THE AREA OF THE MANDELBROT SET AND ZAGIER'S CONJECTURE

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Abstract. We prove Zagier's conjecture regarding the 2-adic valuation of the coefficients $\{b_m\}$ that appear in Ewing and Schober's series formula for the area of the Mandelbrot set in the case where $m \equiv 2 \mod 4$.

1. Introduction

The Mandelbrot set M is defined as the set of complex numbers $c \in \mathbb{C}$ for which the sequence $\{z_n\}$ defined by the recursion

$$z_n = z_{n-1}^2 + c \tag{1}$$

with initial value $z_0 = 0$ remains bounded for all $n \geq 0$. Douady and Hubbard [3] proved that M is connected and Shishikura [11] proved that M has fractal boundary of Hausdorff dimension 2. However, it is unknown whether the boundary of M has positive Lebesgue measure, although Julia sets with positive area are known to exist (Buff and Chéritat [2]).

Ewing and Schober [5] derived a series formula for the area of M by considering its complement, $\tilde{M}$, inside the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, i.e. $\tilde{M} = \overline{\mathbb{C}} - M$. It is known that $\tilde{M}$ is simply connected with mapping radius 1 ([3]). In other words, there exists an analytic homeomorphism

$$\psi(z) = z + \sum_{m=0}^{\infty} b_m z^{-m} \tag{2}$$

which maps the domain $\Delta = \{z : 1 < |z| \leq \infty\} \subset \overline{\mathbb{C}}$ onto $\tilde{M}$. It follows from the classic result of Gronwall [6] that the area of the Mandelbrot set $M = \overline{\mathbb{C}} - \tilde{M}$ is given by

$$A = \pi \left[1 - \sum_{m=1}^{\infty} m |b_m|^2 \right]. \tag{3}$$

The arithmetic properties of the coefficients b_m have been studied in depth, first by Jungreis [7], then independently by Levin [8, 9], Bielefeld, Fisher, and Haeseler [1], Ewing and Schober [4, 5], and more recently by Shimauchi [10]. In particular, Ewing and Schober [5] proved the following formula for the coefficients b_m.

Theorem 1 (Ewing-Schober [5]). *Suppose $m \leq 2^{n+1} - 3$. Define the set of n-tuples*

$$J = \{\mathbf{j} = (j_1, \ldots, j_n) : (2^n - 1)j_1 + \ldots + (2^2 - 1)j_{n-1} + (2 - 1)j_n = m + 1\}$$

and given any $\mathbf{j} \in J$, set

$$\alpha_{\mathbf{j}}(k) := \alpha(k) := \alpha = \frac{m}{2^{n-k+1}} - 2^{k-1} j_1 - 2^{k-2} j_2 - \ldots - 2 j_{k-1}.$$

Then

$$b_m = -\frac{1}{m} \sum_J \prod_{k=1}^{n} C_{j_k}(\alpha(k)) \tag{4}$$

where $C_{j_k}(\alpha(k))$ is the binomial coefficient

$$C_{j_k}(\alpha(k)) = \frac{\alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - (j_k - 1))}{j_k!}. \tag{5}$$

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Using formula (4) to compute b_m is impractical as it requires determining the set of tuples J, which is computationally hard. However, since it is known that each b_m is rational and has denominator equal to a power of 2, it is then useful to find a formula for its 2-adic valuation. Towards this end, Levin [8] gave such a formula when m is odd, and Shimauchi [11] established an upper bound valid for all m with equality if and only if m is odd.

Definition 2. *Let n be a non-negative integer. We define*
(a) $\nu(n)$ to be the 2-adic valuation of n.
(b) $s(n)$ (called the sum-of-digits function) to be the sum of the binary digits of n.

Theorem 3 (Levin [8], Shimauchi [11])**.** *Let m be a non-negative integer. Then*

$$-\nu(b_m) \leq 2(m+1) - s(2(m+1)) \tag{6}$$

Moreover, equality holds precisely when m is odd.

In this paper we prove Zagier's conjecture (see [1]) regarding a formula for the 2-adic valuation of b_m when $m \equiv 2 \mod 4$.

Theorem 4 (Zagier's Conjecture [1])**.** *Suppose $m \equiv 2 \mod 4$. Then*

$$-\nu(b_m) = \left\lfloor \frac{2}{3}(m+1) \right\rfloor - s\left(\left\lfloor \frac{2}{3}(m+1) \right\rfloor \right) + \epsilon(m), \tag{7}$$

where

$$\epsilon(m) = \begin{cases} 0, & \textit{if } m \equiv 22 \mod 24; \\ 1, & \textit{otherwise}. \end{cases} \tag{8}$$

Our proof relies on determining those tuples $\mathbf{j}_{\max} \in J$ that maximize $V(\mathbf{j}) := -\nu\left(\prod_{k=1}^{n} C_{j_k}(\alpha(k))\right)$, i.e., $V(\mathbf{j}) < V(\mathbf{j}_{\max})$ for all $\mathbf{j} \in J$. In particular, we show for $m \equiv 2 \mod 4$ that this largest 2-adic valuation $V(\mathbf{j}_{\max})$ is achieved by exactly one tuple $\mathbf{j}_{\max}$ or else by exactly three tuples $\mathbf{j}_{\max}$, $\mathbf{j}'_{\max}$, $\mathbf{j}''_{\max}$ in the special case where $m \equiv 22 \mod 24$. To prove that $V(\mathbf{j}) < V(\mathbf{j}_{\max})$ for all $\mathbf{j} \in J$, we derive lemmas to compare the values of $V(\mathbf{j})$ for different types of tuples. For example, if $m = 38$, then it holds that

$$V((2,1,0,2)) < V((0,5,1,1)) < V((0,0,13,0)),$$

where $\mathbf{j}_{\max} = (0,0,13,0)$. We refer to the chain of tuples

$$(2,1,0,2) \to (0,5,1,1) \to \mathbf{j}_{\max}$$

as a set of tuple transformations.

As a result of our comparison lemmas (derived in Sections 2 and 3), we have the result

$$-\nu(b_m) = 1 + V(\mathbf{j}_{\max}). \tag{9}$$

This follows from the fact that the 2-adic valuation of the sum of any number of fractions (whose denominators are powers of 2 and whose numerators are odd) is equal to the largest 2-adic valuation of all the fractions, assuming that there are an odd number of fractions with the same largest 2-adic valuation. It remains to calculate $V(\mathbf{j}_{\max})$ in each case, which then establishes Zagier's conjecture.

2. Tuple Transformations

We begin with preliminary definitions.

Definition 5. *Given $\mathbf{j} \in J$, define*

$$\beta_{\mathbf{j}}(k) := \beta(k) := \beta = 2^{n-k+1}\alpha(k) = m - 2^n j_1 - 2^{n-1} j_2 - \cdots - 2^{n-k+2} j_{k-1}$$

and

$$B(k) = \beta(\beta - 2^{n-k+1})(\beta - 2 \cdot 2^{n-k+1}) \cdots (\beta - (j_k - 1) \cdot 2^{n-k+1}).$$

Lemma 6. *We have*

$$\nu(B(k)) = j_k$$

for $1 \leq k \leq n - \nu(m)$.

Proof. First, we establish that $\nu(\beta(k)) = \nu(m)$ for $1 \leq k \leq n - \nu(m)$. This follows from

$$\begin{aligned}
\nu(\beta) &= \nu(m - 2^n j_1 - 2^{n-1} j_2 - \cdots - 2^{n-k+2} j_{k-1}) \\
&= \nu(m - (2^n j_1 - 2^{n-1} j_2 - \cdots - 2^{n-k+2} j_{k-1})) \\
&= \nu(m),
\end{aligned}$$

which holds since $\nu(2^n j_1 - 2^{n-1} j_2 - \cdots - 2^{n-k+2} j_{k-1}) \geq n - k + 2 > \nu(m)$. Then by definition we have

$$B(k) = \beta(\beta - d^{n-k+1})(\beta - 2d^{n-k+1}) \cdots (\beta - (j_k - 1)d^{n-k+1})$$

Taking the 2-adic valuation of both sides and expanding the right-hand side gives

$$\begin{aligned}
\nu(B(k)) &= \nu(\beta(\beta - 2^{n-k+1})(\beta - 2 \cdot 2^{n-k+1}) \cdots (\beta - (j_k - 1)2^{n-k+1})) \\
&= \nu(\beta) + \nu(\beta - 2^{n-k+1}) + \nu(\beta - 2 \cdot 2^{n-k+1}) + \cdots + \nu(\beta - (j_k - 1)2^{n-k+1})
\end{aligned}$$

Since $n - k + 1 > \nu(m)$, $\nu(\beta - p \cdot 2^{n-k+1}) = 1$ for all integers p. Thus

$$\nu(\beta) + \nu(\beta - 2^{n-k+1}) + \nu(\beta - 2(2^{n-k+1})) + \cdots + \nu(\beta - (j_k - 1)2^{n-k+1}) = 1 + 1 + \cdots + 1$$

where there are j_k 1's. Thus, $\nu(B(k)) = j_k$ as desired. $\square$

Lemma 7. *We have*

$$-\nu(C_{j_k}(\alpha(k)) = (n - k + 1)j_k - s(j_k) \tag{10}$$

for $1 \leq k \leq n - \nu(m)$

Proof. It is clear from Definition 5 that

$$\begin{aligned}
C_{j_k}(\alpha(k)) &= \frac{\alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - (j_k - 1))}{j_k!} \\
&= \frac{\beta(\beta - 2^{n-k+1})(\beta - 2 \cdot 2^{n-k+1}) \cdots (\beta - (j_k - 1)2^{n-k+1})}{2^{j_k(n-k+1)} j_k!} \\
&= \frac{B(k)}{2^{j_k(n-k+1)} j_k!}
\end{aligned}$$

and thus

$$\begin{aligned}
-\nu(C_{j_k}(\alpha(k)) &= -\nu\left(\frac{B(k)}{2^{j_k(n-k+1)} j_k!}\right) \\
&= -(\nu(B(k) - \nu(2^{j_k(n-k+1)} j_k!)) \\
&= (n - k + 1)j_k + j_k - s(j_k) - \nu(B(k)) \\
&= (n - k + 1)j_k - s(j_k)
\end{aligned}$$

since we have from Lemma 7 that $\nu(B(k)) = j_k$ for $1 \leq k \leq n - \nu(m)$. $\square$

We now consider the case where $k > n - \nu(m)$. Define $c(x, y)$ to be the number of carries performed when summing two non-negative integers x and y in binary. It is a well known result that

$$c(x, y) = s(x) + s(y) - s(x + y).$$

Lemma 8. *Let* $\mathbf{j} \in J$. *Then for* $k > n - \nu(m)$, *we have*

$$-\nu(C_{j_k}(\alpha(k)))) = \begin{cases} -c(j_k, -\alpha(k) - 1), & \alpha(k) < 0; \\ -\infty, & 0 \leq \alpha(k) \leq j_k; \\ c(j_k, \alpha(k) - j_k), & \alpha(k) > j_k. \end{cases} \tag{11}$$

Proof. First, we demonstrate that $\alpha(k)$ is an intger when $k > n - \nu(m)$. By definition, we have

$$\alpha(k) = \frac{m}{2^{n-k+1}} - 2^{k-1} j_1 - 2^{k-2} j_2 - \ldots - 2j_{k-1}.$$

Since $\nu(m) \geq n - k + 1$, it follows that m is divisible by 2^{n-k+1}. Thus, $\frac{m}{2^{n-k+1}}$ is an integer, and since the remaining terms are all integers, $\alpha(k)$ must be an integer as well.

If $\alpha(k) < 0$, we have

$$\begin{aligned}
-\nu(C_{j_k}(\alpha(k))) &= -\nu\left(\frac{\alpha(\alpha-1)\dots(\alpha-j_k+1)}{j_k!}\right) \\
&= j_k - s(j_k) - \nu((\alpha-j_k+1)\dots(\alpha-1)\alpha) \\
&= j_k - s(j_k) - (\nu((-\alpha-j-k+1)!) - \nu(-\alpha-1)) \\
&= -s(j_k) - s(-\alpha-1) + s(-\alpha-1+j_k) \\
&= -c(j_k, -\alpha-1).
\end{aligned}$$

On the other hand, if $0 \leq \alpha(k)$, then $C_{j_k}(\alpha(k)) = 0$, and therefore $\nu(C_{j_k}) = \infty$. Lastly, if $\alpha(k) > j_k$, then we have

$$\begin{aligned}
-\nu(C_{j_k}(\alpha)) &= -\nu\left(\frac{\alpha!}{(\alpha-j_k)!j_k!}\right) \\
&= \alpha - s(\alpha) - (\alpha-j_k) + s(\alpha-j_k) - j_k + s(j_k) \\
&= s(j_k) + s(\alpha-j_k) - s(\alpha) \\
&= c(j_k, \alpha(k) - j_k)
\end{aligned}$$

as desired. □

Definition 9. *For convenience, define*

$$\gamma(m,k) := \gamma(k) = \begin{cases} -c(j_k, -\alpha(k)-1), & \alpha(k) < 0; \\ \infty, & 0 \leq \alpha(k) \leq j_k; \\ c(j_k, \alpha(k)-j_k), & \alpha(k) > j_k, \end{cases} \tag{12}$$

and for any tuple $\mathbf{j} \in J$*, define*

$$v(m,\mathbf{j}) = \sum_{k=1}^{n-\nu(m)} [(n-k+1)j_k - s(j_k)] \tag{13}$$

and

$$V(m,\mathbf{j}) := V(\mathbf{j}) = -\nu\left(\prod_{k=1}^{n} C_{j_k}(\alpha(k))\right). \tag{14}$$

In the case where $m \equiv 2 \mod 4$ *so that* $\nu(m) = 1$*, we shall simply write*

$$v(\mathbf{j}) := v(m,\mathbf{j}) = \sum_{k=1}^{n-1}[(n-k+1)j_k - s(j_k)]. \tag{15}$$

The next lemma follows immediately from Definition 9 and Lemmas 7 and 8.

Lemma 10. *We have*

$$V(\mathbf{j}) = v(m,\mathbf{j}) + \sum_{k=n-\nu(m)+1}^{n} \gamma(k)$$

and in particular if $m \equiv 2 \mod 4$*, then*

$$V(\mathbf{j}) = v(\mathbf{j}) + \gamma(n). \tag{16}$$

We now consider tuple transformations that allow us to compare $v(m,\mathbf{j})$ for different types of tuples.

Lemma 11. *Suppose* $\nu(m) \geq 1$*. Let* $\mathbf{j}$ *be a* J*-tuple and* $i < n - \nu(m)$ *be such that* $j_i \neq 0$*. Define the tuple* $\mathbf{j}' = (j'_1, \dots, j'_n)$ *by*

$$j'_k = \begin{cases} j_k, & k \neq i, i+1, n \\ j_i - r, & k = i \\ j_{i+1} + p, & k = i+1 \\ j_n + q, & k = n \end{cases}$$

where r is the largest power of 2 less than j_i, and p and q satisfy

$$(2^{n-i}-1)p+q=(2^{n-i+1}-1)r \tag{17}$$

with $q<2^{n-i}-1$. Then

$$v(m,\mathbf{j})<v(m,\mathbf{j}').$$

Proof. It is clear that p and q exist by Euclid's Division Theorem. Then since $j_k=j'_k$ for all $k\neq i,i+1,n$, the corresponding terms will cancel when we compute the difference $v(\mathbf{j}')-v(\mathbf{j})$. If $i<n-2$, then

$$\begin{aligned}
v(m,\mathbf{j}')-v(m,\mathbf{j}) &= (n-i)p-(n-i+1)r+s(j_i)-s(j_i-r)+s(j_{i+1})-s(j_{i+1}+p)\\
&\geq (n-i)p-(n-i+1)r+1-s(p)\\
&> \frac{n-i-1}{2}p-\frac{n-i+1}{2}-\lceil\log_2(p)\rceil\\
&\geq 0
\end{aligned}$$

since $r<(p+1)/2$ and $p\geq 2$. The remaining case, $i=n-2$, can be easily proven by similar means. $\square$

Observe that we can apply Lemma 11 repeatedly to transform any tuple $\mathbf{j}\in J$ containing a non-zero element j_i, $1\leq i\leq n-\nu(m)$, to a tuple $\mathbf{j}'\in J$ with $j'_i=0$. Thus, any tuple $\mathbf{j}\in J$ can be transformed to a tuple $\mathbf{j}'$, where all elements $j'_i=0$ except for $i\geq n-\nu(m)$, with $v(\mathbf{j})<v(\mathbf{j}')$. We will make use of this fact later on.

Lemma 12. *Let $\mathbf{j}$ be a J-tuple where $j_n>2$, and $\mathbf{j}'$ be the tuple such that*

$$j'_k=\begin{cases} j_k, & 1\leq k\leq n-\nu(m)-1;\\ j_{n-\nu(m)}+p, & k=n-\nu(m);\\ 0, & n-\nu(m)<k<n;\\ \sum_{k=n-\nu(m)+1}^{n}(2^{n-k+1}-1)j_k-(2^{\nu(m)+1}-1)p, & k=n,\end{cases}$$

where p is chosen to be as largest as possible so that $j'_n<2^{\nu(m)+1}-1$. Then

$$v(m,\mathbf{j})<v(m,\mathbf{j}').$$

Proof. We have that

$$\begin{aligned}
v(m,\mathbf{j}')-v(m,\mathbf{j}) &= (n-\nu(m)+1)(j_{n-\nu(m)}+p)-s(j_{n-\nu(m)}+p)\\
&\quad -(n-\nu(m)+1)j_{n-\nu(m)}+s(j_{n-\nu(m)})\\
&=(n-\nu(m)+1)p+s(j_{n-\nu(m)})-s(j_{n-\nu(m)}+p)\\
&=(n-\nu(m)+1)p+c(j_{n-\nu(m)},p)-s(p)\\
&\geq (n-\nu(m)+1)p-s(p)\\
&>0.
\end{aligned}$$

$\square$

In particular, when $m\equiv 2\mod 4$, Lemma 12 allows us to transform a tuple $\mathbf{j}\in J$, whose elements are all zero except for j_{n-1} and $j_n>2$, to a tuple $\mathbf{j}'\in J$, whose elements are also all zero but with $j'_n\leq 2$, so that $v(\mathbf{j})<v(\mathbf{j}')$.

3. Zagier's Conjecture

In this section we prove Zagier's conjecture for the case where $m\equiv 2\mod 4$, which we assume throughout this section. In order to do this, we first derive additional lemmas that allow us to compare $V(\mathbf{j})$ for the tuple transformations described in the previous section.

Lemma 13. *If $m+1\equiv 0\mod 3$, then $V(\mathbf{j})<V(\mathbf{j}')$ for all $\mathbf{j}\neq\mathbf{j}'$, where $\mathbf{j}'=(0,0,\ldots,\frac{m+1}{3},0)$.*

Proof. By Lemmas 11 and 12, we can transform $\mathbf{j}$ to a tuple $\mathbf{j}'$ so that $j_i' = 0$ for all $i < n-1$ since $\nu(m) = 1$. Moreover, $j_{n-1}' = (m+1)/3$ and $j_n' = 0$ since $m+1 \equiv 0 \mod 3$. It follows that

$$\begin{aligned}
V(\mathbf{j}) &= \sum_{k=1}^{n-1}[(n-k+1)j_k - s(j_k)] - c(j_n, -\alpha(n)-1) \\
&\leq \sum_{k=1}^{n-1}[(n-k+1)j_k - s(j_k)]] = v(\mathbf{j}) \\
&< v(\mathbf{j}') = V(\mathbf{j}')
\end{aligned}$$

since $c(j_n', -\alpha'(n)-1) = 0$ due to Lemma 10. □

Lemma 14. *If $m+1 \equiv 1 \mod 3$, then $V(\mathbf{j}) < V(\mathbf{j}')$ for all $\mathbf{j} \neq \mathbf{j}'$, where $\mathbf{j}' = (0, 0, \ldots, \frac{m}{3}, 1)$.*

Proof. We have $V(\mathbf{j}) < V(\mathbf{j}')$ by the same reasoning as in the previous lemma. □

Lemma 15. *If $m+1 \equiv 2 \mod 3$ and $m \equiv 2 \mod 8$, then $V(\mathbf{j}) < V(\mathbf{j}')$ for all $\mathbf{j} \neq \mathbf{j}'$, where $\mathbf{j}' = (0, 0, \ldots, \frac{m-1}{3}, 2)$.*

Proof. Again, such a tuple $\mathbf{j}'$ exists because of Lemmas 11 and 12. We first determine the binary representation of $-\alpha(n)-1$. Since

$$\begin{aligned}
-\alpha(n)-1 &= -\frac{j_n - (1 + j_1 + \cdots + j_{n-1})}{2} - 1 \\
&= -\frac{1 - \frac{m-1}{3}}{2} - 1 \\
&= \frac{\frac{m-1}{3} - 1}{2} - 1 \\
&= \frac{\frac{m-1}{3} - 3}{2} \\
&= \frac{m-10}{6}
\end{aligned}$$

and $m \equiv 2 \mod 8$ by assumption, it follows that $-\alpha(n)-1$ has binary representation $b_n \cdots b_3 100$. It follows that $c(2, -\alpha(n)-1) = 0$ and thus $V(\mathbf{j}') = v(\mathbf{j}')$ by Lemma 10. Moreover, we have

$$\begin{aligned}
V(\mathbf{j}') &= \sum_{k=1}^{n-1}[(n-k+1)j_k - s(j_k)] - c(j_n, -\alpha(n)-1) \\
&= \frac{2(m-1)}{3} - s\left(\frac{m-1}{3}\right) - c(2, -\alpha(n)-1) \\
&= \frac{2(m-1)}{3} - s\left(\frac{2(m-1)}{3}\right).
\end{aligned}$$

It remains to be shown that $V(\mathbf{j}) < V(\mathbf{j}')$ for all $\mathbf{j} \neq \mathbf{j}'$. This follows from

$$\begin{aligned}
V(\mathbf{j}) &= \sum_{k=1}^{n-1}[(n-k+1)j_k - s(j_k)] - c(j_n, -\alpha(n)-1) \\
&\leq \sum_{k=1}^{n-1}[(n-k+1)j_k - s(j_k)]] = v(\mathbf{j}) \\
&< v(\mathbf{j}') = V(\mathbf{j}').
\end{aligned}$$

This proves the lemma. □

In order to handle the case $m+1 \equiv 2 \mod 3$ and $m \equiv 6 \mod 8$ (or equivalently $m \equiv 22 \mod 24$), we will need the following lemma. First, we define the following three special tuples, which exist for this case:

$$\begin{aligned}
\mathbf{j}' &= (0, 0, \ldots, \frac{m-1}{3}, 2) \\
\mathbf{j}'' &= (0, 0, \ldots, \frac{m-1}{3} - 1, 5) \\
\mathbf{j}''' &= (0, 0, \ldots, 1, \frac{m-1}{3} - 2, 1).
\end{aligned}$$

Lemma 16. *Suppose $m+1 \equiv 2 \mod 3$ and $m \equiv 6 \mod 8$. Then for all $\mathbf{j} \notin \{\mathbf{j}', \mathbf{j}'', \mathbf{j}'''\}$, we have*

$$V(\mathbf{j}) < V(\mathbf{j}''').$$

Proof. Since $\alpha_{\mathbf{j}'''}(n)$ is odd and $j'''_n = 1$, we have $c(j'''_n, -\alpha(n)-1) = 0$ and thus $V(\mathbf{j}) = v(\mathbf{j})$. Moreover, we have

$$\begin{aligned}
V(\mathbf{j}''') &= \sum_{k=1}^{n-1} [(n-k+1) j'''_k - s(j'''_k)] - c(j'''_n, -\alpha(n)-1) \\
&= 3 \cdot 1 - s(1) + \frac{2(m-7)}{3} - s\left(\frac{m-7}{3}\right) \\
&= \frac{2(m-1)}{3} - 2 - s\left(\frac{m-1}{3} - 2\right) \\
&= \frac{2(m-1)}{3} - s\left(\frac{m-1}{3}\right) - 1.
\end{aligned}$$

Thus, it suffices to show that $v(\mathbf{j}) < v(\mathbf{j}''')$ since this will imply $V(\mathbf{j}) \leq v(\mathbf{j}) < v(\mathbf{j}') = V(\mathbf{j}')$. Note that for any tuple $\mathbf{j}$ containing an element $j_i \neq 0$ such that $1 \leq i \leq n-3$, we have $v(\mathbf{j}) < v(\mathbf{g})$ for some tuple $\mathbf{g}$ with $g_i = 0$ for all $1 \leq i \leq n-3$ and $g_{n-2} = 2^k$ for some k. To construct such a tuple $\mathbf{g}$, we simply apply the tuple transformation in Lemma 11 repeatedly.

We now consider 3 cases. First, if $\mathbf{g} = \mathbf{j}'''$, then the theorem holds trivially. If $g_{n-2} > 1$, we proceed in two steps. Let $7(g_{n-2} - 1) = 3p + q$ where $q < 3$, and let $\mathbf{g}'$ be such that

$$\begin{aligned}
g'_i &= 0 \text{ for } 1 \leq i \leq n-3 \\
g'_{n-2} &= 1 \\
g'_{n-1} &= g_{n-1} + p \\
g'_n &= g_n + q.
\end{aligned}$$

Then we have

$$\begin{aligned}
v(\mathbf{g}') - v(\mathbf{g}) &= 2 + 2(g_{n-1} + p) - s(g_{n+1} + p) - 3g_{n-2} + 1 - 2g_{n-1} + s(g_{n-1}) \\
&\geq 2p - 3g_{n-2} - \lceil \log_2(p) \rceil + 3 \\
&\geq \frac{11p}{7} - \lceil \log_2(p) \rceil + \frac{12}{7} \\
&> 0.
\end{aligned}$$

Then applying Lemma 12 to $\mathbf{g}'$ completes the proof for this case. If $g_{n-2} = 0$, then we proceed as follows. Let $\frac{m-1}{3} = g_{n-1} + p$. Note that because $\mathbf{g} \notin \{\mathbf{j}', \mathbf{j}'', \mathbf{j}'''\}$, we have $p \geq 2$. Thus,

$$\begin{aligned}
v(\mathbf{j}''') - v(\mathbf{g}) &= 2(g_{n-1} + p) - s(g_{n-1} + p) - 1 - 2g_{n-1} + s(g_{n-1}) \\
&\geq 2p - 1 - s(p) \\
&> 0.
\end{aligned}$$

This completes the proof. $\square$

Lemma 17. *If $m+1 \equiv 2 \mod 3$ and $m \equiv 46 \mod 48$, then $V(\mathbf{j}) < V(\mathbf{j}''')$ for all $\mathbf{j} \neq \mathbf{j}'$, where $\mathbf{j}''' = (0, 0, \ldots, 1, \frac{m-7}{3}, 1)$.*

Proof. In light of Lemma 16, it suffices to prove that $V(\mathbf{j}') < V(\mathbf{j}''')$ and $V(\mathbf{j}'') < V(\mathbf{j}''')$. We first consider $\mathbf{j}'$. We have

$$\begin{aligned}\alpha_{\mathbf{j}''}(n) &= m/2 - 2^{n-1}j_1 - 2^{n-2}j_2 - \cdots - 2j_{n-1} \\ &= m/2 - 2(m-1)/3 = (4-m)/6,\end{aligned}$$

which implies $-\alpha(n) - 1 = (m-10)/6$ has binary expansion $b_n \ldots b_3 110$. Thus, $c(j'_n, -\alpha(n) - 1) > 0$ since $j'_n = 2$. It follows that

$$\begin{aligned}V(\mathbf{j}') &= \sum_{k=1}^{n-1}[(n-k+1)j'_k - s(j'_k)] - c(j'_n, -\alpha(n)-1) \\ &< \sum_{k=1}^{n-1}[(n-k+1)j'_k - s(j'_k)] \\ &= \frac{2(m-1)}{3} - s\left(\frac{m-1}{3}\right) \\ &= \frac{2(m-1)}{3} - 2 - s\left(\frac{m-1}{3} - 2\right) \\ &= \frac{2(m-1)}{3} - s\left(\frac{m-1}{3}\right) - 1 \\ &= V(\mathbf{j}''').\end{aligned}$$

As for $\mathbf{j}''$, we have $c(j''_n, -\alpha(n) - 1) > 0$ since $j''_n = 5$ and

$$\begin{aligned}\alpha_{\mathbf{j}''}(n) &= m/2 - 2^{n-1}j_1 - 2^{n-2}j_2 - \cdots - 2j_{n-1} \\ &= m/2 - 2(m-4)/3 = (16-m)/6,\end{aligned}$$

which implies $-\alpha(n) - 1 = (m-22)/6$ has binary expansion $b_n \ldots b_3 100$. It follows that

$$\begin{aligned}V(\mathbf{j}') &= \sum_{k=1}^{n-1}[(n-k+1)j'_k - s(j'_k)] - c(j'_n, -\alpha(n)-1) \\ &< \sum_{k=1}^{n-1}[(n-k+1)j'_k - s(j'_k)] \\ &= \frac{2(m-4)}{3} - s\left(\frac{m-4}{3}\right) \\ &= \frac{2(m-1)}{3} - 2 - s\left(\frac{m-1}{3} - 1\right) \\ &= \frac{2(m-1)}{3} - s\left(\frac{m-1}{3}\right) - 1 \\ &= V(\mathbf{j}''').\end{aligned}$$

This completes the proof. $\square$

Lemma 18. *If $m + 1 \equiv 2 \mod 3$ and $m \equiv 22 \mod 48$, then*

$$V(\mathbf{j}) < V(\mathbf{j}')$$

for all $\mathbf{j} \notin \{\mathbf{j}', \mathbf{j}'', \mathbf{j}'''\}$. Moreover,

$$V(\mathbf{j}') = V(\mathbf{j}'') = V(\mathbf{j}''') = \frac{2(m-1)}{3} - s\left(\frac{2(m-1)}{3}\right) - 1.$$

Proof. Again, in light of Lemma 16, it suffices to prove that $V(\mathbf{j}') = V(\mathbf{j}'') = V(\mathbf{j}''')$. Write $m = 48q + 22$ for $q \in \mathbb{N}$ and so that the elements of $\mathbf{j}'$, $\mathbf{j}''$, and $\mathbf{j}'''$ take the form

$$j'_i = \begin{cases} 0, & 1 \le i \le n-3 \\ 1, & i = n-2 \\ 16q+5 & i = n-1 \\ 1 & i = n, \end{cases} \tag{18}$$

$$j''_i = \begin{cases} 0, & 1 \le i \le n-3 \\ 0, & i = n-2 \\ 16q+7 & i = n-1 \\ 2 & i = n, \end{cases} \tag{19}$$

and

$$j'''_i = \begin{cases} 0, & 1 \le i \le n-3 \\ 0, & i = n-2 \\ 16q+6 & i = n-1 \\ 5 & i = n, \end{cases} \tag{20}$$

It is straightforward to show that

$$\begin{aligned} \alpha_{\mathbf{j}}(n) &= -(8q+3) < 0 \\ \alpha_{\mathbf{j}'}(n) &= -(8q+3) < 0 \\ \alpha_{\mathbf{j}''}(n) &= -(8q+1) < 0. \end{aligned}$$

Then

$$\begin{aligned} V(\mathbf{j}') &= 3j'_{n-2} - s(j'_{n-2}) + 2j'_{n-1} - s(j'_{n-1}) - c(j_n, -\alpha_{\mathbf{j}'}(n) - 1) \\ &= 3(1) - s(1) + 2(16q+5) - s(16q+5) - c(1, 8q+2) \\ &= 32q + 12 - s(q) - s(5) \\ &= 32q + 10 - s(q). \end{aligned}$$

Similarly,

$$\begin{aligned} V(\mathbf{j}'') &= 3j''_{n-2} - s(j''_{n-2}) + 2j''_{n-1} - s(j''_{n-1}) - c(j''_n, -\alpha_{\mathbf{j}''}(n) - 1) \\ &= 3(0) - s(0) + 2(16q+7) - s(16q+7) - c(2, 8q+2) \\ &= 32q + 14 - s(q) - s(7) - c(2, 8q+2) \\ &= 32q + 10 - s(q) \end{aligned}$$

and

$$\begin{aligned} V(\mathbf{j}''') &= 3j'''_{n-2} - s(j'''_{n-2}) + 2j'''_{n-1} - s(j'''_{n-1}) - c(j'''_n, -\alpha_{\mathbf{j}'''}(n) - 1) \\ &= 3(0) - s(0) + 2(16q+6) - s(16q+6) - c(5, 8q) \\ &= 32q + 12 - s(q) - s(6) - c(5, 8q) \\ &= 32q + 10 - s(q). \end{aligned}$$

Thus, $V(\mathbf{j}') = V(\mathbf{j}'') = V(\mathbf{j}''')$. ∎

The following theorem summarizes the form of the maximum tuple $\mathbf{j}_{\max}$ for the case $m \equiv 2 \mod 4$.

Theorem 19. *Suppose $m \equiv 2 \mod 4$. The maximum tuple $\mathbf{j}_{\max}$ occurs in the following form:*

(1) If $m+1 \equiv 0 \mod 3$, then $\mathbf{j}_{\max} = (0, ..., 0, p, 0)$ where $p = (m+1)/3$.
(2) if $m+1 \equiv 1 \mod 3$, then $\mathbf{j}_{\max} = (0, ..., 0, p, 1)$ where $p = m/3$.
(3) If $m+1 \equiv 2 \mod 3$ and
  (a) If $m \equiv 2 \mod 8$, then $\mathbf{j}_{\max} = (0, ..., 0, p, 2)$ where $p = (m-1)/3$.
  (b) If $m \equiv 46 \mod 48$, then $\mathbf{j}_{\max} = (0, ..., 1, p-2, 1)$ where $p = (m-1)/3$.

(c) If $m \equiv 22 \mod 48$*, then*
$$\mathbf{j}_{\max} = (0, ..., 1, (m-7)/3, 1), (0, ..., 0, (m-1)/3, 2), (0, \ldots, 0, (m-4)/3, 5).$$

We now have all the necessary ingredients to prove Zagier's conjecture.

Proof of Theorem 4 (Zagier's Conjecture): We divide the proof into the following cases:

(1) $m+1 \equiv 0 \mod 3$.
(2) $m+1 \equiv 1 \mod 3$.
(3) $m+1 \equiv 2 \mod 3$ and
  (a) $m \equiv 2 \mod 8$.
  (b) $m \equiv 46 \mod 48$.
  (c) $m \equiv 22 \mod 48$.

Case (1): Write $m+1 = 3p$ for some positive integer p. Since $m \equiv 2 \mod 4$, it follows that $3p - 1 \equiv 2 \mod 4$ and so $p \equiv 1 \mod 4$. Now, recall that $\mathbf{j}_{\max} = (j_{n-1}, j_n) = (p, 0)$, we have $\alpha(n) = -(1+p)/2$. Then using the relation

$$c(j_n, -\alpha(n)-1) = s_(j_n) + s(-\alpha(n)-1) - s(j_n - \alpha(n) - 1),$$

we have

$$\begin{aligned}
-\nu(b_{2,m}) &= \nu(m) + \sum_{k=1}^{n-1} [(n-k+1)j_k - s(j_k)] - s(j_n) - s(-\alpha(n)-1) + s(j_n - \alpha(n) - 1) \\
&= 1 + 2j_{n-1} - s(j_{n-1}) \\
&= 1 + 2p - s(p) \\
&= 1 + 2p - s(2p) \\
&= 1 + \lfloor 2p \rfloor - s(\lfloor 2p \rfloor) \\
&= \epsilon(m) + \left\lfloor \frac{2}{3}(m+1) \right\rfloor - s\left(\left\lfloor \frac{2}{3}(m+1) \right\rfloor \right).
\end{aligned}$$

Case (2): Write $m+1 = 3p+1$ for some positive integer p. Since $m \equiv 2 \mod 4$, it follows that $3p \equiv 2 \mod 4$ and so $p \equiv 2 \mod 4$. Since in this case $\mathbf{j}_{\max} = (j_{n-1}, j_n) = (p, 1)$, we have $\alpha(n) = -p/2$. It follows that

$$\begin{aligned}
-\nu(b_{2,m}) &= \nu(m) + \sum_{k=n-1}^{n-1} [(n-k+1)j_k - s(j_k)] - s(j_n) - s(-\alpha(n)-1) + s(j_n - \alpha(n) - 1) \\
&= 1 + 2j_{n-1} - s(j_{n-1}) - s(j_n) - s(p/2 - 1) + s(j_n + p/2 - 1) \\
&= 1 + 2p - s(p) - 1 - s((p-2)/2) + s(p/2) \\
&= 2p - s(p-2) \\
&= 1 + 2p - s(p) \\
&= 1 + 2p - s(2p) \\
&= 1 + \lfloor 2p + 2/3 \rfloor - s(\lfloor 2p + 2/3 \rfloor) \\
&= \epsilon(m) + \left\lfloor \frac{2}{3}(m+1) \right\rfloor - s\left(\left\lfloor \frac{2}{3}(m+1) \right\rfloor \right).
\end{aligned}$$

Case (3)-(a): Write $m+1 = 3p+2$ for some positive integer p. Since $m \equiv 2 \mod 8$, it follows that $3p+1 \equiv 2 \mod 8$ and so $p \equiv 3 \mod 8$. Thus, p has binary representation $b_r \ldots b_3 011$. Since $\mathbf{j}_{\max} = (j_{n-1}, j_n) = (p, 2)$, we have $\alpha(n) = (1-p)/2$. It follows that

$$
\begin{aligned}
-\nu(b_{2,m}) &= \nu(m) + \sum_{k=n-1}^{n-1} [(n-k+1)j_k - s(j_k)] - s(j_n) - s(-\alpha(n)-1) + s(j_n - \alpha(n) - 1) \\
&= 1 + 2j_{n-1} - s(j_{n-1}) - s(j_n) - s((p-1)/2 - 1) + s(j_n + (p-1)/2 - 1) \\
&= 1 + 2p - s(p) - s(2) - s((p-3)/2) + s((p+1)/2) \\
&= 2p - s(p) - s(p-3) + s(p+1) \\
&= 2p - s(p) + s(4) \\
&= 1 + (2p+1) - s(2p+1) \\
&= 1 + \lfloor 2p + 4/3 \rfloor - s(\lfloor 2p + 4/3 \rfloor) \\
&= \epsilon(m) + \left\lfloor \frac{2}{3}(m+1) \right\rfloor - s\left(\left\lfloor \frac{2}{3}(m+1) \right\rfloor \right).
\end{aligned}
$$

Case (3)-(b): Write $m+1 = 3p+2$ for some positive integer p. Since $m \equiv 46 \mod 48$, it follows that $3p+1 \equiv 46 \mod 48$ and so $p \equiv 15 \mod 48$. Thus, p has binary representation $b_r \dots b_5 1111$. Since $\mathbf{j}_{\max} = (j_{n-1}, j_n) = (1, p-2, 1)$, we have $\alpha(n) = (1-p)/2$. It follows that

$$
\begin{aligned}
-\nu(b_{2,m}) &= \nu(m) + \sum_{k=n-1}^{n-1} [(n-k+1)j_k - s(j_k)] - s(j_n) - s(-\alpha(n)-1) + s(j_n - \alpha(n) - 1) \\
&= 1 + 3j_{n-2} - s(j_{n-2}) + 2j_{n-1} - s(j_{n-1}) - s(j_n) - s((p-1)/2 - 1) + s(j_n + (p-1)/2 - 1) \\
&= 1 + 3 \cdot 1 - s(1) + 2(p-2) - s(p-2) - s(1) - s((p-3)/2) + s((p-1)/2) \\
&= 2p - 2 - s(p-2) - s(p-3) + s(p-1) \\
&= 2p - 2 - (s(p) - s(2)) - (s(p) - s(3)) + (s(p) - s(1)) \\
&= 2p - s(p) \\
&= 2p + 1 - s(2p+1) \\
&= 0 + \lfloor 2p + 4/3 \rfloor - s(\lfloor 2p + 4/3 \rfloor) \\
&= \epsilon(m) + \left\lfloor \frac{2}{3}(m+1) \right\rfloor - s\left(\left\lfloor \frac{2}{3}(m+1) \right\rfloor \right).
\end{aligned}
$$

Here, $\epsilon(m) = 0$ since $m = 2m_0$, where $m_0 \equiv -1 \mod 12$.

Case (3)-(c): Write $m+1 = 3p+2$ for some positive integer p. Since $m \equiv 22 \mod 48$, it follows that $3p+1 \equiv 22 \mod 48$ and so $p \equiv 7 \mod 48$. In this case $\mathbf{j}_{\max} = (j_{n-2}, j_{n-1}, j_n) = (1, p-2, 1)$ and thus the same argument applies as in Case (3)-(b). This completes the proof of Zagier's conjecture.

□

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