INTEGRATION
OF A REMARKABLE TYPE
OF DIFFERENTIAL EQUATION
IN ANALYTICAL FUNCTIONS WITH TWO VARIABLES.

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Let z be a function with two variables x and y deriving the following formulæ:

$$P = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$$

$$Q = \frac{\partial^2 z}{\partial x^2} + \frac{2\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2}$$

$$R = \frac{\partial^3 z}{\partial x^3} + \frac{3\partial^3 z}{\partial x^2 \partial y} + \frac{3\partial^3 z}{\partial x \partial y^2} + \frac{\partial^3 z}{\partial y^3} \;\dagger$$

$$S = \frac{\partial^4 z}{\partial x^4} + \frac{4\partial^4 z}{\partial x^3 \partial y} + \frac{6\partial^4 z}{\partial x^2 \partial y^2} + \frac{4\partial^4 z}{\partial x \partial y^3} + \frac{\partial^4 z}{\partial y^4} \;\ddagger$$

and so on.

This proposed, I will give here a rather peculiar method of finding by a single integration the complete integral of this differential equation:

$$Az + BP + CQ + DR + ES + \text{etc.} = 0$$

to what degree the differentials can show.

To this effect, it is necessary to firstly note that all of these formulæ P, Q, R, S, etc. hold a very elegant relation among them; because as $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = P$, it is found that

$$\frac{\partial P}{\partial x} + \frac{\partial P}{\partial y} = \frac{\partial^2 z}{\partial x^2} + \frac{2\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = Q$$

and in the same manner

$$\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial y} = \frac{\partial^3 z}{\partial x^3} + \frac{3\partial^3 z}{\partial x^2 \partial y} + \frac{3\partial^3 z}{\partial x \partial y^2} + \frac{\partial^3 z}{\partial y^3} = R.$$

Similarly, there will be:

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† The third term was erroneously written as $\frac{3\partial^3 x}{\partial x \partial y^2}$ in the original.

‡ The fourth term was erroneously written as $\frac{3\partial^4 z}{\partial x \partial y^3}$ in the original.

$$\frac{\partial R}{\partial x}+\frac{\partial R}{\partial y}=S, \quad \frac{\partial S}{\partial x}+\frac{\partial S}{\partial y}=T;$$

these relations thusly give us the following equalities:

$$\text{I.} \quad \frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=P,$$

$$\text{II.} \quad \frac{\partial P}{\partial x}+\frac{\partial P}{\partial y}=Q,$$

$$\text{III.} \quad \frac{\partial Q}{\partial x}+\frac{\partial Q}{\partial y}=R,$$

$$\text{IV.} \quad \frac{\partial R}{\partial x}+\frac{\partial R}{\partial y}=S,$$

and so on.

After having noted this elegant relation, I consider this differential equation in general: $\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}=nv$, of which it is a matter of finding the complete integral. To this effect, I set $\partial v = p\partial x + q\partial y$, and since $p=\frac{\partial v}{\partial x}$ and $q=\frac{\partial v}{\partial y}$, this differential equation will take the following form: $nv = p+q$ and hence $nv\partial y = p\partial y + q\partial y$, which being subtracted from the assumed equation $\partial v = p\partial x + q\partial y$ produces this: $\partial v - nv\partial y = p(\partial x - \partial y)$ which, being multiplied by e^{-ny} to yield the first integrative member, gives

$$\partial v e^{-ny} = pe^{-ny}(\partial x - \partial y)$$

from where it is seen that the multiplier of the last member pe^{-ny} must necessarily be a function of $x-y$ and then its integral will be similarly one such function; consequently, the integration produces for us $ve^{-ny} = \mathcal{A}(x-y)$ where the letter $\mathcal{A}$ marks some function of the quantity that is joined there, and I will help myself by marking the other functions in the same manner with the following letters $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$. There is thusly an elegant lemma that will assist us toward our proposed goal:

For this differential equation: $nv = \frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}$,

the complete integral is $v = e^{+ny}\mathcal{A}(x-y)$.

Now to find the integral in question, let us assume in this lemma $v = az + bP + cQ + dR$ taking for the proposed differential equation this:

$$Az + BP + CQ + DR + ES = 0$$

from where it is seen that the value of v must contain a term less than the differential equation, and the integral will be in virtue of our lemma:

$$az + bP + cQ + dR = e^{ny}\mathcal{A}(x-y).$$

When this value taken for v is put in the differential equation from the lemma, we will have:

$$naz + nbP + ncQ + ndR = +a\,(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}) + b\,(\frac{\partial P}{\partial x} + \frac{\partial P}{\partial y}) + c\,(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial y}) + d\,(\frac{\partial R}{\partial x} + \frac{\partial R}{\partial y}).$$

Let us thusly put here instead of the differential formulæ their final values marked above, and our derived equation from the lemma will be:

$$naz + nbP + ncQ + ndR = a\,P + b\,Q + c\,R + d\,S$$

which being arranged following the order of the letters P, Q, R, will take this form:

$$naz + (nb - a)P + (nc - b)Q + (nd - c)R - dS = 0.$$

Thus since we just found the integral of this equation:

$$az + bP + cQ + dR = e^{ny}\ \mathcal{A}(x - y)$$

we only had to yield this identical equation with the proposed knowledge

$$Az + BP + CQ + DR + ES = 0$$

and we will have the following equalities:

$$A = na,\ \ B = nb - a,\ \ C = nc - b,\ \ D = nd - c,\ \ E = -d$$

From where we derive the following values:

$$d = -E$$
$$c = -nE - D$$
$$b = -n^2 E - nD - C$$
$$a = -n^3 E - n^2 D - nC - B \text{ and finally}$$
$$n^4 E + n^3 D + n^2 C + nB + A = 0.$$

There is thusly a fourth order equation from where the value of n must be derived, which will thus have four values that we will assume to be α, β, γ, δ, of which each will provide us with an integral equation of which the first will be

$$az + bP + cQ + dR = e^{\alpha y}\ \mathcal{A}(x - y)$$

The other values β, γ, δ, also produce other values for the letters a, b, c, d, that we will distinguish in the customary manner and instead of $\mathcal{A}$, we will use the other characters for the functions of $(x - y)$: this proposed, these other roots will provide the following integral equations:

$$a'z + b'P + c'Q + d'R = e^{\beta y}\ \mathfrak{B}(x - y)$$
$$a''z + b''P + c''Q + d''R = e^{\gamma y}\ \mathfrak{C}(x - y)$$
$$a'''z + b'''P + c'''Q + d'''R = e^{\delta y}\ \mathfrak{D}(x - y).$$

From these four equations it will not be difficult to decide the values of the four quantities z, P, Q, R.

Now it is evident that each of these letters will be expressed by specific multiples of the four formulæ to the right; but we only have need of the first z; thus since the constant multipliers do not change the arbitrary functions we will not take them into account either and hence we will have for z the following value

$$z = e^{\alpha y}\ \mathcal{A}(x - y)\ + e^{\beta y}\ \mathfrak{B}(x - y)\ + e^{\gamma y}\ \mathfrak{C}(x - y)\ + e^{\delta y}\ \mathfrak{D}(x - y)$$

which contains four arbitrary constants that will express the complete integral of the proposed differential equation, which we had assumed to raise to the fourth degree, although it is easy to see what the integral will be for the cases where the proposed equation would be raised to a differential of a higher or lower degree.

Thus, it all returns to resolving this algebraic equation:

$$A + nB + n^2C + n^3D + n^4E + n^5F + etc. = 0$$

of which the roots are assumed to be α, β, γ, δ, etc. We will be first of all assigning the complete integral of all these differential equations to some differential degree to which they can be raised.

Nevertheless, they could occur in the case where the evolution of the integral would cause some difficulty, such for example, where two or more of the roots for the number n will be imaginary or equivalent. For the first case, let us assume that the two roots α and β are imaginary, and that we had found $\alpha = \mu + \nu\sqrt{-1}$ and $\beta = \mu - \nu\sqrt{-1}$ and for truly determining the two members of the integral $e^{\alpha y}\,\mathfrak{A}(x-y) + e^{\beta y}\,\mathfrak{B}(x-y)$ let us pose

$$\mathfrak{A}(x-y) = \mathfrak{H}(x-y) + \mathfrak{G}(x-y) \text{ and}$$
$$\mathfrak{B}(x-y) = \mathfrak{H}(x-y) - \mathfrak{G}(x-y)$$

and we will reach this form:

$$e^{\mu y}\,\mathfrak{H}(x-y)\,\left(e^{\nu y\sqrt{-1}} + e^{-\nu y\sqrt{-1}}\right) + e^{\mu y}\,\mathfrak{G}(x-y)\,\left(e^{\nu y\sqrt{-1}} - e^{-\nu y\sqrt{-1}}\right).$$

Now we know by the reduction of the imaginaries that

$$e^{\nu y\sqrt{-1}} + e^{-\nu y\sqrt{-1}} = 2\cos\nu y \quad \text{and} \quad e^{\nu y\sqrt{-1}} - e^{-\nu y\sqrt{-1}} = 2\sqrt{-1}\sin\nu y$$

thus since we can ignore the constant factors, the two members that would respond to the two values of α and β reduce to this real form:

$$e^{\mu y}\cos\nu y\;\mathfrak{H}(x-y) + e^{\mu y}\sin\nu y\;\mathfrak{G}(x-y).$$

For the other case where two or more of the roots α, β, γ, become equivalent, let us assume first of all $\beta = \alpha$, and since then the two first members would reunite in a single term, we would no longer have as many of the arbitrary functions as the degree of the proposed differential equation requires, so let us assume $\beta = \alpha + \omega$ in taking ω to mark an infinitely small quantity. Since $e^{\omega y} = 1 + \omega y + \frac{1}{2}\omega^2 y^2 + etc.$ we will have $e^{\beta y} = e^{\alpha y}(1 + \omega y)$, and since it is allowable to put $\mathfrak{B}(x-y)$ instead of $\omega\,\mathfrak{B}(x-y)$ we will have instead of the two first terms that correspond to α and β these two new terms $e^{\alpha y}\,\mathfrak{A}(x-y) + e^{\alpha y}\,\mathfrak{B}(x-y)$. By the same reasoning, we easily will convince ourselves that if there were three equal roots $\alpha = \beta = \gamma$, we would have instead of three members that correspond to these letters, these three others:

$$e^{\alpha y}\,\mathfrak{A}(x-y) + e^{\alpha y}\,y\,\mathfrak{B}(x-y) + e^{\alpha y}\,y^2\,\mathfrak{C}(x-y)$$

and if there were a fourth equal root, we would only have to add to the three previous terms this fourth term $e^{\alpha y}y^3\,\mathfrak{D}(x-y)$, from where we could resolve the particular problems that follow.

Problem I.

Find the complete integral of this particular equation:

$$P = 0, \text{ or } \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0.$$

Solution.

Since $P = 0$ here, we will have in the general equation:

$$A = 0, B = 1, C = D = E = 0,$$

from where the equation for finding the number n will be $n = 0$; from where it is derived that $\alpha = 0$, and hence the complete integral will be $z = \mathfrak{A}(x - y)$.

Problem II.

Find the complete integral of this equation:

$$Q = 0, \text{ or } \frac{\partial^2 z}{\partial x^2} + \frac{2\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0.$$

Solution.

Since $Q = 0$ here, we will have in the general equation:

$$A = 0, B = 0, C = 1, D = E = F = 0,$$

from where the equation for the number n will be $n^2 = 0$; thus the two roots $\alpha = 0$ and $\beta = 0$ and hence equivalent. Consequently, the complete integral will be:

$$z = \mathfrak{A}(x - y) + y\,\mathfrak{B}(x - y).$$

Problem III.

Find the complete integral of this equation:

$$\frac{\partial^3 z}{\partial x^3} + \frac{3\partial^3 z}{\partial x^2 \partial y} + \frac{3\partial^3 z}{\partial x \partial y^2} + \frac{\partial^3 z}{\partial y^3} = 0.$$

Solution.

Since $R = 0$ here, we will have in the general equation:

$$A = 0, B = 0, C = 0, D = 1, E = F = 0,$$

from where the equation for the number n will be $n^3 = 0$, and hence $\alpha = \beta = \gamma = 0$. Consequently, the complete integral will be:

$$z = \mathfrak{A}(x - y) + y\,\mathfrak{B}(x - y) + y^2\,\mathfrak{C}(x - y).$$