Calculating $\frac{(n+m)^2}{(n-m)^2}\sin a\sin(\tfrac{1}{2}\omega)\cos^2 a\tan^2(\tfrac{1}{2}\omega)$ gives 4″, thus z = 48′, 29″ – 4″ = 48′, 25″ at the moment when the centers are closest. This allows the setup of the following ratio:

radius of the Moon : 6 :: z of the greatest obscuration : magnitude of the Eclipse

Euler measured the magnitude of Eclipses using digits, in which 12 digits represent the diameter of the disk being covered, in this case the Moon. This ratio is solved by:

1004 : 6 :: 839 : 5.014

thus the magnitude of the Eclipse is 5.014 digits, or the Eclipse covers 5.014 / 12 = 0.418 of the diameter of the Moon.

XVII. Euler now wants to find the beginning and ending times for the Eclipse. At these times he knows that z = 62′, 24″, or the sum of the radii of the umbra and the Moon. Euler must now solve $\cos\varphi=\frac{\sin a\sin(\tfrac{1}{2}\omega)}{\sin(\tfrac{1}{2}z)}$ so he can use his formula from section XII. All of the variables on the right-hand side are known, so finding φ is easy. φ = 39°, 1′, 8″.

Next, Euler calculates x using the formula (1.21):

$$x=\frac{\sin a\tan(\tfrac{1}{2}\omega)}{n-m}\left(2\tan\varphi-\frac{2(n+m)}{n-m}\cos a\tan(\tfrac{1}{2}\omega)-\frac{(n+m)^2}{(n-m)^2}\cos(2a)\tan^2(\tfrac{1}{2}\omega)\tan\varphi\right),$$

and when the leading value is distributed, there are three terms to calculate. Euler finds their values to be: 1.1104^{h}, 0.0713, and 0.0015 respectively.

XVIII. The first and last terms both have $\tan\varphi$ in them, and to find the start time, this value is negative, while finding the end time makes this value positive. Thus:

I. x = 1.1104 – 0.0713 – 0.0015 = +1.0376
II. x = –1.1104 – 0.0713 + 0.0015 = –1.1802

To find the starting and ending times, adjust the time of opposition by these values to get:

Time of the opposition, Aug. 8^{d}, 12^{h}, 14′, 39″			
	–	1,	10, 49
	+	1,	2, 16
Beginning of the Eclipse		11,	3, 50

End of the Eclipse	13, 16, 55

XIX. In conclusion, Euler compiles all of the values he has determined about the Eclipse and is excited that they agree so well with observations:

The beginning	11^h, 3′, 50″	observed at 11^h, 5′
The largest obscuration	12, 10, 22	
The opposition in orbit	12, 14, 39	
The end of the Eclipse	13, 16, 55	observed at 13^h, 18′
The magnitude of the Eclipse	5.014 digits	
and the duration of the Eclipse	2^h, 13′, 05″	observation 2^h, 13′

Appendix C contains a short Mathematica program that compares Euler's approximations with the original equations to see how much error was introduced into the calculations in these last few sections. In section XV, Euler uses an approximation of x to find the moment of greatest obscuration. His approximation results in the time = August 8^d, 12^h, 10′, 22″ in 1748 true time. This approximation results in an error of only 1 second. In section XVI, Euler uses his formula for finding z at x to get z = 48′, 25″ at the moment when the centers are closest. The error here is negligible, about 0.5 sixtieths of an arc-second. The rest of the paper is devoted to finding the starting and ending times of the Eclipse. Euler's calculations here give the start to be 11^h, 3′, 50″, where the exact equation gives 11, 3, 43, and he gives the end to be 13, 16, 55, where the exact equation gives 13, 17, 3. Even though this approximation causes greater error than the others, it is still a good approximation since predicting the start and end of an eclipse only have to be accurate to the minutes.

References

[1] A. Fabian, Translation of Leonard Euler's E141 Paper: *Sur l'accord des deux dernieres eclipses du soleil et de la lune avec mes tables, pour trouver les vrais momens des pleni-lunes et novi-lunes* (On the agreement of the last two eclipses of the Sun and Moon with my tables, for finding the actual times of the half-moon and new moon), Memoires de l'academie des sciences de Berlin **4** (1750) pp. 86-98 (*Opera Omnia*: Series 2, Volume 30, pp. 89 – 100). Available online at the Euler Archive: http://www.math.dartmouth.edu/~euler/

Andrew Fabian
Department of Mathematics
Rowan University
Glassboro, NJ 08028
fabian78@students.rowan.edu

Hieu D. Nguyen
Department of Mathematics
Rowan University
Glassboro, NJ 08028
nguyen@rowan.edu

Appendix A

The following is a step-by-step justification of (1.12) in section XI.

We begin with equation (1.10),

$$\begin{aligned}4\sin^2(\tfrac{1}{2}z) &= (2\sin a + (n+m)x\cos a)^2\sin^2(\tfrac{1}{2}\omega) + (n-m)^2x^2\cos^2(\tfrac{1}{2}\omega)\\ &\quad -(n+m)^2x^2\sin^2 a\sin^2(\tfrac{1}{2}\omega)\end{aligned},$$

and substitute formula (1.11) for x into each of the three terms on the right hand side. The first term can be rewritten as follows:

$$\begin{aligned}&(2\sin a + (n+m)x\cos a)^2\sin^2(\tfrac{1}{2}\omega)\\
&= (2\sin a + (n+m)\ \frac{-(n+m)\sin(2a)\sin^2(\tfrac{1}{2}\omega)}{(n-m)^2\ \cos^2(\tfrac{1}{2}\omega) + (n+m)^2\ \cos(2a)\ \sin^2(\tfrac{1}{2}\omega)}\ \cos a)^2\sin^2(\tfrac{1}{2}\omega)\\
&= (2\sin a - \frac{(n+m)^2\sin(2a)\cos a\ \sin^2(\tfrac{1}{2}\omega)}{(n-m)^2\ \cos^2(\tfrac{1}{2}\omega) + (n+m)^2\ \cos(2a)\ \sin^2(\tfrac{1}{2}\omega)})^2\sin^2(\tfrac{1}{2}\omega)\\
&= (2\sin a - \frac{2(n+m)^2\sin a\cos^2 a\ \sin^2(\tfrac{1}{2}\omega)}{(n-m)^2\ \cos^2(\tfrac{1}{2}\omega) + (n+m)^2\ \cos(2a)\ \sin^2(\tfrac{1}{2}\omega)})^2\sin^2(\tfrac{1}{2}\omega)\\
&= (2\sin a\ (1 - \frac{(n+m)^2\cos^2 a\ \sin^2(\tfrac{1}{2}\omega)}{(n-m)^2\ \cos^2(\tfrac{1}{2}\omega) + (n+m)^2\ \cos(2a)\ \sin^2(\tfrac{1}{2}\omega)}))^2\sin^2(\tfrac{1}{2}\omega)\\
&= 4\sin^2 a\sin^2(\tfrac{1}{2}\omega)\ (1 - \frac{(n+m)^2\cos^2 a\ \sin^2(\tfrac{1}{2}\omega)}{(n-m)^2\ \cos^2(\tfrac{1}{2}\omega) + (n+m)^2\ \cos(2a)\ \sin^2(\tfrac{1}{2}\omega)})^2\end{aligned}$$

Next, create a common denominator and use the *Double Angle Formula* for cosine to simplify the last line above as

$$\begin{aligned}&(2\sin a + (n+m)x\cos a)^2\sin^2(\tfrac{1}{2}\omega)\\
&= 4\sin^2 a\sin^2(\tfrac{1}{2}\omega)(\frac{(n-m)^2\ \cos^2(\tfrac{1}{2}\omega) - (n+m)^2\sin^2 a\ \sin^2(\tfrac{1}{2}\omega)}{(n-m)^2\ \cos^2(\tfrac{1}{2}\omega) + (n+m)^2\ \cos(2a)\ \sin^2(\tfrac{1}{2}\omega)})^2\end{aligned} \tag{1.24}$$

The second and third terms in (1.10) can be simplified as follows:

$$\begin{aligned}&x^2((n-m)^2\cos^2(\tfrac{1}{2}\omega) - (n+m)^2\sin^2 a\sin^2(\tfrac{1}{2}\omega))\\
&= \frac{4(n+m)^2\sin^2 a\cos^2 a\ \sin^4(\tfrac{1}{2}\omega)}{((n-m)^2\ \cos^2(\tfrac{1}{2}\omega) + (n+m)^2\ \cos(2a)\ \sin^2(\tfrac{1}{2}\omega))^2}\cdot\\
&\qquad ((n-m)^2\cos^2(\tfrac{1}{2}\omega) - (n+m)^2\sin^2 a\sin^2(\tfrac{1}{2}\omega))\\
&= \frac{4(n+m)^2\sin^2 a\cos^2 a\ \sin^4(\tfrac{1}{2}\omega)}{(n-m)^2\cos^2(\tfrac{1}{2}\omega) - (n+m)^2\sin^2 a\sin^2(\tfrac{1}{2}\omega)}\cdot\\
&\qquad (\frac{(n-m)^2\cos^2(\tfrac{1}{2}\omega) - (n+m)^2\sin^2 a\sin^2(\tfrac{1}{2}\omega)}{(n-m)^2\ \cos^2(\tfrac{1}{2}\omega) + (n+m)^2\ \cos(2a)\ \sin^2(\tfrac{1}{2}\omega)})^2\end{aligned} \tag{1.25}$$

Now sum up all three reduced terms in (1.24) and (1.25), then combine like terms to get

$$4\sin^2(\tfrac{1}{2}z) = 4\sin^2 a\sin^2(\tfrac{1}{2}\omega)\left(\frac{(n-m)^2\cos^2(\tfrac{1}{2}\omega)-(n+m)^2\sin^2 a\sin^2(\tfrac{1}{2}\omega)}{(n-m)^2\cos^2(\tfrac{1}{2}\omega)+(n+m)^2\cos(2a)\sin^2(\tfrac{1}{2}\omega)}\right)^2\cdot$$

$$\left(1+\frac{(n+m)^2\cos^2 a\sin^2(\tfrac{1}{2}\omega)}{(n-m)^2\cos^2(\tfrac{1}{2}\omega)-(n+m)^2\sin^2 a\sin^2(\tfrac{1}{2}\omega)}\right)$$

Again, create a common denominator for the last factor on the right-hand side above to get

$$4\sin^2(\tfrac{1}{2}z) = 4\sin^2 a\sin^2(\tfrac{1}{2}\omega)\left(\frac{(n-m)^2\cos^2(\tfrac{1}{2}\omega)-(n+m)^2\sin^2 a\sin^2(\tfrac{1}{2}\omega)}{(n-m)^2\cos^2(\tfrac{1}{2}\omega)+(n+m)^2\cos(2a)\sin^2(\tfrac{1}{2}\omega)}\right)^2\cdot$$

$$\frac{(n-m)^2\cos^2(\tfrac{1}{2}\omega)-(n+m)^2\sin^2 a\sin^2(\tfrac{1}{2}\omega)+(n+m)^2\cos^2 a\sin^2(\tfrac{1}{2}\omega)}{(n-m)^2\cos^2(\tfrac{1}{2}\omega)-(n+m)^2\sin^2 a\sin^2(\tfrac{1}{2}\omega)}$$

Next, use the *Double Angle Formula* for cosine to simplify the last factor:

$$4\sin^2(\tfrac{1}{2}z) = 4\sin^2 a\sin^2(\tfrac{1}{2}\omega)\left(\frac{(n-m)^2\cos^2(\tfrac{1}{2}\omega)-(n+m)^2\sin^2 a\sin^2(\tfrac{1}{2}\omega)}{(n-m)^2\cos^2(\tfrac{1}{2}\omega)+(n+m)^2\cos(2a)\sin^2(\tfrac{1}{2}\omega)}\right)^2\cdot$$

$$\frac{(n-m)^2\cos^2(\tfrac{1}{2}\omega)+(n+m)^2\cos(2a)\sin^2(\tfrac{1}{2}\omega)}{(n-m)^2\cos^2(\tfrac{1}{2}\omega)-(n+m)^2\sin^2 a\sin^2(\tfrac{1}{2}\omega)}$$

Notice that this expression is of the form $\kappa\psi^2\psi^{-1}$, which reduces to $\kappa\psi$. Thus,

$$4\sin^2(\tfrac{1}{2}z) = 4\sin^2 a\sin^2(\tfrac{1}{2}\omega)\frac{(n-m)^2\cos^2(\tfrac{1}{2}\omega)-(n+m)^2\sin^2 a\sin^2(\tfrac{1}{2}\omega)}{(n-m)^2\cos^2(\tfrac{1}{2}\omega)+(n+m)^2\cos(2a)\sin^2(\tfrac{1}{2}\omega)},$$

which simplifies to the promised result:

$$\sin(\tfrac{1}{2}z) = \sin a\sin(\tfrac{1}{2}\omega)\sqrt{\frac{(n-m)^2\cos^2(\tfrac{1}{2}\omega)-(n+m)^2\sin^2 a\sin^2(\tfrac{1}{2}\omega)}{(n-m)^2\cos^2(\tfrac{1}{2}\omega)+(n+m)^2\cos(2a)\sin^2(\tfrac{1}{2}\omega)}}.$$

Appendix B

The following is a step-by-step justification of (1.14) in section XI.

$$x \approx \alpha(1+r) = -\frac{(n+m)}{(n-m)^2}\sin(2a)\,\tan^2(\tfrac{1}{2}\omega)(1-(\frac{n+m}{n-m})^2\cos(2a)\tan^2(\tfrac{1}{2}\omega))$$

We begin with equation (1.9):

$$\begin{aligned} 4\sin^2(\tfrac{1}{2}z) &= ((n-m)^2\cos^2(\tfrac{1}{2}\omega)+(n+m)^2\cos(2a)\sin^2(\tfrac{1}{2}\omega))x^2 \\ &\quad + 4(n+m)x\sin a\cos a\sin^2(\tfrac{1}{2}\omega)+4\sin^2 a\sin^2(\tfrac{1}{2}\omega) \end{aligned} \tag{1.26}$$

Substituting (1.13) into the first term on the right-hand side of (1.26) gives

$$\begin{aligned} &((n-m)^2\cos^2(\tfrac{1}{2}\omega)+(n+m)^2\cos(2a)\sin^2(\tfrac{1}{2}\omega))x^2 \\ &= 4\sin^2 a\sin^2(\tfrac{1}{2}\omega)(1-(\frac{n+m}{n-m})^2\cos(2a)\tan^2(\tfrac{1}{2}\omega))^2\cdot \\ &\quad \frac{(n+m)^2}{(n-m)^4}\cos^2 a\,\frac{\tan^2(\tfrac{1}{2}\omega)}{\cos^2(\tfrac{1}{2}\omega)}((n-m)^2\cos^2(\tfrac{1}{2}\omega)+(n+m)^2\cos(2a)\sin^2(\tfrac{1}{2}\omega)) \\ &= 4\sin^2 a\sin^2(\tfrac{1}{2}\omega)(1-(\frac{n+m}{n-m})^2\cos(2a)\tan^2(\tfrac{1}{2}\omega))^2\cdot \\ &\quad ((\frac{n+m}{n-m})^2\cos^2 a\tan^2(\tfrac{1}{2}\omega)+(\frac{n+m}{n-m})^4\cos^2 a\cos(2a)\tan^4(\tfrac{1}{2}\omega)) \end{aligned}$$

The second term on the right-hand side of (1.26) reduces to

$$\begin{aligned} &4(n+m)\sin a\cos a\sin^2(\tfrac{1}{2}\omega)x \\ &= -2\frac{(n+m)}{(n-m)^2}\sin a\cos a\,\tan^2(\tfrac{1}{2}\omega)(1-(\frac{n+m}{n-m})^2\cos(2a)\tan^2(\tfrac{1}{2}\omega))\cdot \\ &\quad 4(n+m)\sin a\cos a\sin^2(\tfrac{1}{2}\omega) \\ &= 4\sin^2 a\sin^2(\tfrac{1}{2}\omega)(1-(\frac{n+m}{n-m})^2\cos(2a)\tan^2(\tfrac{1}{2}\omega))(-2)(\frac{n+m}{n-m})^2\cos^2 a\,\tan^2(\tfrac{1}{2}\omega) \end{aligned}$$

Combining both terms gives

$$\begin{aligned} &4\sin^2(\tfrac{1}{2}z) \\ &= 4\sin^2 a\sin^2(\tfrac{1}{2}\omega)[\,(1-(\frac{n+m}{n-m})^2\cos(2a)\tan^2(\tfrac{1}{2}\omega))^2((\frac{n+m}{n-m})^2\cos^2 a\tan^2(\tfrac{1}{2}\omega) \\ &\quad +(\frac{n+m}{n-m})^4\cos^2 a\cos(2a)\tan^4(\tfrac{1}{2}\omega)) \\ &\quad +(1-(\frac{n+m}{n-m})^2\cos(2a)\tan^2(\tfrac{1}{2}\omega))(-2)(\frac{n+m}{n-m})^2\cos^2 a\,\tan^2(\tfrac{1}{2}\omega)+1\,] \end{aligned}$$

To simplify the notation, we denote by

$$\kappa = 4\sin^2 a\sin^2(\tfrac{1}{2}\omega)$$

$$\sigma = 1-(\frac{n+m}{n-m})^2\cos(2a)\tan^2(\tfrac{1}{2}\omega)$$

We then simplify

$$4\sin^2(\tfrac{1}{2}z)$$

$$= \kappa\,[\,1+(\frac{n+m}{n-m})^2\cos^2 a\tan^2(\tfrac{1}{2}\omega)(\sigma-2)\sigma] + \kappa\,\sigma^2(\frac{n+m}{n-m})^4\cos^2 a\cos(2a)\tan^4(\tfrac{1}{2}\omega))$$

$$= \kappa\,[1-(\frac{n+m}{n-m})^2\cos^2 a\tan^2(\tfrac{1}{2}\omega) + (\frac{n+m}{n-m})^6\cos^2 a\cos^2(2a)\tan^6(\tfrac{1}{2}\omega))]$$

$$+\kappa\,[\,(\frac{n+m}{n-m})^4\cos^2 a\cos(2a)\tan^4(\tfrac{1}{2}\omega)) - 2(\frac{n+m}{n-m})^6\cos^2 a\cos^2(2a)\tan^6(\tfrac{1}{2}\omega)$$

$$+(\frac{n+m}{n-m})^8\cos^2 a\cos^3(2a)\tan^8(\tfrac{1}{2}\omega))\,]$$

$$= 4\sin^2 a\sin^2(\tfrac{1}{2}\omega)\,[1-(\frac{n+m}{n-m})^2\cos^2 a\tan^2(\tfrac{1}{2}\omega) + (\frac{n+m}{n-m})^4\cos^2 a\cos(2a)\tan^4(\tfrac{1}{2}\omega)$$

$$-(\frac{n+m}{n-m})^6\cos^2 a\cos^2(2a)\tan^6(\tfrac{1}{2}\omega) + (\frac{n+m}{n-m})^8\cos^2 a\cos^3(2a)\tan^8(\tfrac{1}{2}\omega)\,]$$

Euler's Approximation

Euler recognizes the following factorization:

$$1-(\frac{n+m}{n-m})^2\cos^2 a\tan^2(\tfrac{1}{2}\omega) + \tfrac{1}{4}(\frac{n+m}{n-m})^4\cos^4 a\tan^4(\tfrac{1}{2}\omega) =$$

$$(1-\tfrac{1}{2}(\frac{n+m}{n-m})^2\cos^2 a\tan^2(\tfrac{1}{2}\omega))^2$$

He then drops the last two terms from the previous result since they contain higher-order powers of $\tan(\tfrac{1}{2}\omega)$, thus negligible, to get

$$4\sin^2(\tfrac{1}{2}z)$$

$$= 4\sin^2 a\sin^2(\tfrac{1}{2}\omega)\,[1-(\frac{n+m}{n-m})^2\cos^2 a\tan^2(\tfrac{1}{2}\omega) + (\frac{n+m}{n-m})^4\cos^2 a\cos(2a)\tan^4(\tfrac{1}{2}\omega)]$$

The last term above is also negligible, and we are led to assume that Euler replaces it by a different negligible term, namely $\tfrac{1}{4}(\frac{n+m}{n-m})^4\cos^4 a\tan^4(\tfrac{1}{2}\omega)$ (we see no other way to justify for this step):

$$4\sin^2(\tfrac{1}{2}z)$$

$$= 4\sin^2 a\sin^2(\tfrac{1}{2}\omega)\,[1-(\frac{n+m}{n-m})^2\cos^2 a\tan^2(\tfrac{1}{2}\omega) + \tfrac{1}{4}(\frac{n+m}{n-m})^4\cos^4 a\tan^4(\tfrac{1}{2}\omega)]$$

$$= 4\sin^2 a\sin^2(\tfrac{1}{2}\omega)\,(1-\tfrac{1}{2}(\frac{n+m}{n-m})^2\cos^2 a\tan^2(\tfrac{1}{2}\omega))^2$$

We thus arrive at our desired relation:

$$\sin(\tfrac{1}{2}z) = \sin a\sin(\tfrac{1}{2}\omega)(1-\frac{(n+m)^2\cos^2 a\tan^2(\tfrac{1}{2}\omega)}{2(n-m)^2})\,.$$

Appendix C

```
(* This Mathematica program checks Euler's approximation in E141 *)

a = 0.153812;
n = 0.0110392;
m = 0.0007369;
w = 0.0920807;
```

$$eq = 4*Sin[a]^2*Sin[w/2]^2\left(\frac{(n-m)^2 Cos[w/2]^2-(n+m)^2 Sin[a]^2 Sin[w/2]^2}{(n-m)^2 Cos[w/2]^2+(n+m)^2 Cos[2*a] Sin[w/2]^2}\right)$$

0.000198339

$$eqapprox = 4*Sin[a]^2*Sin[w/2]^2\left(1-\frac{(n+m)^2*Cos[a]^2*Tan[w/2]^2}{2*(n-m)^2}\right)^2$$

0.000198338

$$z1 := \sqrt{eq}\ \frac{180}{\pi}$$

```
z1sign := Floor[z1 / 30]
z1degwr := z1 - z1sign * 30
z1deg := Floor[z1degwr]
z1mwr := (z1degwr - z1deg) * 60
z1min := Floor[z1mwr]
z1swr := (z1mwr - z1min) * 60
z1sec := Floor[z1swr]
z1sswr := (z1swr - z1sec) * 60
Print["EXACT:  ", z1sign, " signs ", z1deg, " degrees ", z1min,
 " minutes ", z1sec, " seconds ", z1sswr, " sixtieths of a second"]
```

$$z2 := \sqrt{eqapprox}\ \frac{180}{\pi}$$

```
z2sign := Floor[z2 / 30]
z2degwr := z2 - z2sign * 30
z2deg := Floor[z2degwr]
z2mwr := (z2degwr - z2deg) * 60
z2min := Floor[z2mwr]
z2swr := (z2mwr - z2min) * 60
z2sec := Floor[z2swr]
z2sswr := (z2swr - z2sec) * 60
Print["APPROX: ", z2sign, " signs ", z2deg, " degrees ", z2min,
 " minutes ", z2sec, " seconds ", z2sswr, " sixtieths of a second"]
```

```
EXACT:  0 signs 0 degrees 48 minutes 24 seconds 53.2738 sixtieths of a second
APPROX: 0 signs 0 degrees 48 minutes 24 seconds 52.8101 sixtieths of a second
```

$$\text{time} = \frac{-(n+m)\,\text{Sin}[2*a]\,\text{Sin}[w/2]^{\wedge}2}{(n-m)^{\wedge}2\,\text{Cos}[w/2]^{\wedge}2 + (n+m)^{\wedge}2\,\text{Cos}[2*a]\,\text{Sin}[w/2]^{\wedge}2}$$

$$\text{tapprox} = \frac{-(n+m)\,\text{Sin}[2*a]\,\text{Tan}[w/2]^{\wedge}2}{(n-m)^{\wedge}2}\left(1 - \frac{(n+m)^{\wedge}2\,\text{Cos}[2*a]\,\text{Tan}[w/2]^{\wedge}2}{(n-m)^{\wedge}2}\right)$$

$$\text{tapprox2} = \frac{-(n+m)\,\text{Sin}[2*a]\,\text{Tan}[w/2]^{\wedge}2}{(n-m)^{\wedge}2}$$

```
-0.0711257

-0.0711252

-0.0713137
```

```
x1 := Abs[time]
x1h := Floor[x1]
x1mwr := (x1 - x1h) * 60
x1m := Floor[x1mwr]
x1swr := (x1mwr - x1m) * 60
If[time < 0, Print["EXACT:   -", x1h, " hours ", x1m, " minutes ", x1swr, " seconds "],
 Print["EXACT:    ", x1h, " hours ", x1m, " minutes ", x1swr, " seconds "]]

x2 := Abs[tapprox]
x2h := Floor[x2]
x2mwr := (x2 - x2h) * 60
x2m := Floor[x2mwr]
x2swr := (x2mwr - x2m) * 60
If[tapprox < 0, Print["APPROX:  -", x2h, " hours ", x2m, " minutes ", x2swr, " seconds "],
 Print["APPROX:    ", x2h, " hours ", x2m, " minutes ", x2swr, " seconds "]]

x3 := Abs[tapprox2]
x3h := Floor[x3]
x3mwr := (x3 - x3h) * 60
x3m := Floor[x3mwr]
x3swr := (x3mwr - x3m) * 60
If[tapprox2 < 0, Print["APPROX2: -", x3h, " hours ", x3m, " minutes ", x3swr, " seconds "],
 Print["APPROX2:  ", x3h, " hours ", x3m, " minutes ", x3swr, " seconds "]]
```

```
EXACT:   -0 hours 4 minutes 16.0526 seconds

APPROX:  -0 hours 4 minutes 16.0508 seconds

APPROX2: -0 hours 4 minutes 16.7294 seconds
```

```
A := (n - m)^2 Cos[w / 2]^2 + (n + m)^2 Cos[2 * a] Sin[w / 2]^2
B := 2 (n + m) Sin[2 * a] Sin[w / 2]^2
F := -4 Sin[a]^2 Sin[w / 2]^2 Tan[φ]^2
```

$$\text{start} := \frac{-B - \sqrt{B^{\wedge}2 - 4*A*F}}{2*A}$$

```
startN1 := start /. a → 0.153812
startN2 := startN1 /. n → 0.0110392
startN3 := startN2 /. m → 0.0007369
```

```
startN4 := startN3 /. w → 0.0920807
startN5 := startN4 /. φ → 0.6810081

stapprox :=
 Sin[a] Tan[w / 2] / (n - m) (-2 Tan[φ] - 2 (n + m) / (n - m) Cos[a] Tan[w / 2] + (n + m) ^2 / (n - m) ^2 Cos[2 * a] Tan[w / 2] ^2 Tan[φ])
saN1 := stapprox /. a → 0.153812
saN2 := saN1 /. n → 0.0110392
saN3 := saN2 /. m → 0.0007369
saN4 := saN3 /. w → 0.0920807
saN5 := saN4 /. φ → 0.6810081

end := (-B + √(B^2 - 4 * A * F)) / (2 * A)
endN1 := end /. a → 0.153812
endN2 := endN1 /. n → 0.0110392
endN3 := endN2 /. m → 0.0007369
endN4 := endN3 /. w → 0.0920807
endN5 := endN4 /. φ → 0.6810081

endapprox :=
 Sin[a] Tan[w / 2] / (n - m) (2 Tan[φ] - 2 (n + m) / (n - m) Cos[a] Tan[w / 2] - (n + m) ^2 / (n - m) ^2 Cos[2 * a] Tan[w / 2] ^2 Tan[φ])
eaN1 := endapprox /. a → 0.153812
eaN2 := eaN1 /. n → 0.0110392
eaN3 := eaN2 /. m → 0.0007369
eaN4 := eaN3 /. w → 0.0920807
eaN5 := eaN4 /. φ → 0.6810081

st := -startN5
sth := Floor[st]
stmwr := (st - sth) * 60
stm := Floor[stmwr]
stswr := (stmwr - stm) * 60
Print["START EXACT:  -", sth, " hours ",
 stm, " minutes ", stswr, " seconds before opposition"]

sta := -saN5
stah := Floor[sta]
stamwr := (sta - stah) * 60
stam := Floor[stamwr]
staswr := (stamwr - stam) * 60
Print["START APPROX: -", stah, " hours ",
 stam, " minutes ", staswr, " seconds before opposition"]

doneh := Floor[endN5]
donemwr := (endN5 - doneh) * 60
donem := Floor[donemwr]
doneswr := (donemwr - donem) * 60
Print["END EXACT:    +", doneh, " hours ",
 donem, " minutes ", doneswr, " seconds after opposition"]
```

```
dah := Floor[eaN5]
damwr := (eaN5 - dah) * 60
dam := Floor[damwr]
daswr := (damwr - dam) * 60
Print["END APPROX:   +", dah, " hours ",
 dam, " minutes ", daswr, " seconds after opposition"]
```

```
START EXACT:  -1 hours 10 minutes 56.4262 seconds before opposition

START APPROX: -1 hours 10 minutes 48.8896 seconds before opposition

END EXACT:    +1 hours 2 minutes 24.3211 seconds after opposition

END APPROX:   +1 hours 2 minutes 15.4308 seconds after opposition
```