A q-DIGITAL BINOMIAL THEOREM

TOUFIK MANSOUR AND HIEU D. NGUYEN

Abstract. We present a multivariable generalization of the digital binomial theorem from which a q-analog is derived as a special case.

1. Introduction

The classical binomial theorem is an important fundamental result in mathematics:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

In 2006, Callan [5] found a digital version of the binomial theorem (see also [8]) and one of us [9] recently found a non-binary extension of Callan's result.

Theorem 1 (digital binomial theorem [5, 8]). *Let n be a non-negative integer. Then*

$$(x+y)^{s(n)} = \sum_{\substack{0\le m\le n \\ (m,n-m)\ \text{carry-free}}} x^{s(m)} y^{s(n-m)}. \tag{1}$$

In (1), $s(m)$ denotes the binary sum-of-digits function. Moreover, a pair of non-negative integers (j,k) is said to be carry-free if their sum in binary involves no carries.

In this paper, we establish a q-analog of the digital binomial theorem:

Theorem 2 (q-digital binomial theorem). *Let n be a non-negative integer with binary expansion $n = n_{N-1}2^{N-1} + \cdots + n_0 2^0$. Then*

$$\prod_{i=0}^{N-1} \binom{x+q^i y + n_i - 1}{n_i} = \sum_{\substack{0\le m\le n \\ (m,n-m)\ \text{carry-free}}} q^{z_n(m)} x^{s(m)} y^{s(n-m)}. \tag{2}$$

In (2), $z_n(m)$ is a function that counts (in a weighted manner) those digits in the N-bit binary expansion of m that are less than the corresponding digits of n (see Definition 7).

In the special case where $n = 2^N - 1$, equation (2) simplifies to

$$(x+y)(x+qy)\cdots(x+q^{N-1}y) = \sum_{m=0}^{n} q^{z_n(m)} x^{s(m)} y^{s(n-m)}. \tag{3}$$

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Observe that (3) is a variation of the well-known terminating q-binomial theorem attributed to H. A. Rothe [10] (see [2, Cor. 10.2.2(c), p. 490] or [3, Eq. 1.13, p. 15]):

$$(x+y)(x+qy)\cdots(x+q^{N-1}y) = \sum_{k=0}^{N} q^{k(k-1)/2} \binom{N}{k}_q x^{N-k} y^k, \tag{4}$$

where $\binom{N}{k}_q$ denotes the q-binomial coefficient given by (see [1, Def. 3.1, p. 35] or [6, Exercise 1.2, p. 24])

$$\binom{N}{k}_q = \frac{(1-q^N)(1-q^{N-1})\cdots(1-q^{N-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)}$$

for $0 \le k \le N$. As an application, we set $x = 1$ in both (3) and (4) and equate coefficients corresponding to like powers of y to obtain the following formula for q-binomial coefficients:

Theorem 3. *For any* $0 \le k \le N$,

$$q^{k(k-1)/2} \binom{N}{k}_q = \sum_{\substack{0 \le m \le n \\ s(m)=k}} q^{z_n(n-m)}. \tag{5}$$

For example, if $N = 3$ so that $n = 2^N - 1 = 7$, then we obtain from (5) known values for the q-binomial coefficients

$$\begin{aligned}
\binom{3}{0}_q &= q^{z_7(7)} = 1 \\
\binom{3}{1}_q &= q^{z_7(6)} + q^{z_7(5)} + q^{z_7(3)} = 1 + q + q^2 \\
\binom{3}{2}_q &= \frac{1}{q}(q^{z_7(4)} + q^{z_7(2)} + q^{z_7(1)}) = \frac{1}{q}(q + q^2 + q^3) = 1 + q + q^2 \\
\binom{3}{3}_q &= \frac{1}{q^3} \cdot q^{z_7(0)} = 1.
\end{aligned}$$

Other identities can be derived from (3) as follows:

1. Setting $x = y = 1$, we obtain

$$\sum_{m=0}^{n} q^{z_n(m)} = 2(1+q)\cdots(1+q^{N-1}).$$

2. Differentiating respect to x and setting $x = y = 1$, we obtain

$$\sum_{m=0}^{n} s(m) q^{z_n(m)} = 2(1+q)\cdots(1+q^{N-1}) \sum_{j=0}^{N-1} \frac{1}{1+q^i}.$$

Then setting $q = 1$ gives

$$\sum_{m=0}^{n} s(m) = N \cdot 2^{N-1}.$$

3. Differentiating respect to q and setting $q=1$, we obtain

$$\sum_{m=0}^{n} z_m(m) x^{s(m)} y^{s(n-m)} = \binom{N}{2} y(x+y)^{N-1}.$$

We shall derive Theorem 2 as a special case of the following multivariable generalization of the digital binomial theorem, which extends the main result in [9]. Our notation for the generalized binomial coefficients appearing in (6) is given in Definition 8.

Theorem 4. *Let n be a non-negative integer with base b expansion $n = n_{N-1}b^{N-1} + \cdots + n_0 b^0$. Then*

$$\prod_{i=0}^{N-1} \binom{x_i + y_i; r_i}{n_i} = \sum_{0 \le m \preceq_b n} \left(\prod_{i=0}^{N-1} \binom{x_i; r_i}{m_i} \prod_{i=0}^{N-1} \binom{y_i; r_i}{n_i - m_i} \right), \tag{6}$$

where $m = m_{N-1}b^{N-1} + \cdots + m_0 b^0$ and $m \preceq_b n$ denotes the fact that each digit of m is less or equal to each corresponding digit of n in base b (see Definition 6).

In particular, Theorem 2 now follows by setting $b = 2$, $x_i = x$, $y_i = q^i y$, and $r_i = 1$ in (6).

By making the substitutions $x_i = p^i x$, $y_i = q^i y$, and $r_i = r$ in (6), we also obtain the following three-parameter version of the digital binomial theorem.

Theorem 5. *Let n be a non-negative integer with base b expansion $n = n_{N-1}b^{N-1} + \cdots + n_0 b^0$. Then*

$$\prod_{i=0}^{N-1} \binom{p^i x + q^i y; r}{n_i} = \sum_{0 \le m \preceq_b n} \left(\prod_{i=0}^{N-1} \binom{p^i x; r}{m_i} \prod_{i=0}^{N-1} \binom{q^i y; r}{n_i - m_i} \right), \tag{7}$$

where $m = m_{N-1}b^{N-1} + \cdots + m_0 b^0$.

The proof of Theorem 4 will be given in Section 2, where we develop a multivariable generalization of the Sierpiński matrix and use its multiplicative property to derive (6).

2. Multivariable Sierpinski Matrices

We begin with some preliminary definitions and assume throughout this paper that b is an integer greater than 1. Our first definition involves the notion of digital dominance (see [4, 9]).

Definition 6. *Let m and n be non-negative integers with base b expansions $m = m_{N-1}b^{N-1} + \cdots + m_0 b^0$ and $n = n_{N-1}b^{N-1} + \cdots + n_0 b^0$, respectively. We denote $m \preceq_b n$ to mean that m is digitally less than n in base b, i.e., $m_k \le n_k$ for all $k = 0, \ldots, N-1$.*

Observe that m digitally less than n is equivalent to the pair $(m, n-m)$ being carry-free, i.e., the sum of m and $n-m$ involves no carries when performed in base b. This is also equivalent to the equality (see [4])

$$s_b(m) + s_b(n-m) = s_b(n),$$

where $s_b(m)$ is the base b sum-of-digits function.

Definition 7. *Let m and n be non-negative integers and assume $m \preceq_b n$. We define*

$$z_n(m; b) = \sum_{k=0}^{N-1} k(1 - \delta(n_k, m_k)), \tag{8}$$

where δ is the Kronecker delta function: $\delta(i,j)=1$ if $i=j$; otherwise, $\delta(i,j)=0$.

NOTE: If $b=2$ (binary), we denote $z_n(m) := z_n(m;2)$.

Definition 8. *We define the generalized binomial coefficient $\binom{x;r}{d}$ by*

$$\binom{x;r}{d} = \frac{x(x+r)\cdots(x+(d-1)r)}{d!}, \tag{9}$$

where we set

$$\binom{x;r}{0} = 1.$$

Observe that if $r=1$, then $\binom{x;1}{d}$ gives the ordinary binomial coefficient $\binom{x+d-1}{d}$.

The following identity will be useful later in our paper.

Lemma 9. *Let p and q be non-negative integers with $q \le p$. Then*

$$\sum_{v=q}^{p} \binom{x;r}{p-v}\binom{y;r}{v-q} = \binom{x+y;r}{p-q}. \tag{10}$$

Proof. This identity can be obtained as a special case of the Chu-Vandemonde convolution formula for ordinary binomial coefficients (see [7, Eq. (3), p. 84] or [9, Lemma 7]):

$$\sum_{v=q}^{p} \binom{x+p-v-1}{p-v}\binom{y+v-q-1}{v-q} = \binom{x+y+p-q-1}{p-q}. \tag{11}$$

It suffices to make the change of variables $x \to x/r$ and $y \to y/r$ in (11) and use the relation

$$\binom{x/r+d-1}{d} = \frac{1}{r^d}\binom{x;r}{d}$$

to obtain (10). $\square$

Next, we define a multivariable analog of the Sierpiński matrix.

Definition 10. *Let N be a non-negative integer. Denote $\mathbf{x}_N = (x_0,\ldots,x_{N-1})$ and $\mathbf{r}_N = (r_0,\ldots,r_{N-1})$. For $N>0$, we define the N-variable Sierpiński matrix*

$$S_{b,N}(\mathbf{x}_N,\mathbf{r}_N) = (\alpha_N(j,k,\mathbf{x}_N,\mathbf{r}_N))$$

of dimension $b^N \times b^N$ by

$$\alpha_N(j,k,\mathbf{x}_N,\mathbf{r}_N) = \begin{cases} \prod_{i=0}^{N-1} \binom{x_i;r_i}{d_i} & \text{if } 0 \le k \le j \le b^N-1 \\ & \text{and } k \preceq_b j; \\ 0, & \text{otherwise,} \end{cases} \tag{12}$$

where $j-k = d_0b^0 + d_1b^1 + \ldots + d_{N-1}b^{N-1}$ is the base-b expansion of $j-k$. If $N=0$, we set $S_{b,0}(\mathbf{x}_0,\mathbf{r}_0)=1$.

The following lemma gives a recurrence for $S_{b,N}(\mathbf{x}_N,\mathbf{r}_N)$.

Lemma 11. *The Sierpiński matrix $S_{b,N}(\mathbf{x}_N, \mathbf{r}_N)$ satisfies the recurrence*

$$S_{b,N+1}(\mathbf{x}_{N+1}, \mathbf{r}_{N+1}) = S_{b,1}(x_N, r_N) \otimes S_{b,N}(\mathbf{x}_N, \mathbf{r}_N), \tag{13}$$

where we define

$$S_{b,1}(x,r) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \binom{x;r}{1} & 1 & 0 & \cdots & 0 \\ \binom{x;r}{2} & \binom{x;r}{1} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{x;r}{b-1} & \binom{x;r}{b-2} & \binom{x;r}{b-3} & \cdots & 1 \end{pmatrix} = \begin{cases} \binom{x;r}{j-k}, & \textit{if } 0 \le k \le j \le b-1; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We argue by induction on N. The recurrence clearly holds for $N = 0$. Next, assume that the recurrence holds for arbitrary N. We shall prove that the recurrence holds for $N+1$. Since $S_{N+1}(\mathbf{x}_{N+1}, \mathbf{r}_{N+1})$ is a square matrix of size b^{N+1}, we can write it as a $b \times b$ matrix of blocks $(A_{p,q})_{0 \le p,q \le b-1}$, that is

$$S_{N+1}(\mathbf{x}_{N+1}, \mathbf{r}_{N+1}) = \begin{pmatrix} A_{0,0} & \dots & A_{0,b-1} \\ \vdots & \ddots & \vdots \\ A_{b-1,0} & \dots & A_{b-1,b-1} \end{pmatrix},$$

where each $A_{p,q}$ is a square matrix of size b^N. Let $A_{p,q}$ be an arbitrary block. We consider two cases:

Case 1. $p < q$. Then by definition of $S_{b,N+1}(\mathbf{x}_N, \mathbf{r}_N)$ we have that $A_{p,q} = 0$.

Case 2. $p \ge q$. Let $\alpha_{N+1}(j, k, \mathbf{x}_N, \mathbf{r}_N)$ be an arbitrary entry of $A_{p,q}$. Then $pb^N \le j \le (p+1)b^N - 1$ and $qb^N \le k \le (q+1)b^N - 1$. Set $j' = j - pb^N$ and $k' = k - qb^N$. If $j < k$, then $\alpha_{N+1}(j, k, \mathbf{x}_N, \mathbf{r}_N) = 0$ by definition. Therefore, assume $j \ge k$. Let $j - k = d_0 b^0 + d_1 b^1 + \cdots + d_N b^N$, where $d_N = p - q$. Then $j' - k' = d_0 b^0 + d_1 b^1 + \cdots + d_{N-1} b^{N-1}$. Since $k \preceq_b j$ if and only if $k' \preceq_b j'$, it follows that

$$\begin{aligned} \alpha_{N+1}(j, k, \mathbf{x}_{N+1}, \mathbf{r}_{N+1}) &= \begin{cases} \prod_{i=0}^{N} \binom{x_i; r_i}{d_i} & \text{if } 0 \le k \le j \le b^{N+1} - 1 \text{ and } k \preceq_b j; \\ \\ 0 & \text{otherwise.} \end{cases} \\ &= \binom{x_N; r_N}{d_N} \alpha_{b,N}(j', k', \mathbf{x}_N, \mathbf{r}_N). \end{aligned}$$

Thus,

$$A_{p,q} = \binom{x_N; r_N}{p-q} S_{b,N}(\mathbf{x}_N, \mathbf{r}_N), \tag{14}$$

or equivalently, $S_{b,N+1}(\mathbf{x}_{N+1}, \mathbf{r}_{N+1}) = S_{b,1}(x_N, r_N) \otimes S_{b,N}(\mathbf{x}_N, \mathbf{r}_N)$. ☐

We now demonstrate that our Sierpiński matrices are multiplicative.

Theorem 12. *Let N be a non-negative integer. Then*

$$S_{b,N}(\mathbf{x}_N, \mathbf{r}_N) S_{b,N}(\mathbf{y}_N, \mathbf{r}_N) = S_{b,N}(\mathbf{x}_N + \mathbf{y}_N, \mathbf{r}_N), \tag{15}$$

where we define

$$\mathbf{x}_N + \mathbf{y}_N = (x_0 + y_0, x_1 + y_1, \ldots, x_{N-1} + y_{N-1}).$$

Proof. We argue by induction on N. By definition, (15) clearly holds for $N = 0$. For $N = 1$, let $\beta(j,k)$ denote the (j,k)-entry of $T = S_{b,1}(\mathbf{x}_1, \mathbf{r}_1)S_{b,1}(\mathbf{y}_1, \mathbf{r}_1)$. Since T is lower-triangular, we have that $\beta(j,k) = 0$ if $j < k$. Therefore, we assume $j \geq k$. Then

$$\beta(j,k) = \sum_{i=k}^{j} \binom{x_0; r_0}{j-i}\binom{y_0; r_0}{i-k} = \binom{x_0 + y_0; r_0}{j-k}, \tag{16}$$

which follows from Lemma 9. Thus,

$$S_{b,1}(\mathbf{x}_1, \mathbf{r}_1)S_{b,1}(\mathbf{y}_1, \mathbf{r}_1) = S_{b,1}(\mathbf{x}_1 + \mathbf{y}_1, \mathbf{r}_1),$$

which shows that (15) holds for $N = 1$.

Next, assume that (15) holds for arbitrary N. We intend to prove that (15) holds for $N + 1$. By Lemma 11 and the mixed-property of a Kronecker product, we have

$$\begin{aligned}
&S_{b,N+1}(\mathbf{x}_{N+1}, \mathbf{r}_{N+1})S_{b,N+1}(\mathbf{y}_{N+1}, \mathbf{r}_{N+1}) \\
&\quad = (S_{b,1}(x_N, r_N) \otimes S_{b,N}(\mathbf{x}_N, \mathbf{r}_N))(S_{b,1}(y_N, r_N) \otimes S_{b,N}(\mathbf{y}_N, \mathbf{r}_N)) \\
&\quad = (S_{b,1}(x_N, r_N)S_{b,1}(y_N, r_N)) \otimes (S_{b,N}(\mathbf{x}_N, \mathbf{r}_N)S_{b,N}(\mathbf{y}_N, \mathbf{r}_N)).
\end{aligned}$$

Hence, by the induction hypothesis and Lemma 11 again, we obtain

$$\begin{aligned}
&S_{b,N+1}(\mathbf{x}_{N+1}, \mathbf{r}_{N+1})S_{b,N+1}(\mathbf{y}_{N+1}, \mathbf{r}_{N+1}) \\
&\quad = S_{b,1}(x_N + y_N, r_N) \otimes S_{b,N}(\mathbf{x}_N + \mathbf{y}_N, \mathbf{r}_N) \\
&\quad = S_{b,N+1}(\mathbf{x}_{N+1} + \mathbf{y}_{N+1}, \mathbf{r}_{N+1}).
\end{aligned}$$

This proves that (15) holds for $N + 1$. $\square$

Proof of Theorem 4. We equate the matrix entries at position $(n, 0)$ in both sides of (15) to obtain

$$\sum_{0 \leq m \preceq_b n} \left(\prod_{i=0}^{N-1} \binom{x_i; r_i}{n_i - m_i} \prod_{i=0}^{N-1} \binom{y_i; r_i}{m_i} \right) = \prod_{i=0}^{N-1} \binom{x_i + y_i; r_i}{n_i}.$$

It remains to switch the roles of x and y to obtain (6) as desired. $\square$

Proof of Theorem 2. We set $b = 2$, $x_i = x$, $y_i = q^i y$, and $r_i = 1$ in (6) to obtain

$$\prod_{i=0}^{N-1} \binom{x + q^i y + n_i - 1}{n_i} = \sum_{\substack{0 \leq m \leq n \\ (m, n-m) \text{ carry-free}}} \left(\prod_{i=0}^{N-1} \binom{x + m_i - 1}{m_i} \prod_{i=0}^{N-1} \binom{q^i y + n_i - m_i - 1}{n_i - m_i} \right).$$

Observe that since the digits m_i and n_i can only take on the values 0 or 1 with $m_i \leq n_i$, we have

$$\prod_{i=0}^{N-1} \binom{x + m_i - 1}{m_i} = x^{\sum_{i=0}^{N-1} m_i} = x^{s(m)}$$

and

$$\prod_{i=0}^{N-1} \binom{q^i y + n_i - m_i - 1}{n_i - m_i} = q^C y^{s(n-m)},$$

where

$$C = \sum_{i=0}^{N-1} i(n_i - m_i) = z_n(m).$$

It follows that

$$\prod_{i=0}^{N-1}\binom{x+q^iy+n_i-1}{n_i}=\sum_{\substack{0\le m\le n\\ (m,n-m)\text{ carry-free}}} q^{z_n(m)}x^{s(m)}y^{s(n-m)}.$$

This proves (2). □

We conclude by observing that many other variations of equation (3) can be obtained by making appropriate substitutions for the variables in Theorems 2 and 4. For example, equation (3) can be extended to obtain a p,q-analog of the digital binomial theorem by replacing y with yp^{1-N}, q with pq, and then multiplying through by $p^{(N-1)N/2}$:

$$(p^{N-1}x+y)(p^{N-2}x+qy)\cdots(x+q^{N-1}y)=\sum_{m=0}^{n}p^{w_n(m)}q^{z_n(m)}x^{s(m)}y^{s(n-m)}.$$

Here,

$$w_n(m)=\sum_{k=0}^{N-1}(N-k-1)\delta(n_k,m_k)=(N-1)s(m)-z_n(n-m)$$

is a reverse-weighted sum of the positions of the digit 1 in the N-bit binary expansion of m.

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Department of Mathematics, University of Haifa, 31905 Haifa, Israel
E-mail address: `toufik@math.haifa.ac.il`

Department of Mathematics, Rowan University, Glassboro, NJ, USA
E-mail address: `nguyen@rowan.edu`