NEW APPROXIMATIONS FOR THE AREA OF THE MANDELBROT SET

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Abstract. Due to its fractal nature, much about the area of the Mandelbrot set M remains to be understood. While a series formula has been derived by Ewing and Schober to calculate the area of M by considering its complement inside the Riemann sphere, to date the exact value of this area remains unknown. This paper presents new improved upper bounds for the area based on a parallel computing algorithm and for the 2-adic valuation of the series coefficients in terms of the sum-of-digits function.

1. Introduction

The Mandelbrot set (hereafter M) is defined as the set of complex numbers $c \in \mathbb{C}$ such that the sequence $\{z_n\}$ defined by the recursion

$$z_n = z_{n-1}^2 + c$$

with initial value $z_0 = 0$ remains bounded for all $n \geq 0$. Douady and Hubbard [3] proved that M is connected and Shishikura [11] proved that M has fractal boundary of Hausdorff dimension 2. However, it is unknown whether the boundary has positive Lebesgue measure.

In [6] Ewing and Schober derived a series formula for the area of M by considering its complement, $\tilde{M}$, inside the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, i.e. $\tilde{M} = \overline{\mathbb{C}} - M$. It is known that $\tilde{M}$ is simply connected with mapping radius 1 ([3]). In other words, there exists an analytic homeomorphism

$$\psi(z) = z + \sum_{m=0}^{\infty} b_m z^{-m} \tag{1}$$

which maps the domain $\Delta = \{z : 1 < |z| \leq \infty\} \subset \overline{\mathbb{C}}$ onto $\tilde{M}$. It follows from the classic result of Gronwall [8] that the area of the Mandelbrot set $M = \overline{\mathbb{C}} - \tilde{M}$ is given by

$$A = \pi \left[1 - \sum_{m=1}^{\infty} m |b_m|^2 \right] \tag{2}$$

In order to calculate the coefficients b_m, Ewing and Schober considered the expansion

$$p_n(\psi(z)) = \sum_{m=0}^{\infty} \beta_{n,m} z^{2^n - m} \tag{3}$$

where $b_m = \beta_{0m+1}$ and the polynomials $p_n(w)$ are defined recursively by

$$\begin{aligned} p_0(w) &= w \\ p_n(w) &= p_{n-1}^2(w) + w \end{aligned} \tag{4}$$

They were able to prove in [5] that the polynomials $p_n(w)$ are Faber polynomials of degree 2^n for M, i.e. $p_n(\psi(z)) = z^{2^n} + o(1)$ as $z \to \infty$. It follows that $\beta_{n,m} = 0$ for $n \geq 1$ and $1 \leq m \leq 2^n$. Moreover, this range of zero values can be extended to $1 \leq m \leq 2^{n+1} - 2$ because of the recursion

$$\beta_{n,m} = 2\beta_{n-1,m} + \sum_{k=1}^{m-1} \beta_{n-1,k} \beta_{m-1,m-k} \tag{5}$$

Date: 10-5-2014.
2000 *Mathematics Subject Classification.* Primary 37F45.
Key words and phrases. Mandelbrot set, sum of digits.

which they obtained by substituting (3) into (4) and equating coefficients. Formula (5) can then be manipulated to obtain the following backward recursion formula ([6]):

$$\beta_{n,m} = \frac{1}{2}\left[\beta_{n+1,m} - \sum_{k=2^{n+1}-1}^{m-2^{n+1}+1} \beta_{n,k}\beta_{n,m-k} - \beta_{0,m-2^{n+1}+1}\right] \tag{6}$$

where $\beta_{n,0} = 1$ and $\beta_{0,m} = b_{m-1}$ for $m \geq 1$.

No explicit formula is known for $\beta_{n,m}$, except those at certain positions. However, it is clear from (6) that $\beta_{n,m}$ is rational and that its denominator equals a power of 2 when expressed in lowest terms. In [6] Ewing and Schober established the following upper bound on its 2-adic valuation.

Theorem 1 (Ewing-Schober [6]). *Let $n \geq 0$ and $m \geq 1$. Then $2^{2m+3-2^{n+2}}\beta_{n,m}$ is an integer, i.e.*

$$|\nu(\beta_{n,m})| \leq 2m + 3 - 2^{n+2} \tag{7}$$

for non-zero $\beta_{n,m}$.

Here, the 2-adic valuation $\nu(x)$ of a positive integer x is defined to be the greatest integer for which $2^{\nu(x)}$ divides x and if x/y is a fraction in lowest terms, then we define $\nu(x/y) = \nu(x) - \nu(y)$. Observe that in the special case $b_m = \beta_{0,m+1}$, (7) reduces to

$$|\nu(b_m)| \leq 2m - 1 \tag{8}$$

In the same paper [6], Ewing and Schober used (6) to compute the first 240,000 coefficients for b_n by computer. Since

$$A \leq A_N \equiv \pi\left[1 - \sum_{m=1}^{N} m|b_m|^2\right] \tag{9}$$

this yielded an upper bound of 1.7274 for the area of M. They were able to slightly improve their result to 1.72 by extending their computations to the first 500,000 coefficients as reported by Ewing in [4]. They also calculated a crude lower bound of $7\pi/16 \approx 1.3744$ by estimating the size of the main cardioid ($3\pi/8$) and the main bulb ($\pi/16$). However, they reported a discrepancy with their approximation of 1.52 obtained by pixel counting. More recent calculations by Förstemann [7] provide an estimate of 1.50659 based on a resolution of almost 88 trillion pixels. Thus, as noted by Ewing and Schober, either the series (2) converges so slowly that the approximation $A_{500,000} \approx 1.72$ is poor or else the pixel counting method fails to account for the boundary of M. Recently, Buff and Chéritat [2] found Julia sets with positive area. Therefore, coupled with Shishikura's result that the boundary of M has Hausdorff dimension 2, it is not far-fetched to suspect that the boundary of M may have positive area.

In this paper, we report on progress in obtaining new upper bounds for A and new results involving the two-dimensional sequence $\beta_{n,m}$. In particular, we were able to compute the first five million coefficients for b_n by developing a parallel processing implementation of (6). As a result, we obtained the new upper bound

$$A_{5,000,000} \approx 1.68288 \tag{10}$$

Moreover, we were able to improve on (7) by establishing the tighter bound (Theorem 7)

$$|\nu(\beta_{n,m})| \leq 2m - 2^{n+2} + 4 - s(n,m) \tag{11}$$

for non-zero $\beta_{n,m}$ where $s(n,m)$ is the base-2 sum-of-digits function of degree n (Definition 2). In the special case $b_m = \beta_{0,m+1}$, we obtain as a corollary

$$|\nu(b_m)| \leq 2(m+1) - s(0, m+1) \tag{12}$$

It is observed (yet unproven) that equality in (12) is achieved when m is odd. In comparison, equality in (8) appears to hold only when $m+1$ equals a power of 2. Thus, our result is significant on two levels. First, from a computational perspective, inequality (11) allows the values of $\beta_{n,m}$ to be calculated by integer arithmetic (as discussed by Ewing and Schober in [6]) using less memory than (7). Such an approach would increase the accuracy in which upper bounds for the area of the M are calculated over floating-point arithmetic where the values of $\beta_{n,m}$ are stored as truncated decimals. Secondly, (12) is significant mathematically in that it reveals the sum-of-digits function to be a crucial ingredient in determining the exact area of M by using the series formula (2). This we believe to be an important advance in understanding the nature of the Mandelbrot set.

2. Two-adic Valuation of $\beta_{n,m}$

In this section we consider the 2-adic valuation of $\beta_{n,m}$ and prove the bound in (11), which is a refinement of (7). We begin by defining the sum-of-digits function and present a series of lemmas on properties of this function that will be utilized in the proof.

Definition 2. Let m be a non-negative integer with base-2 expansion $m = d_L 2^L + d_{L-1}2^{L-1} + ... + d_0 2^0$ where $d_L = 1$ and $d_i \in \{0, 1\}$ for $i < L$. We define the *base 2 sum-of-digits function* $s(n, m)$ of *degree* n by

$$s(n,m) = \sum_{i=n}^{L} d_i$$

Lemma 3. *s(n,m) is sub-additive, i.e.*

$$s(n, l + m) \leq s(n, l) + s(n, m)$$

for all $l, m, n \in \mathbb{N}$.

Proof. We follow the proof in [10]. Let $l = c_K 2^K + c_{K-1}2^{K-1} + ... + c_0 2^0$ and $m = d_L 2^L + d_{L-1}2^{L-1} + ... + d_0 2^0$. Since $s(n, m + 2^i) = s(n, m)$ for $i < n$ and $s(n, m + 2^i) \leq s(n, m) + 1$ for $i \geq n$, it follows that

$$\begin{aligned}
s(n, l + m) &= s(n, m + \sum_{i=0}^{K} c_i 2^i) \\
&= s(n, m + \sum_{i=n}^{K} c_i 2^i) \\
&\leq s(n, m) + \sum_{i=n}^{K} c_i \\
&\leq s(n, m) + s(n, l)
\end{aligned}$$

□

Lemma 4. *For all* $m, n \in \mathbb{N}$, *we have*

a) $0 \leq s(n, m) - s(n + 1, m) \leq 1$

b) $s\left(n, 2^{n+1} - 1\right) = 1$

c) $s(0, m) \leq 2s\left(0, \frac{m}{2}\right) - 1$ *for positive even integers* m.

Proof. (a) We express m as in Definition 2. It follows that

$$\begin{aligned}
s(n, m) - s(n + 1, m) &= \sum_{i=n}^{L} d_i - \sum_{i=n+1}^{L} d_i \\
&= d_n + \sum_{i=n+1}^{L} d_i - \sum_{i=n+1}^{L} d_i \\
&= d_n
\end{aligned}$$

where d_n must equal either 0 or 1. This completes the proof for part a).

b) The result follows immediately from the fact that $2^{n+1} - 1 = 2^0 + ... + 2^{n-1} + 2^n$.

c) We shall prove this inequality by induction on m. It is clear that the inequality is true for $m = 0$ since $s(0,2) = s(0,1) = 1$. Then from the induction hypothesis and Lemma 3, we have

$$\begin{aligned} s(0, m+2) &\leq s(0,m) + s(0,2) \\ &\leq 2s\left(0, \frac{m}{2}\right) - 1 + s(0,2) = 2s\left(0, \frac{m}{2}\right) \\ &\leq 2s\left(0, \frac{m}{2} + 1\right) - 2s(0,1) = 2s\left(0, \frac{m+2}{2}\right) - 2 \\ &\leq 2s\left(0, \frac{m+2}{2}\right) - 1 \end{aligned}$$

This complete the proof. □

Next, we present a lemma regarding the convolution described in equation (6).

Lemma 5. *Let $m \geq 2^{n+2} - 2$.*

a) For m even, we have

$$\sum_{k=2^{n+1}-1}^{m-2^{n+1}+1} \beta_{n,k}\beta_{n,m-k} = 2\left[\sum_{k=2^{n+1}-1}^{m/2-1} \beta_{n,k}\beta_{n,m-k}\right] + \left(\beta_{n,m/2}\right)^2 \tag{13}$$

b) For m is odd, we have

$$\sum_{k=2^{n+1}-1}^{m-2^{n+1}+1} \beta_{n,k}\beta_{n,m-k} = 2\left[\sum_{k=2^{n+1}-1}^{(m-1)/2} \beta_{n,k}\beta_{n,m-k}\right] \tag{14}$$

Proof. When m is even, we have

$$\sum_{k=2^{n+1}-1}^{m-2^{n+1}+1} \beta_{n,k}\beta_{n,m-k} = \sum_{k=2^{n+1}-1}^{m/2-1} \beta_{n,k}\beta_{n,m-k} + \sum_{m/2}^{m/2} \beta_{n,k}\beta_{n,m-k} + \sum_{m/2+1}^{m-2^{n+1}+1} \beta_{n,k}\beta_{n,m-k}$$

Letting $h = m - k$, we obtain

$$\begin{aligned} \sum_{k=2^{n+1}-1}^{m-2^{n+1}+1} \beta_{n,k}\beta_{n,m-k} &= \sum_{k=2^{n+1}-1}^{m/2-1} \beta_{n,k}\beta_{n,m-k} + \left(\beta_{n,m/2}\right)\left(\beta_{n,m/2}\right) + \sum_{h=m/2-1}^{2^{n+1}-1} \beta_{n,m-h}\beta_{n,h} \\ &= 2\left[\sum_{k=2^{n+1}-1}^{m/2-1} \beta_{n,k}\beta_{n,m-k}\right] + \left(\beta_{n,m/2}\right)^2 \end{aligned}$$

This proves part a).

On the other hand, when m is odd,

$$\sum_{k=2^{n+1}-1}^{m-2^{n+1}+1} \beta_{n,k}\beta_{n,m-k} = \sum_{k=2^{n+1}-1}^{(m-1)/2} \beta_{n,k}\beta_{n,m-k} + \sum_{k=(m+2)/2}^{m-2^{n+1}+1} \beta_{n,k}\beta_{n,m-k}$$

Letting $l = m - k$, we have

$$\begin{aligned} \sum_{k=2^{n+1}-1}^{m-2^{n+1}+1} \beta_{n,k}\beta_{n,m-k} &= \sum_{k=2^{n+1}-1}^{(m-1)/2} \beta_{n,k}\beta_{n,m-k} + \sum_{l=(m-1)/2}^{2^{n+1}-1} \beta_{n,m-l}\beta_{n,l} \\ &= 2\left[\sum_{k=2^{n+1}-1}^{(m-1)/2} \beta_{n,k}\beta_{n,m-k}\right] \end{aligned}$$

This justifies part b). □

We now present one final lemma involving the right hand side of (11).

Lemma 6. *Define*

$$p(n,m) = 2m - 2^{n+2} + 4 - s(n,m) \tag{15}$$

for $m, n \in \mathbb{N}$. Then

a) $p(n,m) - 1 \geq p(n+1,m)$

b) $p(n,m) \geq p(n,k) + p(n,m-k)$ for $0 \leq k \leq m$.

c) $p(0,m) \geq 2p(0,m/2)$ for m is even.

d) $p(n,m) - 1 \geq p(0, m - 2^{n+1} + 1)$

Proof. a) Consider the following chain of equivalent inequalities:

$$\begin{aligned} p(n,m) - 1 &\geq p(n+1,m) \\ 2m - 2^{n+2} + 4 - s(n,m) - 1 &\geq 2m - 2^{n+3} + 4 - s(n,m) \\ 2^{n+3} - 2^{n+2} - 1 &\geq s(n,m) - s(n+1,m) \\ 2^{n+2} - 1 &\geq s(n,m) - s(n+1,m) \end{aligned}$$

Therefore, it suffices to prove the last inequality above, which follows from transitivity of the two inequalities $2^{n+2} - 1 > 1$ and $1 \geq s(n,m) - s(n-1,m)$, the former being clearly true and the latter as a result of part a) in Lemma 4.

b) The following inequalities are clearly equivalent:

$$\begin{aligned} p(n,m) &\geq p(n,k) + p(n,m-k) \\ 2m - 2^{n+2} + 4 - s(n,m) &\geq 2k - 2^{n+2} + 4 - s(n,k) + 2(m-k) - 2^{n+2} + 4 - s(n,m-k) \\ s(n,m-k) + s(n,k) + 2^{n+2} - 4 &\geq s(n,m) \end{aligned}$$

Therefore, it suffices to prove the last inequality above, which follows from transitivity of the two inequalities

$$2^{n+2} - 4 \geq 0$$

whenever $n \geq 0$ and

$$s(n,m-k) + s(n,k) \geq s(n,m)$$

because of the sub-additive property of the sum of digits function (Lemma 3). This establishes part b).

c) This part follows from the equivalent inequalities

$$\begin{aligned} p(0,m) - 1 &> 2p\left(0, \frac{m}{2}\right) \\ 2m - 2^2 + 4 - s(0,m) &> 2\left(2\left(\frac{m}{2}\right) - 2^2 + 4 - s\left(0, \frac{m}{2}\right)\right) \\ 2s\left(0, \frac{m}{2}\right) - 1 &> s(0,m) \end{aligned}$$

where the last inequality above follows from Lemma 4.

d) It is straightforward to verify that following inequalities are equivalent:

$$\begin{aligned} p(0, m - 2^{n+1} + 1) &\leq p(n,m) - 1 \\ 2(m - 2^{n+1} + 1) - 2^2 + 4 - s(0, m - 2^{n+1} + 1) &\leq 2m - 2^{n+2} + 4 - s(n,m) - 1 \\ s(n,m) &\leq s(0, m - 2^{n+1} + 1) + 1 \end{aligned}$$

But the last inequality above follows from Lemmas 3 and 4 since

$$\begin{aligned} s(n,m) &= s(n, m-2^{n+1}+1+2^{n+1}-1) \\ &\leq s(n, m-2^{n+1}+1) + s(n, 2^{n+1}-1) \\ &\leq s(0, m-2^{n+1}+1)+1 \end{aligned}$$

Therefore part d) is true. $\square$

We now have presented all lemmas needed to prove the following theorem.

Theorem 7. *Let $m, n \in \mathbb{N}$. Then $2^{p(n,m)}\beta_{n,m}$ is an integer, i.e.*

$$|\nu(\beta_{n,m})| \leq p(n,m) \tag{16}$$

Proof. From (6) we have

$$\begin{aligned} 2^{p(n,m)}\beta_{n,m} &= 2^{p(n,m)-1}\left[\beta_{n+1,m} - \sum_{k=2^{n+1}-1}^{m-2^{n+1}+1} \beta_{n,k}\beta_{n,m-k} - \beta_{0,m-2^{n+1}+1}\right] \\ &= 2^{p(n,m)-1}\beta_{n+1,m} - \sum_{k=2^{n+1}-1}^{m-2^{n+1}+1} 2^{p(n,m)-1}\beta_{n,k}\beta_{n,m-k} - 2^{p(n,m)-1}\beta_{0,m-2^{n+1}+1} \end{aligned} \tag{17}$$

It suffices to show that each term on the right-hand side of (17) is an integer by induction on m, which we will do so using properties of $p(n,m)$ established in Lemma 6. Assume that the values of $\beta_{n,m}$ are arranged in a two-dimensional array where the rows are indexed by n and the columns indexed by m. Since $\beta_{n,m} = 0$ for $n \geq 1$ and $1 \leq m \leq 2^{n+1}-2$, it follows that each column has at most a finite number of non-zero entries. Therefore, we shall apply induction by moving upwards along each column from left to right as employed by Ewing and Schober in their induction arguments in [6]. In particular, given m and n, we assume that $2^{p(j,k)}\beta_{j,k}$ is an integer for $0 \leq j \leq n$ and $2^{n+1}-1 \leq k \leq m-1$ and also $2^{p(j,m)}\beta_{j,m}$ is an integer for $j \geq n+1$.

Let us now consider the first term $2^{p(n,m)-1}\beta_{n+1,m}$ on the right-hand side of (17). Since $p(n,m)-1 \geq p(n+1,m)$ (due to part a) in Lemma 6) and $2^{p(n+1,m)}\beta_{n+1,m}$ is an integer by induction, it follows that $2^{p(n,m)-1}\beta_{n+1,m}$ is an integer.

Next, we rewrite the summation in (17) according to whether m is even or odd by using Lemma 5. If m is odd, then

$$\sum_{k=2^{n+1}-1}^{m-2^{n+1}+1} 2^{p(n,m)-1}\beta_{n,k}\beta_{n,m-k} = \sum_{k=2^{n+1}-1}^{(m-1)/2} 2^{p(n,m)}\beta_{n,k}\beta_{n,m-k}$$

Since $p(n,m) \geq p(n,k) + p(n,m-k)$ for $0 \leq k \leq m$ from part b) of Lemma 6 and

$$(2^{p(n,k)}\beta_{n,k})(2^{p(n,m-k)}\beta_{n,m-k})$$

is an integer by induction, it follows that each term $2^{p(n,m)-1}\beta_{n,k}\beta_{n,m-k}$ in the summation must be an integer. On the other hand, if m is even, then

$$\sum_{k=2^{n+1}-1}^{m-2^{n+1}+1} 2^{p(n,m)-1}\beta_{n,k}\beta_{n,m-k} = \sum_{k=2^{n+1}-1}^{m/2-1} 2^{p(n,m)}\beta_{n,k}\beta_{n,m-k} + 2^{p(n,m)-1}\left(\beta_{n,m/2}\right)^2$$

By the same argument as before, we have that $2^{p(n,m)}\beta_{n,k}\beta_{n,m-k}$ is an integer. Morever, since $p(n,m)-1 \geq 2p(n,m/2)$ (due to part c) in Lemma 6) and $2^{p(n,m/2)}\beta_{n,m/2}$ is an integer by induction, it follows that $2^{p(n,m)-1}\left(\beta_{n,m/2}\right)^2$ must also be an integer. Thus, each term $2^{p(n,m)-1}\beta_{n,k}\beta_{n,m-k}$ in the summation must also be an integer.

As for the last term $2^{p(n,m)-1}\beta_{0,m-2^{n+1}+1}$ in (17), we know from part d) of Lemma 6 that $p(n,m)-1 \geq p(0, m-2^{n+1}+1)$. Since $2^{p(0,m-2^{n+1}-1)}\beta_{0,m-2^{n+1}+1}$ is an integer by induction, it follows by the same reasoning that $2^{p(n,m)-1}\beta_{0,m-2^{n+1}+1}$ must be an integer. This finishes the proof of Theorem 7. $\square$

3. Special Values of $\beta_{n,m}$

In this section we derive recurrences for special values of $\beta_{n,m}$ where m is restricted to a certain interval. Recall that $\beta_{n,m} = 0$ for $1 \leq m \leq 2^{n+1} - 2$. We therefore begin with an unpublished result by Malik Ahmed and one of the authors regarding $\beta_{n,m}$ in the interval $2^{n+1} - 1 \leq m \leq 2^{n+2} - 3$.

Theorem 8 (Ahmed-Nguyen)**.** *Let n and m be integers such that $n \geq 0$ and $2^{n+1} - 1 \leq m \leq 2^{n+2} - 3$. Then for every $p \in \mathbb{N}$, we have*

$$\beta_{n,m} = \beta_{n+p,m+2^{n+1}(2^p-1)} = -\frac{1}{2}\beta_{0,m-2^{n+1}+1} = -\frac{1}{2}b_{m-2^{n+1}} \tag{18}$$

Proof. It follows from (5) that

$$\beta_{n,m} = -\frac{1}{2}\beta_{0,m-2^{n+1}+1} = -\frac{1}{2}b_{m-2^{n+1}} \tag{19}$$

Next, set

$$n' = n + p, m' = m + 2^{n+1}(2^p - 1)$$

Then

$$m' - 2^{n'+1} + 1 = m - 2^{n+1} + 1$$

which proves

$$\beta_{n,m} = \beta_{n',m'} \tag{20}$$

□

As a corollary of Theorem 8, we establish a special case of (7).

Corollary 9. *Let $n \geq 1$ and $2^{n+1} \leq m \leq 2^{n+2} - 3$. Then $2^{2m+2-2^{n+2}}\beta_{n,m}$ is an integer.*

Proof. We know from (7) that

$$2^{2(m-2^{n+1}+1)+3-2^2}\beta_{0,m-2^{n+1}+1} = 2^{2m+1-2^{n+2}}\beta_{0,m-2^{n+1}+1}$$

is an integer. It follows from Theorem 8 that

$$2^{2m+2-2^{n+2}}\beta_{n,m} = 2^{2m+2-2^{n+2}}\left(-\frac{1}{2}\beta_{0,m-2^{n+1}+1}\right) = -2^{2m+1-2^{n+2}}\beta_{0,m-2^{n+1}+1} \tag{21}$$

must also be an integer. □

NOTE: Observe that the corollary above fails for $m = 2^{n+1} - 1$. By Theorem 8 we have $\beta_{n,2^{n+1}-1} = -\frac{1}{2}\beta_{0,0}$. But (7) doesn't apply to $\beta_{0,0} = 1$.

We next focus on deriving recurrences for special values of $\beta_{n,m}$ where $2^{n+2} - 2 \leq m \leq 2^{n+2} + 6$.

Lemma 10. *Let n be a non-negative integer. Then*

$$\beta_{n,2^{n+2}-2} = -\frac{1}{2}\left(\beta_{0,2^{n+1}-1} + \frac{1}{4}\right) \tag{22}$$

$$\beta_{n,2^{n+2}-1} = -\frac{1}{2}\left(\beta_{0,2^{n+1}} + \frac{1}{4}\right) \tag{23}$$

Proof. Recall that $\beta_{n,m} = -\frac{1}{2}\beta_{0,m-2^{n+1}+1}$ for $n{\geq}0$ and $2^{n+1} - 1 \leq m \leq 2^{n+2} - 3$. We have

$$
\begin{aligned}
\beta_{n,2^{n+2}-2} &= \frac{1}{2}\left[\beta_{n+1,2^{n+2}-2} - \sum_{k=2^{n+1}-1}^{2^{n+1}-1} \beta_{n,k}\beta_{n,2^{n+2}-2-k} - \beta_{0,2^{n+1}-1}\right] \\
&= \frac{1}{2}\left[0 - \beta_{n,2^{n+1}-1}^2 - \beta_{0,2^{n+1}-1}\right] \\
&= \frac{1}{2}\left[-\frac{1}{4}\beta_{0,0}^2 - \beta_{0,2^{n+1}-1}\right] \\
&= -\frac{1}{2}\left[\beta_{0,2^{n+1}-1} + \frac{1}{4}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
\beta_{n,2^{n+2}-1} &= \frac{1}{2}\left[\beta_{n+1,2^{n+2}-1} - \sum_{k=2^{n+1}-1}^{2^{n+1}} \beta_{n,k}\beta_{n,m-k} - \beta_{0,2^{n+1}}\right] && (24) \\
&= \frac{1}{2}\left[\beta_{n+1,2^{n+2}-1} - 2\left(\beta_{n,2^{n+1}-1}\beta_{n,2^{n+1}}\right) - \beta_{0,2^{n+1}}\right] && (25) \\
&= \frac{1}{2}\left[\left(-\frac{1}{2}\beta_{0,0}\right) - 2\left(\left(-\frac{1}{2}\beta_{0,0}\right)\left(-\frac{1}{2}\beta_{0,1}\right)\right) - \beta_{0,2^{n+1}}\right] && (26) \\
&= -\frac{1}{2}\left[\beta_{0,2^{n+1}} + \frac{1}{4}\right] && (27)
\end{aligned}
$$

□

In the case where $m = 2^{n+2}$, we find that $\beta_{n,m}$ is constant.

Lemma 11. *Let n be a non-negative integer. Then $\beta_{n,2^{n+2}} = 1/16$.*

Proof. Recall that $\beta_{n,m} = -\frac{1}{2}\beta_{0,m-2^{n+1}+1}$ for $n \geq 0$ and $2^{n+1} - 1 \leq m \leq 2^{n+2} - 3$. We have

$$
\begin{aligned}
\beta_{n,2^{n+2}} &= \frac{1}{2}\left[\beta_{n+1,2^{n+2}} - \sum_{k=2^{n+1}-1}^{2^{n+1}+1} \beta_{n,k}\beta_{n,2^{n+2}-k} - \beta_{0,2^{n+1}+1}\right] \\
&= \frac{1}{2}\left[-\frac{1}{2}\beta_{0,1} - 2\beta_{n,2^{n+1}-1}\beta_{n,2^{n+1}+1} - \beta_{n,2^{n+1}}^2 - b_{0,2^{n+1}}\right] \\
&= \frac{1}{2}\left[-\frac{1}{2}\beta_{0,1} - \frac{1}{2}\beta_{0,0}\beta_{0,2} - \frac{1}{4}\beta_{0,1}^2 - 0\right] \\
&= \frac{1}{2}\left[-\frac{1}{2}(-1/2) - \frac{1}{2}(1)(1/8) - \frac{1}{4}(-1/2)^2\right] \\
&= 1/16
\end{aligned}
$$

□

We end this section by considering three other special cases.

Lemma 12.

a) Let $n \geq 2$ be an integer. Then $\beta_{n,2^{n+2}+2} = -\frac{1}{2}\beta_{0,2^{n+1}+3}$.
b) Let $n \geq 2$ be an integer. Then $\beta_{n,2^{n+2}+4} = -\frac{1}{2}\beta_{0,2^{n+1}+5}$.
c) Let $n \geq 3$ be an integer. Then $\beta_{n,2^{n+2}+6} = -\frac{1}{2}\beta_{0,2^{n+1}+7}$.

Proof. We have

$$\begin{aligned}
\beta_{n,2^{n+2}+2} &= \frac{1}{2}\left[\beta_{n+1,2^{n+2}+2} - \sum_{k=2^{n+1}-1}^{2^{n+1}+3} \beta_{n,k}\beta_{n,2^{n+2}+2-k} - \beta_{0,2^{n+1}+3}\right] \\
&= \frac{1}{2}\left[\beta_{n+1,2^{n+2}+2} - 2\sum_{k=2^{n+1}-1}^{2^{n+1}} \beta_{n,k}\beta_{n,2^{n+2}+2-k} - \beta^2_{n,2^{n+1}+1} - \beta_{0,2^{n+1}+3}\right] \\
&= \frac{1}{2}\left[-\frac{1}{2}\beta_{0,3} - \frac{1}{2}\sum_{j=0}^{1}\beta_{0,j}\beta_{0,4-j} - \frac{1}{4}\beta^2_{0,2} - \beta_{0,2^{n+1}+3}\right] \\
&= \frac{1}{2}\left[-\frac{1}{2}(-1/4) - \frac{1}{2}(\beta_{0,0}\beta_{0,4} + \beta_{0,1}\beta_{0,3}) - \frac{1}{4}(1/8)^2 - \beta_{0,2^{n+1}+3}\right] \\
&= \frac{1}{2}\left[-\frac{1}{2}(-1/4) - \frac{1}{2}[(1)(15/128) + (-1/2)(-1/4)] - \frac{1}{4}(1/8)^2 - \beta_{0,2^{n+1}+3}\right] \\
&= -\frac{1}{2}\beta_{0,2^{n+1}+3}
\end{aligned}$$

This proves part a). Parts b) and c) can be proven in a similar manner. □

4. New Area Approximations

In this section we describe a parallel processing algorithm to compute the values of $\beta_{n,m}$ and present new upper bounds for the area of M that were calculated using this algorithm. Assume as before that the values of $\beta_{n,m}$ are arranged in a two-dimensional array with the rows indexed by n and columns indexed by m. We recall Ewing and Schober's backwards algorithm for computing the non-trivial values of $\beta_{n,m}$ recursively one at a time by moving upwards along each column from left to right as described in our induction proof of Theorem 7. Thus, the order of computation would be:

$$\beta_{0,1}, \beta_{0,2}, \beta_{1,3}, \beta_{0,3}, \beta_{1,4}, \beta_{0,4}, ...$$

Our new method is as follows: we calculate values of $\beta_{n,m}$ across multiple columns simultaneously in a parallel fashion while moving up along them as before until we reach a critical row near the top where from this point on, all remaining column values must be computed one at a time. This is then repeated for the next set of columns, etc.

To illustrate this method, consider for example the calculation of $\beta_{1,7}$ and $\beta_{1,8}$ in row $n = 1$ using the backward recursion given in (6):

$$\beta_{1,7} = \frac{1}{2}\left[\beta_{2,7} - \sum_{k=3}^{4}\beta_{1,k}\beta_{1,7-k} - \beta_{0,4}\right] = \frac{1}{2}\left[\beta_{2,7} - 2\beta_{1,3}\beta_{1,4} - \beta_{0,4}\right]$$

$$\beta_{1,8} = \frac{1}{2}\left[\beta_{1,8} - \sum_{k=3}^{4}\beta_{1,k}\beta_{1,8-k} - \beta_{0,5}\right] = \frac{1}{2}\left[\beta_{2,8} - 2\beta_{1,3}\beta_{1,5} - \beta^2_{1,4} - \beta_{0,4}\right]$$

Observe that these two values do not depend on each other and can be computed in parallel. However, this is not the case for $\beta_{0,7}$ and $\beta_{0,8}$ in the top row ($n = 0$) where the latter depends on the former:

$$\beta_{0,7} = \frac{1}{2}\left[\beta_{1,7} - \sum_{k=1}^{6}\beta_{0,k}\beta_{0,7-k} - \beta_{0,6}\right] = \frac{1}{2}\left[\beta_{1,7} - 2\beta_{0,1}\beta_{0,6} - 2\beta_{0,2}\beta_{0,5} - 2\beta_{0,3}\beta_{0,4} - \beta_{0,4}\right]$$

$$\beta_{0,8} = \frac{1}{2}\left[\beta_{1,8} - \sum_{k=1}^{7}\beta_{0,k}\beta_{0,8-k} - \beta_{0,7}\right] = \frac{1}{2}\left[\beta_{1,8} - 2\beta_{0,1}\beta_{0,7} - 2\beta_{0,2}\beta_{0,6} - 2\beta_{0,3}\beta_{0,5} - \beta^2_{0,4} - \beta_{0,7}\right]$$

In general, the values $\beta_{n,m}$, $\beta_{n,m+1}$ and $\beta_{n,m+2}$ in three consecutive columns can be calculated in parallel as long as $n \geq 1$. This is because $\beta_{n,m+1}$ depends only on the values $\beta_{n,k}$ in row n, where $k = 3, 4, ..., m-2$, which are prior to $\beta_{n,m}$. Similarly, $\beta_{n,m+2}$ depends only on $\beta_{n,k}$ where $k = 3, 4, ..., m-1$. Since the number

Table 1. New Upper Bounds for the Area of the Mandelbrot Set

N (millions)	A_N
0.5	1.72 (Ewing-Schober)
1	1.70393
1.5	1.69702
2	1.69388
2.5	1.69096
3	1.68895
3.5	1.6874
4	1.68633
4.5	1.68447
5	1.68288

Table 2. Run-Times for Calculating b_m in Batches of 500,000

Range of m (millions)	Run-time to compute b_m (days)
2.5-3	9
3-3.5	10.8
3.5-4	12.5
4-4.5	14.4
4.5-5	16.2

of non-zero values in each column increases as m increases, this parallel algorithm becomes more effective and asymptotically three times as fast in comparison to that of calculating $\beta_{n,m}$ one at a time. Moreover, this approach can be extended to calculate the values $\beta_{n,m}$, $\beta_{n,m+1}$, ..., $\beta_{n,m+6}$ in seven consecutive columns simultaneously as long as $n \geq 2$. More generally, if $n \geq N$, then up to $2^(N+1)-1$ columns can be computed in parallel.

We were able to use this parallel algorithm to calculate the first five million terms of b_m and obtain a new upper bound of $A_{5\cdot 10^6} \approx 1.68288$ for the area of the Mandelbrot set. This algorithm was implemented using the programming language C++ and message passing interface Open MPI and programmed to calculate the values of $\beta_{n,m}$ across four columns in parallel for $n \geq 2$ beginning with the first group of columns $\beta_{n,8}$, $\beta_{n,9}$, $\beta_{n,10}$, $\beta_{n,11}$ (we initialized columns $\beta_{n,0}$, ..., $\beta_{n,7}$ with their known values). Our code was executed on a Linux cluster and required four processors (1.05 Ghz AMD Opteron 2352 quad-core processors) to execute it since each column was computed using a different processor. We note that each processor was required to store all values of $\beta_{m,n}$ (generated from all four processors) separately in its own RAM. This we believed improved the performance of our implementation slightly, but at the cost of quadrupling our memory requirements.

Table 1 gives values for the approximations A_N, where N ranges from 500,000 to 5 million in increments of 500,000, based on our computed values of $\beta_{n,m}$, and thus $b_m = \beta_{0,m+1}$. These values were computed in batches over a five-month period between August-December, 2014, although the actual total run-time was approximately 3 months. Table 2 gives the reader a sense of the run-time required to compute b_m in batches of 100,000 starting at $m = 250,000$.

To check the accuracy of our calculations, we compared our calculated values of b_m (in double-precision floating point format) with known exact values at certain positions. For example, we found our calculated values to satisfy $b_m = 0$ for all $m = (2k+1)2^\nu$, where k and ν are non-negative integers satisfying $k+3 \leq 2^v$. This is in exact agreement with Ewing and Schober's result in [6]. Table 3 gives non-trivial values of b_m between $m = 500,000$ and $m = 5,000,000$ in increments of $500,000$ so that the reader may verify our calculations.

Figure 1 shows a plot of Table 1 that clearly reveals the slow convergence of A_N. If the exact value of A lies closer to 1.50659 as computed by pixel counting, then certainly using A_N to closely approximate A is

Table 3. Calculated values of b_m

m	b_m
500,000	$5.5221313 \cdot 10^{-8}$
1,000,000	$-4.713883 \cdot 10^{-8}$
1,500,000	$8.4477641 \cdot 10^{-8}$
2,000,000	$-6.437866 \cdot 10^{-9}$
2,500,000	$1.6594295 \cdot 10^{-8}$
3,000,000	$8.150385 \cdot 10^{-9}$
3,500,000	$-3.911993 \cdot 10^{-9}$
4,000,000	$2.315128 \cdot 10^{-9}$
4,500,000	$-8.87746 \cdot 10^{-9}$
5,000,000	$8.0532 \cdot 10^{-11}$

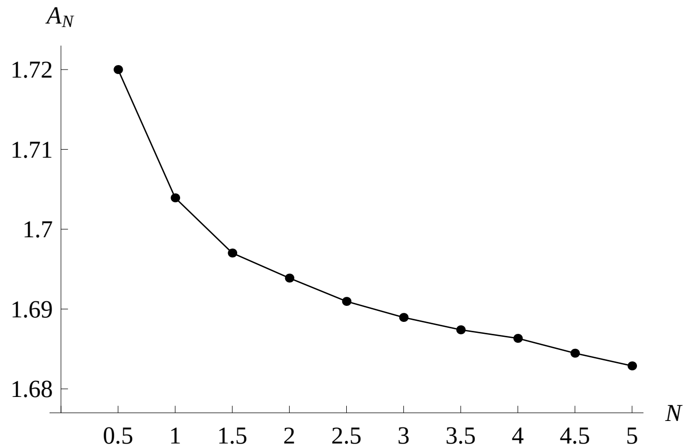

Figure 1. Plot of A_N

impractical due to the extremely large number of terms required. On the other hand, if the exact value lies closer to 1.68, then this would indicate that the boundary of the Mandelbrot set may have positive area.

5. Conclusions

In this paper we presented new results which improve on known upper bounds for the area of the Mandelbrot set and 2-adic valuations of the series coefficients $\beta_{n,m}$ given by Ewing and Schober in [6]. Of course, our calculations of the first five million terms of b_m were performed using more powerful computers that those available to Ewing and Schober two decades ago. Therefore, it would be interesting to find out in the next two decades what improvements can be made to our results by using computers that will be even more powerful, unless we are fortunate enough to see the exact area be found before then.

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