# EXPLANATION OF CERTAIN PARADOXES
## IN INTEGRAL CALCULUS

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*The First Paradox*

### I.

Here I intend to explain a paradox in integral calculus that will seem rather strange: this is that we sometimes encounter differential equations in which it would seem very difficult to find the integrals by the rules of integral calculus yet are still easily found, not by the method of integration, but rather in differentiating the proposed equation again; so in these cases, a repeated differentiation leads us to the sought integral. This is undoubtedly a very surprising accident, that differentiation can lead us to the same goal, to which we are accustomed to find by integration, which is an entirely opposite operation.

II. To get a better feel for the importance of this paradox, we only have to remember that integral calculus holds the natural method for finding integrals from differential quantities: and from this it seems that for a proposed differential equation, there is no other way to arrive at its integral than to attempt its integration. And if we would, instead of integrating this equation, differentiate it once more, we would need to believe that we would further distance ourselves from the proposed goal; considering that we would then have a differential equation of the second degree, it would need two integrations before we reach the proposed goal.

III. It must therefore be very surprising that a repeated differentiation does not distance us only further from the integral that we proposed to find, but it can even give us this integral. This would undoubtedly be a great advantage, if this accident were general and always held true, since then the study of integrals, which are often impossible, would no longer pose the least difficulty: but it is only found in some very particular cases in which I will relate some examples: the other cases always follow the ordinary method of integration. Therefore, here are some problems that serve to clarify this paradox.

PROBLEM I

*Given point A, find the curve EM such that the perpendicular AV, derived from point A onto some tangent of the curve MV, is the same size everywhere. (Fig. 1)*

IV. Taking for the axis some straight line *AP* derived from the given point *A*, we derive the perpendicular *MP* there from some point *M* on the sought curve and another infinitely close line *mp*. Also, let us call $AP = x$, $PM = y$, and the given length of the line $AV = a$. Furthermore, let the element of the curve $Mm = ds$, and having derived $M\pi$ parallel to the axis *AP*, we will have $Pp = M\pi = dx$ and $\pi m = dy$; therefore

$ds = \sqrt{(dx^2 + dy^2)}$. We extend from the point $P$ also onto the tangent $MV$ the perpendicular $PS$ and onto this line from the point $A$ the perpendicular $AR$, which will be parallel to the tangent $MV$. Now, since the triangles $PMS$ and $APR$ are similar to the triangle $Mm\pi$, we can derive: $PS = \frac{M\pi \cdot PM}{Mm} = \frac{ydx}{ds}$ and $PR = \frac{m\pi \cdot AP}{Mm} = \frac{xdy}{ds}$: from where, because of $AV = PS - PR$, we will have this equation, $a = \frac{ydx - xdy}{ds}$ or $ydx - xdy = a\,ds = a\sqrt{(dx^2 + dy^2)}$, which will express the nature of the sought curve.

V. Therefore, here is a differential equation for the curve we seek: and if we want to handle it according to the ordinary method, it is necessary to first of all remove the differentials from the radical sign: therefore taking the squares, we will have:

$$y^2 dx^2 - 2xydxdy + x^2 dy^2 = a^2 dx^2 + a^2 dy^2$$

and hence:

$$dy^2 = \frac{-2xydxdy - a^2 dx^2 + y^2 dx^2}{a^2 - x^2} \quad \dagger$$

from which extracting the root produces:

$$dy = \frac{-xydx + a\,dx\sqrt{(x^2 + y^2 - a^2)}}{a^2 - x^2} \quad \dagger$$

or

$$a^2 dy - x^2 dy + xydx = a\,dx\sqrt{(x^2 + y^2 - a^2)}$$

from which it is now necessary to find the integral to know the curve in question.

VI. To integrate this equation, let us call $y = u\sqrt{(a^2 - x^2)}$, to have $\sqrt{(x^2 + y^2 - a^2)} = \sqrt{(a^2 - x^2)(u^2 - 1)}$, and $dy = du\sqrt{(a^2 - x^2)} - \frac{uxdx}{\sqrt{(a^2 - x^2)}}$, in which $a^2 dy - x^2 dy = du\,(a^2 - x^2)^{3/2} - uxdx\sqrt{(a^2 - x^2)}$. These values being substituted give:

$$du\,(a^2 - x^2)^{3/2} = a\,dx\sqrt{(a^2 - x^2)(u^2 - 1)}$$

or

$$\frac{du}{\sqrt{(u^2 - 1)}} = \frac{a\,dx}{a^2 - x^2},$$

an equation where the variables $x$ and $u$ are separated.

VII. Since this equation is separated, I first of all note that its conditions are satisfied if we set $\sqrt{(u^2 - 1)} = 0$, or $u^2 = 1$; because in this case both

† The negative signs at the front of these numerators were erroneously placed in front of the whole fraction in the original paper.

$adx\sqrt{(a^2-x^2)(u^2-1)}$ and $du\,(a^2-x^2)^{3/2}$, since $du=0$, vanish. Hence, again we have an integral value $u^2=1$, or $u=\pm 1$, from where we derive $y=\pm\sqrt{(a^2-x^2)}$, or $y^2+x^2=a^2$; this is the equation for a circle described around the center $A$ with radius $a$. Now it is clear that this circle would satisfy the problem, since the perpendicular $AV$ becomes equal to the radius of the circle and falls on the tangent point $M$, as is known by the properties of circles.

VIII. But this case is still not the extent of the differential equation $\frac{du}{\sqrt{(u^2-1)}}=\frac{a\,dx}{a^2-x^2}$; let us therefore search for its integral, which will be by logarithms

$$\log[u+\sqrt{(u^2-1)}\,]=\tfrac{1}{2}\log\frac{n^2(a+x)}{a-x}$$

so that we would have:

$$u+\sqrt{(u^2-1)}=n\sqrt{\frac{a+x}{a-x}}.$$

From this, we will find,

$$-1=n^2\,\frac{a+x}{a-x}-2nu\sqrt{\frac{a+x}{a-x}}$$

and hence

$$u=\frac{n}{2}\sqrt{\frac{a+x}{a-x}}+\frac{1}{2n}\sqrt{\frac{a-x}{a+x}}.$$

Resulting in

$$y=u\sqrt{(a^2-x^2)}=\frac{n}{2}(a+x)+\frac{1}{2n}(a-x),$$

the equation for a straight line derived in such a way that the perpendicular that is derived from the given point $A$ to this line is equal to $a$.

IX. Therefore, here is the solution of the proposed problem that we would find by the ordinary method where it is firstly necessary to separate the variables and then integrate the separated differential equation. Now it is clear that this operation is not only awkward, but it would become impossible if instead of the irrational formula $\sqrt{(dx^2+dy^2)}$, we would have a more complicated one. For example, if we were to encounter this equation

$$ydx-xdy=a\sqrt[3]{(dx^3+dy^3)}$$

In taking the cubes, it would be a pain to extract the root for finding the relation between the differentials $dx$ and $dy$, and if the root were higher, this extraction would become impossible.

X. Well, now I say that this same equation that contains the solution of the problem

$$ydx - xdy = a\sqrt{(dx^2 + dy^2)}$$

can reduce to a finite equation, and even algebraic, between $x$ and $y$, without $y$ using the ordinary integration just seen: but, in what consists of the crux of the paradox, by a subsequent differentiation of this equation. Or this will be the same differentiation that we will apply to the integral equation, which will make known to us the nature of the sought curve. What I just advanced will put the crux of the paradox that I had proposed to disentangle here in all its glory.

XI. In order that the differentials do not cause us any trouble in using a subsequent differentiation, let us assume $dy = p\ dx$, and we will have $\sqrt{(dx^2 + dy^2)} = dx\sqrt{(1+p^2)}$. By this substitution our equation, being divided by $dx$, will take this form,

$$y - px = a\sqrt{(1+p^2)} \text{ or } y = px + a\sqrt{(1+p^2)}$$

where it does well to note that although we no longer perceive the differentials here, this equation is still differential because of the letter $p$, which has the value of $\frac{dy}{dx}$; so if we were to replace it, we would return to the first differential equation.

XII. Presently, instead of integrating this differential equation, I differentiate it once more to have,

$$dy = p\,dx + x\,dp + \frac{ap\,dp}{\sqrt{(1+p^2)}}.$$

Now, having assumed $dy = p\ dx$, replacing $dy$ with this value first of all gives us:

$$0 = x\,dp + \frac{ap\,dp}{\sqrt{(1+p^2)}},$$

from where in dividing by $dp$ we first of all derive:

$$x = -\frac{ap}{\sqrt{(1+p^2)}}$$

and since $y = px + a\sqrt{(1+p^2)}$, in substituting this value of $x = -\frac{ap}{\sqrt{(1+p^2)}}$ here, we will have:

$$y = -\frac{ap^2}{\sqrt{(1+p^2)}} + a\sqrt{(1+p^2)} \text{ or } y = \frac{a}{\sqrt{(1+p^2)}}.$$

XIII. Therefore, here are the values, and even algebraic, for the two coordinates $x$ and $y$, these only containing the single variable $p$: and as presently there is no longer a question of the assumed value of $p = \frac{dy}{dx}$, the problem is solved by this repeated differentiation. Because we only have to eliminate the variable $p$ from these two equations

$$x = -\frac{ap}{\sqrt{(1+p^2)}} \text{ and } y = \frac{a}{\sqrt{(1+p^2)}}$$

this can be easily done by adding together the squares $x^2$ and $y^2$, from where we will first of all have

$$x^2 + y^2 = \frac{a^2p^2 + a^2}{1+p^2} = a^2$$

which is the equation for the circle that would satisfy the proposed problem.

XIV. It is true that besides the circle there are still infinitely many straight lines that equally satisfy the question that this method does not seem to produce. But it nevertheless contains them, and visibly still more than the other ordinary method. We only have to regard the equation $0 = x\,dp + \frac{ap\,dp}{\sqrt{(1+p^2)}}$, which we arrived at by differentiation, and which, since it is divisible by *dp*, also contains the solution *dp* = 0. Now from this we immediately derive *p* = const. = *n*, and hence $y = nx + a\sqrt{(1+n^2)}$; where all of the straight lines that fulfill the conditions of the problem are comprised.

XV. Having already noted that this equation:

$$ydx - xdy = a\sqrt[3]{(dx^3 + dy^3)}$$

is a pain to solve by the ordinary method, we will first of all produce its integral by differentiation. From calling *dy* = *p dx*, we will have $\sqrt[3]{(dx^3+dy^3)} = dx\sqrt[3]{(1+p^3)}$, and hence

$$y - px = a\sqrt[3]{(1+p^3)} \text{ or } y = px + a\sqrt[3]{(1+p^3)}$$

which being differentiated gives

$$dy = p\,dx = p\,dx + x\,dp + \frac{ap^2\,dp}{\sqrt[3]{(1+p^3)^2}}$$

from where we derive $0 = x\,dp + \frac{ap^2\,dp}{\sqrt[3]{(1+p^3)^2}}$, or

$$x = \frac{-ap^2}{\sqrt[3]{(1+p^3)^2}} \text{ and } y = \frac{a}{\sqrt[3]{(1+p^3)^2}}.$$

XVI. If we want to eliminate *p* here, we only have to add the cubes to have $y^3 + x^3 = \frac{a^3(1-p^6)}{(1+p^3)^2} = \frac{a^3(1-p^3)}{1+p^3} = -a^3 + \frac{2a^3}{1+p^3}$ so that $\frac{1}{1+p^3} = \frac{a^3+x^3+y^3}{2a^3}$, and hence

$$y = \frac{a}{\sqrt[3]{(1+p^3)^2}} = \frac{(a^3+x^3+y^3)^{2/3}}{a\sqrt[3]{4}}.$$

Therefore

$$4a^3y^3 = (a^3 + x^3 + y^3)^2$$

or

$$0 = a^6 + 2a^3x^3 - 2a^3y^3 + x^6 + 2x^3y^3 + y^6$$

for a sixth order curve. But besides this, it would also satisfy $dp = 0$, or $p = n$, because of the division by $dp$; and this case gives infinitely many straight lines contained in this equation

$$y = nx + a\sqrt[3]{(1+n^3)}.$$

XVII. We see that by the same method we will easily solve all the problems which would lead to such equations:

$$y\,dx - x\,dy = a\sqrt[n]{(\alpha dx^n + \beta dx^{n-\nu} dy^\nu + \gamma dx^{n-\mu} dy^\mu + etc.)}$$

Because of calling $dy = p\,dx$, we would have

$$y = px + a\sqrt[n]{(\alpha + \beta p^\nu + \gamma p^\mu + etc.)}$$

and differentiating and dividing by $dp$,

$$x = \frac{-\nu a\beta p^{\nu-1} - \mu a\gamma p^{\mu-1} - etc.}{n\sqrt[n]{(\alpha + \beta p^\nu + \gamma p^\mu + etc.)^{n-1}}}$$

$$\text{and } y = \frac{na\alpha + (n-\nu)a\beta p^\nu + (n-\mu)a\gamma p^\mu + etc.}{n\sqrt[n]{(\alpha + \beta p^\nu + \gamma p^\mu + etc.)^{n-1}}}$$

from where, in eliminating $p$, we will derive an algebraic equation between $x$ and $y$. Now, also since $dp = 0$ and $p$ = const. = $m$, the straight lines contained in this formula:

$$y = mx + a\sqrt[n]{(\alpha + \beta m^\nu + \gamma m^\mu + etc.)}$$

are equally satisfying. I therefore move to another problem.

## PROBLEM II

*On the axis AB, find the curve AMB such that having derived from one of its points M the tangent TMV, it intersects the two straight lines AE and BF, derived perpendicularly to the axis AB at the two given points A and B, so that the rectangle formed by the lines AT and BV is the same size everywhere. (Fig. 2)*

XVIII. Let the given interval $AB = 2a$, the abscissa $AP = x$,† the ordinate $PM = y$, and having derived the infinitely close line $pm$, we will have $Pp = M\pi = dx$, and $\pi m = dy$. We derive the straight lines $MR$ and $MS$ parallel to the axis $AB$, and the relation of the triangles $M\pi m$, $TRM$, and $MSV$, because of $PB = MS = 2a - x$, will produce:

$$RM = \frac{x\,dy}{dx}\ddagger \text{ and } SV = \frac{(2a-x)\,dy}{dx}$$

from where we will have:

† Originally written $AB = x$.

‡ Originally written $PM = \dfrac{x\,dy}{dx}$.

$$AT = y - \frac{x\,dy}{dx} \text{ and } BV = y + \frac{(2a-x)\,dy}{dx}$$

of which the upcoming product is a constant $c^2$ :

$$(y - \frac{x\,dy}{dx})(y - \frac{x\,dy}{dx} + \frac{2a\,dy}{dx}) = c^2.$$

XIX. If we would like to treat this equation by the ordinary method, we would surely encounter difficulties, and maybe would only arrive at the integral equation after numerous detours. But to assist us in the other method, let us call $dy = p\,dx$ to have

$$(y - px)(y - px + 2ap) = c^2$$

or:

$$y^2 + 2(a-x)py - 2ap^2x + p^2x^2 = c^2$$

or

$$y^2 + 2(a-x)py + (a-x)^2p^2 = c^2 + a^2p^2$$

from where the extraction of the root produces:

$$y + (a-x)p = \sqrt{(c^2 + a^2p^2)}$$

or

$$y = -(a-x)p + \sqrt{(c^2 + a^2p^2)}.$$

XX. Let us now differentiate this equation, instead of searching for the integral, and we will obtain:

$$dy = p\,dx = -(a-x)\,dp + p\,dx + \frac{a^2p\,dp}{\sqrt{(c^2 + a^2p^2)}}$$

where the terms $p\,dx$ eliminating each other and the division by $dp$ will give:

$$a - x = \frac{a^2p}{\sqrt{(c^2 + a^2p^2)}} \text{ or } x = a - \frac{a^2p}{\sqrt{(c^2 + a^2p^2)}}$$

and substituting the value of $a - x$ in the one of $y$, we will have

$$y = \frac{-a^2p^2}{\sqrt{(c^2 + a^2p^2)}} + \sqrt{(c^2 + a^2p^2)} \text{ or } y = \frac{c^2}{\sqrt{(c^2 + a^2p^2)}}.$$

XXI. Having therefore:

$$\frac{a-x}{a} = \frac{ap}{\sqrt{(c^2 + a^2p^2)}} \text{ and } \frac{y}{c} = \frac{c}{\sqrt{(c^2 + a^2p^2)}}$$

The elimination of the quantity $p$ will occur in adding the squares of these two formulæ, which will give:

$$\frac{(a-x)^2}{a^2} + \frac{y^2}{c^2} = \frac{a^2p^2 + c^2}{c^2 + a^2p^2} = 1,$$

therefore:

$$\frac{y^2}{c^2} = \frac{2ax - x^2}{a^2} \text{ or } y = \frac{c}{a}\sqrt{(2ax - x^2)}$$

from where we see that the sought curve is an ellipse described on the axis *AB*, and of which the conjugate semi-axis is *c*, so that in such an ellipse the rectangle of the tangents *AT* and *BV* is always equal to the square of the conjugate semi-axis.

XXII. But it is clear that besides this curve, the problem would also be satisfied by infinitely many straight lines *TV* derived such that the rectangle $AT \cdot BV$ is $c^2$. These lines are found by setting the divisor *dp* equal to 0, which gives *p* = const. = *n*, from where we will have: $y = -n\,(a-x) + \sqrt{(c^2 + n^2a^2)}$. From this, if *x* = 0, we derive $AT = -\,na + \sqrt{(c^2 + n^2a^2)}$, and if *x* = 2*a*, $BV = na + \sqrt{(c^2 + n^2a^2)}$, so that we always have

$$AT \cdot BV = c^2$$

some value that is able to have the number *n*.

## PROBLEM III

*Given two points A and C, find the curve EM such that if we derive some tangent MV, which the perpendicular AV is directed towards from the first point A, and we join the straight line CV to V from the other point C, this line CV is the same size everywhere. (Fig. 3)*

XXIII. Let us set the given distance *AC* equal to *b*, and taking this line for the axis, the ordinate *MP* is directed towards it from the point *M* and is infinitely close to the line *pm*. Let *AP* = *x*, and *PM* = *y*; and because of $Pp = M\pi = dx$ and $\pi m = dy$, let $Mm = \sqrt{(dx^2 + dy^2)} = ds$. This proposed, we have seen in the solution of the first problem that we will have: $AV = \dfrac{ydx - xdy}{ds}$. Let us also extend the line *VX* from the point *V* perpendicular to the axis, and because of the similarity of the triangles *Mmπ* and *VAX*, we will have:

$$VX = \frac{dx\,(ydx - xdy)}{ds^2} \text{ and } AX = \frac{dy\,(ydx - xdy)}{ds^2}$$

and hence:

$$CX = b + \frac{dy\,(ydx - xdy)}{ds^2}.$$

XXIV. Now let the given length *CV* equal *a*, and because $CV^2 = CX^2 + XV^2$ we will have:

$$a^2 = b^2 + \frac{2b\,dy\,(ydx - xdy)}{ds^2} + \frac{(ydx - xdy)^2}{ds^2}$$

and furthermore, because $dx^2 + dy^2 = ds^2$:

$$\frac{(ydx - xdy)^2}{ds^2} + \frac{2b\,dy\,(ydx - xdy)}{ds^2} + \frac{b^2dy^2}{ds^2} = a^2 - b^2 + \frac{b^2dy^2}{ds^2} = a^2 - \frac{b^2dx^2}{ds^2}$$

of which the square root is

$$\frac{y\,dx - x\,dy}{ds} + \frac{b\,dy}{ds} = \sqrt{(a^2 - \frac{b^2 dx^2}{ds^2})}$$

or from multiplying by *ds*

$$y\,dx - x\,dy + b\,dy = \sqrt{(a^2 ds^2 - b^2 dx^2)}.$$

XXV. Here it is evident enough that we would dive into a very annoying calculation, if we would want to attempt to solve this equation by the ordinary method. I therefore set $dy = p\,dx$, and because $ds^2 = dx^2(1+p^2)$, our differential equation will take this form

$$y - px + bp = \sqrt{(a^2(1+p^2) - b^2)}$$

which I differentiate again, placing $p\,dx$ in for $dy$, and I will have:

$$p\,dx - p\,dx - x\,dp + b\,dp = \frac{a^2 p\,dp}{\sqrt{(a^2(1+p^2) - b^2)}}$$

which being divided by $dp$ gives:

$$b - x = \frac{a^2 p}{\sqrt{(a^2(1+p^2) - b^2)}} \text{ or } x = b - \frac{a^2 p}{\sqrt{(a^2(1+p^2) - b^2)}}$$

and

$$y = -(b-x)p + \sqrt{(a^2(1+p^2) - b^2)} = \frac{a^2 - b^2}{\sqrt{(a^2(1+p^2) - b^2)}}.$$

XXVI. From this, to eliminate $p$, I form these equations:

$$\frac{b-x}{a} = \frac{ap}{\sqrt{(a^2(1+p^2) - b^2)}} \text{ and } \frac{y}{\sqrt{(a^2 - b^2)}} = \frac{\sqrt{(a^2 - b^2)}}{\sqrt{(a^2(1+p^2) - b^2)}}$$

and adding the squares of these formulæ, I find:

$$\frac{(b-x)^2}{a^2} + \frac{y^2}{a^2 - b^2} = \frac{a^2(1+p^2) - b^2}{a^2(1+p^2) - b^2} = 1$$

which is the equation for an ellipse whose center is $C$[†], one of its foci at $A$, and the semi-major axis equal to $CV$. But besides this ellipse, the divisor $dp = 0$ still gives infinitely many straight lines comprised in this equation

$$y = -n(b-x) + \sqrt{(a^2(1+n^2) - b^2)}.$$

## PROBLEM IV

*Given two points A and B, find the curve EM such that having derived some tangent VMX, if the perpendiculars AV and BX are directed towards it from the points A and B, the rectangle of these lines* $AV \cdot BX$ *is the same size everywhere. (Fig. 4)*

XXVII. Let the distance of the given points $AB = 2b$ on which we derive the perpendicular $MP$ and an infinitely close line $mp$: and we call the coordinates: $AP = x$,

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[†] Originally mislabeled $D$.

$PM = y$, to have $Pp = M\pi = dx$, $\pi m = dy$, and $Mm = \sqrt{(dx^2 + dy^2)} = ds$. This proposed, we have seen that we will have $AV = \dfrac{ydx - xdy}{ds}$. Furthermore, we derive $AR$ perpendicular to $BX$, and the similarity of the triangles $Mm\pi$ and $ABR$ will produce $BR = \dfrac{2bdy}{ds}$, and in adding to it $RX = AV = \dfrac{ydx - xdy}{ds}$, we will have $BX = \dfrac{ydx + (2b - x)dy}{ds}$. Therefore, let $c^2$ be the rectangle of the lines $AV$ and $BX$, and we will have for the curve $EM$ this equation:

$$(ydx - xdy)(ydx - xdy + 2bdy) = c^2 ds^2.$$

XXVIII. Without troubling ourselves with the ordinary method, let us call $dy = p\,dx$, so that $ds^2 = dx^2(1 + p^2)$, and we will have:

$$(y - px)(y - px + 2bp) = c^2(1 + p^2)$$

which reduces to:

$$y^2 + 2(b - x)py - 2bp^2x + p^2x^2 = c^2(1 + p^2)$$

or to

$$y^2 + 2(b - x)py + (b - x)^2 p^2 = c^2(1 + p^2) + b^2p^2$$

of which the square root is:

$$y + (b - x)p = \sqrt{(c^2 + (b^2 + c^2)p^2)}$$

and hence

$$y = -(b - x)p + \sqrt{(c^2 + (b^2 + c^2)p^2)}.$$

XXIX. Let us differentiate this differential equation again, and because $dy = p\,dx$, we will have:

$$p\,dx = -(b - x)\,dp + p\,dx + \frac{(b^2 + c^2)p\,dp}{\sqrt{(c^2 + (b^2 + c^2)p^2)}}$$

which being divided by $dp$ first of all gives:

$$b - x = \frac{(b^2 + c^2)p}{\sqrt{(c^2 + (b^2 + c^2)p^2)}}$$

or $b - x = \dfrac{a^2 p}{\sqrt{(c^2 + a^2p^2)}}$ calling $b^2 + c^2 = a^2$ for brevity. From this we will derive:

$$y = -(b - x)p + \sqrt{(c^2 + a^2p^2)} = \frac{c^2}{\sqrt{(c^2 + a^2p^2)}}.$$

Therefore, having:

$$\frac{b - x}{a} = \frac{ap}{\sqrt{(c^2 + a^2p^2)}} \text{ and } \frac{y}{c} = \frac{c}{\sqrt{(c^2 + a^2p^2)}}$$

We will have when adding the squares

$$\frac{(b-x)^2}{a^2}+\frac{y^2}{c^2}=1.$$

XXX. This equation, as is evident, is an ellipse whose foci are at the points *A* and *B*; and hence the center is at the point in the middle *C*. The semi-minor axis will therefore be *c*; and this squared is from which the rectangle $AV\cdot BX$ will be equal everywhere: this being also a property of ellipses. Now there are also straight lines that satisfy this same problem that the divisor $dp = 0$ will provide us. Since calling $p = n$, the equation for all these straight lines will be $y=-n\,(b-x)+\sqrt{(c^2+n^2a^2)}$. I would still add many similar problems to confirm this paradox, but these four will entirely suffice to prove it true.

*The Second Paradox*

## XXXI.

The second paradox that I will put forth is no less surprising, since it is also contrary to the common ideas of integral calculus. We usually imagine that having some differential equation, we only need to find its integral and to render it in its full extent by adding to it an undefined constant to have all the cases that are comprised in the differential equation. Or when this differential equation is resultant from a solution of a problem, we have no doubt that the integral equation found by the ordinary rules contains every possible solution of the problem: this is understood when we have not neglected the addition of a constant that all integration demands.

XXXII. However, there are cases where ordinary integration gives us a finite equation that does not contain all that would be contained in the proposed differential equation, still not neglecting the aforementioned constant. This would seem much more paradoxical since we are accustomed to being convinced of the accuracy of the idea explained in the previous paragraph. Because if the integral equation, which we will have found after all prescribed precautions, does not exhaust the extent of the differential equation, the problem will allow solutions that the integration will not produce, and hence we will arrive at a defective solution that undoubtedly seems to upset the ordinary principles of integral calculus.

XXXIII. Now it is very easy to propose infinitely many differential equations that show a certain relation between the variable quantities that is impossible to find by the scope of ordinary integration. Let us propose, for example, this differential equation:

$$xdx+ydy=dy\sqrt{(x^2+y^2-a^2)}$$

and it is evident that the finite equation $x^2+y^2-a^2=0$ would entirely satisfy this. Because of having from that $xdx+ydy=0$, both members of the differential equation vanish by themselves: this is an undoubted sign that this finite equation $x^2+y^2=a^2$ is contained in the proposed differential equation: or that a circle resolves the problems that lead to this differential equation.

XXXIV. However, when we integrate this differential equation, we will not find this relation $x^2 + y^2 = a^2$ in the least: because, dividing our equation by $\sqrt{(x^2 + y^2 - a^2)}$, we have:

$$\frac{xdx + ydy}{\sqrt{(x^2 + y^2 - a^2)}} = dy.$$

The integral is evident and when completely expanded is

$$\sqrt{(x^2 + y^2 - a^2)} = y + c$$

having introduced the undefined constant $c$. Now it is clear that the equation we already found $x^2 + y^2 = a^2$ is absolutely not contained in this integral equation for whatever value that we give to the constant $c$.

XXXV. Taking the squares of our integral equation we found, we will have:

$$x^2 - a^2 = 2cy + c^2 \text{ and } y = \frac{x^2 - a^2 - c^2}{2c}$$

and hence we would believe that the proposed problem, which led us to this equation, only satisfies infinitely many parabolas, contained in the equation $y = \dfrac{x^2 - a^2 - c^2}{2c}$, according to the different values of $c$. And since we found infinitely many parabolas, we will doubt much less that we had not arrived at a complete solution. However, we just saw that the same problem would also be satisfied by the circle contained in the equation $x^2 + y^2 = a^2$.

XXXVI. I have encountered several other cases of this type in my Treatise on movement, where I had already noted this same paradox in which a differential equation sometimes contains solutions that are not comprised in the integrated equation: I had also given a sure rule there, by the way of which we can find these solutions contained in the differential equations that we would not be able to derive from the integrated equation. However, as I had not created enough of a feel there for the evident importance of this paradox, we could believe that this is some bizarreness in mechanical problems that would have no place in Geometric problems, or that this would not be a reproach that we could directly make to the same Analysis.

XXXVII. For the example I just made here, as it is formed from fancy, we could also doubt if this case is ever encountered in the solution of a real problem. But the same examples that I related to clarify the first paradox will also serve to clarify this one. Because the first problem asks for a curve such that if perpendicular lines head from a given point to all its tangents, all its perpendiculars will be equivalent; this problem, I say, being proposed, we first of all see that a circle described about a point given as its center with a radius equal to the straight line to which all the mentioned perpendiculars must be equal will satisfy the problem.

XXXVIII. However, having been lead to this differential equation:

$$a^2\,dy - x^2\,dy + xy\,dx = a\,dx\sqrt{(x^2+y^2-a^2)}$$

where the variables $x$ and $y$ are mixed together, we saw that by way of this substitution $y = u\sqrt{(a^2-x^2)}$, it changes to this separated form,

$$\frac{du}{\sqrt{(u^2-1)}} = \frac{a\,dx}{a^2-x^2}$$

of which the integral in its full extent would be

$$u+\sqrt{(u^2-1)} = u\sqrt{\frac{a+x}{a-x}}$$

from where I derived this equation:

$$y = \frac{n}{2}(a+x)+\frac{1}{2n}(a-x)$$

which only contains straight lines, so that the circle seems at this hour entirely excluded from the solution of the proposed problem.

XXXIX. It is the same with the second problem, which is solved, as we have seen, by an ellipse expressed by this equation $y = \frac{c}{a}\sqrt{(2ax-x^2)}$; this is also clear by the known properties of the ellipse. Now having found this differential equation:

$$(y-\frac{x\,dy}{dx})(y-\frac{x\,dy}{dx}+\frac{2a\,dy}{dx}) = c^2,$$

we will derive by the extraction of the root:

$$\frac{dy}{dx} = \frac{(a-x)y+\sqrt{(a^2y^2-c^2(2ax-x^2))}}{2ax-x^2}$$

$$(2ax-x^2)\,dy-(a-x)\,y\,dx = dx\sqrt{(a^2y^2-c^2(2ax-x^2))}.$$

Now it is evident that the equation $a^2y^2-c^2(2ax-x^2)=0$ would satisfy this formula, because we will have from this $y=\frac{c}{a}\sqrt{(2ax-x^2)}$, and hence in differentiating their logarithms:

$$\frac{dy}{y} = \frac{dx\,(a-x)}{2ax-x^2},\text{ or } (2ax-x^2)\,dy-(a-x)\,y\,dx = 0,$$

so that in this case, both members of the differential equation vanish.

XL. But, if we treat this differential equation according to the ordinary method, and we call $y = u\sqrt{(2a-x^2)}$, to have

$$\sqrt{(a^2y^2-c^2(2ax-x^2))} = \sqrt{(2ax-x^2)(a^2u^2-c^2)}$$

and

$$dy = du\sqrt{(2ax-x^2)}+\frac{u\,(a-x)dx}{\sqrt{(2ax-x^2)}}$$

These substituted values will change our equation to this form:

$$du\,(2ax-x^2)^{3/2}+u\,(a-x)\,dx\,\sqrt{(2ax-x^2)}-u\,(a-x)\,dx\,\sqrt{(2ax-x^2)}=$$
$$dx\,\sqrt{(2ax-x^2)(a^2u^2-c^2)}$$

which now reduces to this separated form,

$$\frac{du}{\sqrt{(a^2u^2-c^2)}}=\frac{dx}{2ax-x^2}\ \text{ or }\ \frac{a\,du}{\sqrt{(a^2u^2-c^2)}}=\frac{a\,dx}{2ax-x^2}$$

of which the general integral is

$$\log\frac{au+\sqrt{(a^2u^2-c^2)}}{b}=\tfrac{1}{2}\log\frac{x}{2a-x}$$

or

$$au+\sqrt{(a^2u^2-c^2)}=b\sqrt{\frac{x}{2a-x}}=\sqrt{\frac{bx}{(2ax-x^2)}}.$$

XLI. From this we will easily find the value of $u$, which will be:

$$u=\frac{c^2\sqrt{(2ax-x^2)}}{2bx}+\frac{bx}{2\sqrt{(2ax-x^2)}},$$

and since $y=u\sqrt{(2a-x^2)}$, we will obtain:

$$y=\frac{c^2\sqrt{(2ax-x^2)}}{2bx}+\frac{bx}{2}=\frac{ac^2}{b}+\frac{(b^2-c^2)\,x}{2b},$$

and it is evident that this integral equation, somewhat general as it is because of the undefined constant $b$, does not contain the ellipse we already found. This same accident will also take place in the two other related problems, where we will treat the differential equations we find by the ordinary method for finding integrals, where the ellipse, which produces a valid solution, will not be comprised.

XLII. But here is the general rule by which we can easily find these cases where the integral of a proposed differential equation escapes ordinary integration. Let $z$ be some function of two variables $x$ and $y$, and let $Z$ be some function of $z$. Furthermore, let $P$, $Q$, $V$ also be some functions of the variables $x$ and $y$, and let us assume that we could reach this differential equation

$$V\,dz=Z\,(P\,dx+Q\,dy)$$

and it is clear that the value $Z = 0$ would satisfy this equation; from this we will derive $z$ = const. and hence $dz = 0$, so that in the cases $Z = 0$ the two members of the proposed equation vanish.

XLIII. By this rule, we will easily find the ellipse that contains a solution of the second problem; because of being reached by this differential equation:

$$\frac{du}{\sqrt{(a^2u^2-c^2)}}=\frac{dx}{2ax-x^2}\ \text{ or }\ du\,(2ax-x^2)=dx\,\sqrt{(a^2u^2-c^2)}$$

replacing $z$ with $u$ and the function $\sqrt{(a^2u^2 - c^2)}$ for $Z$, the proposed equation will be satisfied by the equality $Z = 0$ or $a^2u^2 - c^2 = 0$ from where we derive $u = \frac{c}{a}$ and hence $y = \frac{c}{a}\sqrt{(2a - x^2)}$, which is the equation for the ellipse in question that is found to be excluded from the integrated equation.

XLIV. Here it is worth noting that these same cases that are inaccessible by ordinary integration are precisely those that a repeated differentiation produced for us in the clarifications of the first paradox. And in briefly reflecting here, we will notice that this agreement is not due to mere chance, and we could claim in general that every time a differential equation is differentiated again, it immediately leads to a finite equation that never would be found by the ordinary scope of integration; but to find it, it is necessary to apply the rule that I just explained. From this, we therefore see that the two explained paradoxes are bound together as such, one must contain the other.

XLV. The rule therefore, following what we ordinarily judge, if a differential equation is fully integrated or not, is not general. We commonly believe that when we integrated a differential equation in sort, the integral equation contains an undefined constant that is not found in the differential, and then the integral equation is complete to the same extent as the differential. But we see by the examples related here that although the equations found by integration contain such a constant, which seems to make it general, the differential equations still hold a solution that is not comprised in the integral. This circumstance on the criterion of complete integral equations could provide us with a third paradox, if it were not already so tightly bound with the preceding ones.

XLVI. It can therefore often occur that it is even absolutely impossible to integrate, or still to separate a proposed differential equation, and of which we could nevertheless by the given rule find a finite equation that would satisfy the question. Thus, if we were to find in the solution of a problem an equation such as

$$a^2(a^2 - x^2)\,dy + a^2xy\,dx = (a^2 - x^2)(y\,dx - x\,dy)\sqrt{(y^2 + x^2 - a^2)}$$

for which we would uselessly attempt integration, we would yet be sure that this finite equation $y^2 + x^2 = a^2$ is comprised in it, because in setting $y^2 + x^2 - a^2 = 0$, both members of the equation vanish; this becomes clearer when we set $y = z\sqrt{(a^2 - x^2)}$, because then the equation will take this form: $a^2dz = (y\,dx - x\,dy)\sqrt{(z^2 - 1)}$. Then, calling $Z = \sqrt{(z^2 - 1)}$ we will have by the given rule $\sqrt{(z^2 - 1)} = 0$, or $z = 1$, and hence $y^2 + x^2 = a^2$.

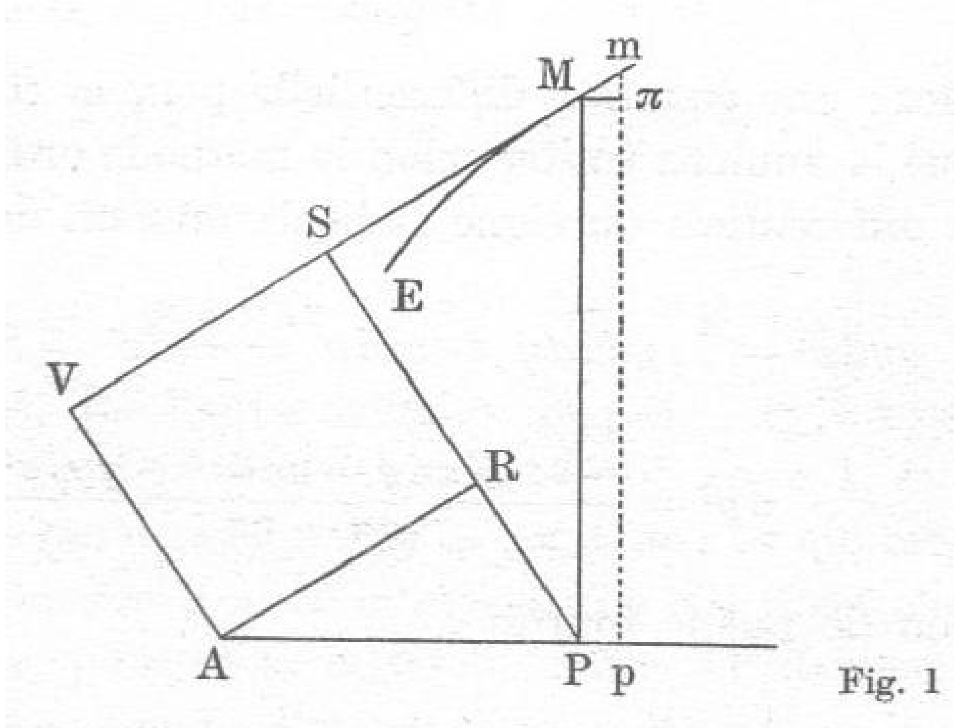


Fig. 1

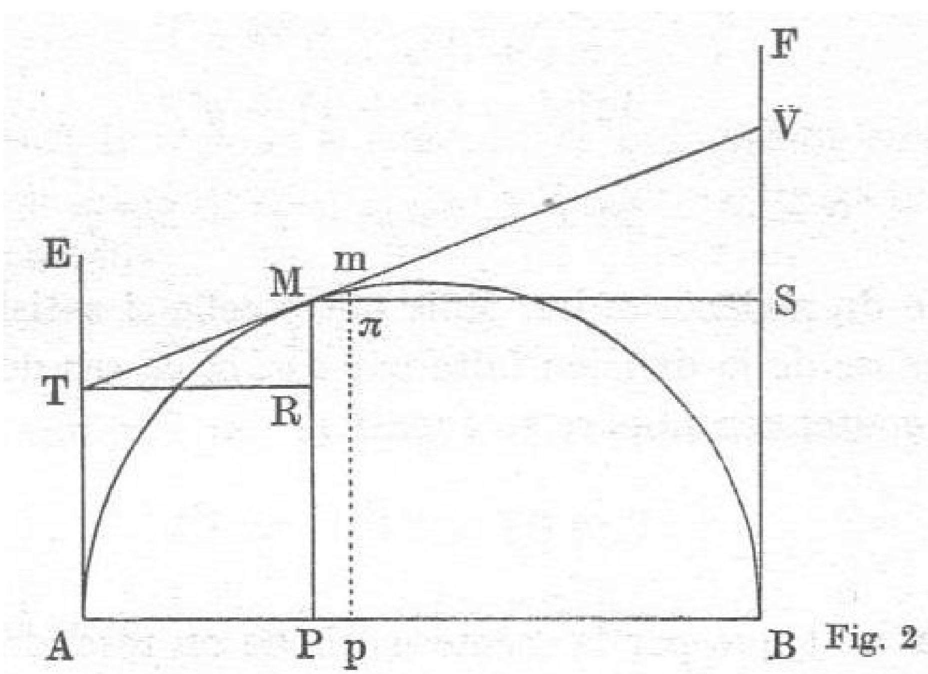


Fig. 2

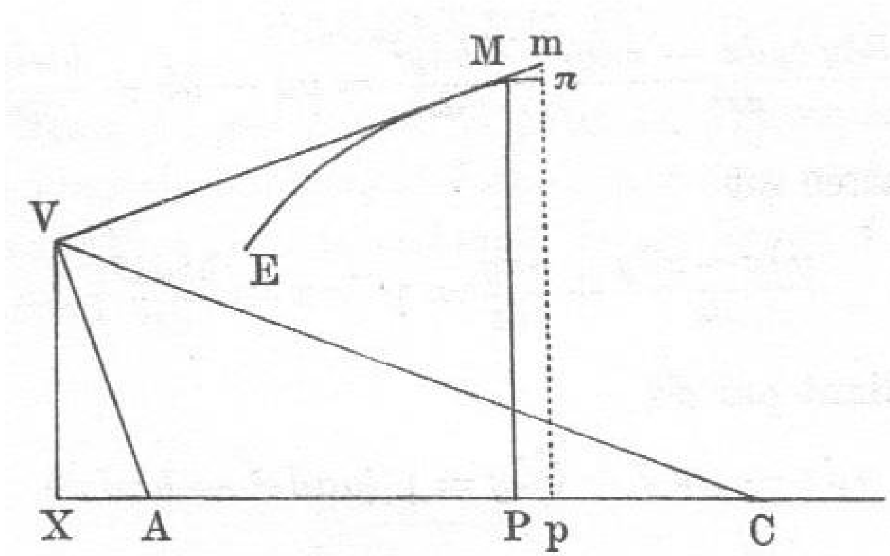


Fig. 3

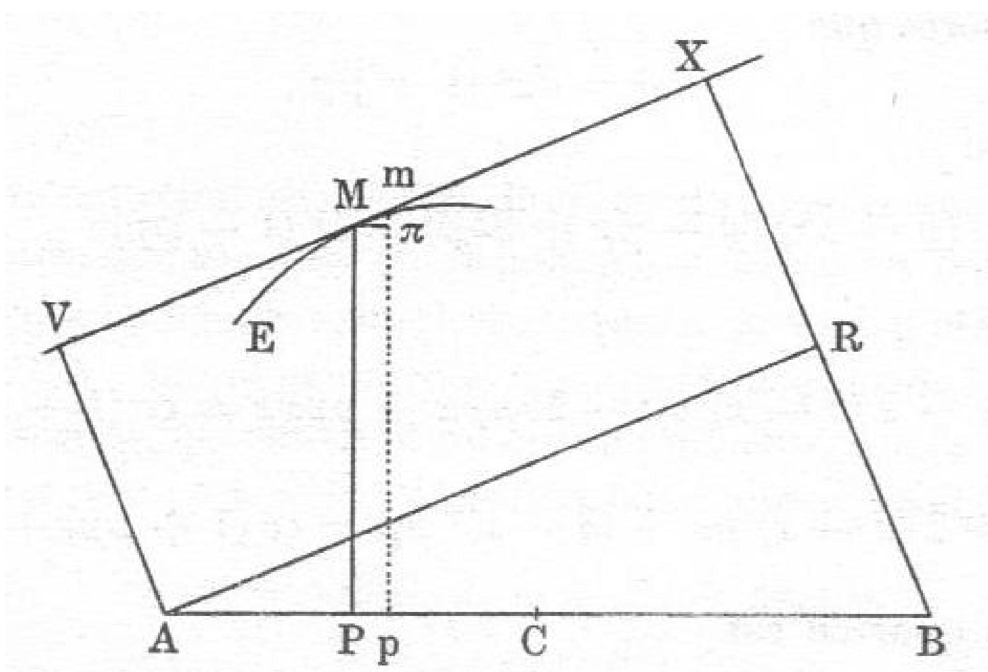


Fig. 4