Row-Correlation Function: A New Approach to Complementary Code Matrices

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***Abstract*—New tools are developed for testing Complementary Code Matrix (CCM) existence, first for binary CCMs and then for more general p-phase classes, by formulating a CCM in terms of its row-correlation function. It is shown that a result of Lam and Leung on vanishing sums of roots of unity is useful in this approach for developing p-phase CCM existence tests in terms of the prime factorization of p. In addition, other existence tests and properties are found by using the row-correlation function to derive new dot product and congruence relations between rows of a CCM.**

Keywords: Complementary code matrix, row-correlation function, binary code, unimodular code, autocorrelation sidelobes, Hadamard matrix.

I. Introduction

Complementary Code Matrices (CCMs) provide a useful matrix formulation for the study of complementary code sets, which find uses in waveform design for enhanced detection in radar systems [1] and in communication systems [2][3]. While complementary code sets have yet to be widely used for radar waveform designs due to certain design challenges [1][14], some recent work has shown a number of concepts for overcome key practical concerns [13][14][15][16][17].

A set of K binary or polyphase codes of length N is complementary if corresponding sidelobes of the autocorrelations of the separate codes sum to zero; the associated CCM is the $N \times K$ matrix whose k^{th} column is the k^{th} code in the set for $k = 1, \ldots, K$. Classes of CCMs are often specified by the set from which matrix elements may be drawn. Many classes of interest are special cases of unimodular CCMs whose elements are drawn from the unit circle. A special case of the unimodular CCMs are the p-phase CCMs, whose elements are drawn from the set of complex values $\{e^{2\pi i/p}, e^{4\pi i/p}, \ldots, 1\}$, that is, the p^{th}-roots of unity for a given positive integer p. Corresponding to $p = 2$ are the binary CCMs, whose elements are restricted to $\{-1, 1\}$. Hadamard matrices are a subclass of CCMs [4], arising in a number of important applications [5]; each class of CCMs (binary, unimodular, p-phase) includes the corresponding subclass of the Hadamard matrices.

A growing body of literature is focused on properties of CCMs [4][6][7][8][9]. Among the many questions to be asked about CCMs is whether, for a given pair (N, K), any $N \times K$ CCMs of the dimension exist. The answer to this question determines the existence or non-existence of complementary sets of K codes of length N.

This paper develops several new tools for testing CCM existence, first for binary CCMs and then for more general p-phase classes, by introducing the notion of a row-correlation function, an analogue of the auto-correlation function, but applied to the rows of CCM as opposed to its columns. It is shown that results by Lam and Leung [11] are particularly useful in this approach for developing existence tests in terms of the prime factorization of p. In addition, other CCM existence tests and properties are found by using the row-correlation function to obtain new dot product and congruence formulas, in line with recent results by one of the authors and J. Russo [10] where this row-based approach was employed to develop efficient exhaustive searches for low-order CCMs [10].

II. Complementary Code Sets

Define a p-phase code z of length N as a sequence of elements z_i where $z_i^p = 1$ for $i = 1, \ldots, N$. It will be useful to think of z as a column vector of length N:

$$z = [z_1, \ldots, z_N]^T.$$

The autocorrelation of a code z is defined as the sequence of length $2N - 1$

$$\mathrm{ACF}_z = z * \mathrm{rev}(\overline{z})$$

where $*$ represents aperiodic convolution, $\overline{z}$ means complex conjugation of vector z, and $\mathrm{rev}(z)$ indicates reversal. The elements $\mathrm{ACF}_z(k)$ may be written explicitly as sums of pairwise products of the elements of z:

$$\mathrm{ACF}_z(k) = \sum_{i=1}^{N-k} z_i \overline{z}_{i+k}$$

for $k = 0, ..., N-1$. For $k = -(N-1), ..., -1$, we define $\mathrm{ACF}_z(k) = \overline{\mathrm{ACF}_z(-k)}$.

Definition. A set of K length N codes $\{z_1, \ldots, z_K\}$ is a **complementary code set** if

$$\sum_{j=1}^{K} \mathrm{ACF}_{z_j}(k) = 0$$

for $k = -(N-1), ..., -1$ and $1, ..., N-1$.

Note that $\mathrm{ACF}_{z_j}(0) = N$ for $j = 1, \ldots, K$, by the definition of the autocorrelation. Hence

$$\sum_{j=1}^{K} \mathrm{ACF}_{z_j}(0) = KN \tag{1}$$

for any set of K length-N codes.

Definition. An $N \times K$ matrix Q is a **complementary code matrix**, or CCM, if its columns form a complementary code set.

III. Diagonal Regularity and Row-Correlation Functions

In this section we present two other equivalent formulations of a complementary code matrix: one in terms of its row Gramian and the other in terms of its rows. The latter allows us to express definition (1) in terms of a function that we shall refer to as the row-correlation function.

Definition. Given an $N \times K$ matrix Q, the matrix product QQ^* is called its **row Gramian**, where Q^* is the conjugate transpose of Q.

Definition. An $N \times N$ matrix is called **diagonally regular** if the elements of each of its diagonals, with the exception of the main diagonal, sum to 0.

In [4], one of the authors and W. Haloupek characterized CCMs in terms of its rows. We use this idea to introduce the row-correlation function, which will be useful for deriving new properties of CCMs as we demonstrate later in the paper.

Definition. Let Q be an $N \times K$ matrix consisting of rows $\{r_1, r_2, \ldots, r_N\}$. The **row-correlation function** (RCF) of Q

is defined to be

$$\mathrm{RCF}_Q(k) = \sum_{j=1}^{N-k} r_j \cdot \overline{r}_{j+k} \tag{2}$$

for $k = 0, 1, ..., N-1$. For $k = -(N-1), ..., -1$, we define $\mathrm{RCF}_Q(k) = \overline{\mathrm{RCF}_Q(-k)}$. In (2), the notation $r_i \cdot \overline{r}_j$ represents the dot, or scalar, product, of rows r_i and r_j.

Definition. A **p-phase** matrix Q is one whose columns are p-phase codes. A p-phase matrix with $p = 2$ is called a binary matrix.

The next theorem, due to Coxson and Haloupek [4], completely characterizes CCMs in terms of the row-correlation function.

Theorem 3.1 (Coxson-Haloupek [4]). An $N \times K$ p-phase matrix Q consisting of rows denoted by $\{r_1, r_2, \ldots, r_N\}$ and columns denoted by $\{z_1, z_2, ..., z_K\}$ is a complementary code matrix (CCM) if and only if its row Gramian is diagonally regular, that is,

$$\mathrm{RCF}_Q(k) = \sum_{j=1}^{K} \mathrm{ACF}_{z_j}(k) = NK\delta_k \tag{3}$$

for $k = -(N-1), ..., 0, \ldots, N-1$ where δ_k is the Kronecker delta function.

IV. Several Results for Unimodular CCMs

In this section we demonstrate how existence tests for unimodular CCMs can be obtained from the row-correlation function. These tests generalize those established by Coxson, Haloupek, and Russo in [18] where the notion of imbalance was introduced to measure the difference in the number of 1's and -1's along a column of a CCM. We reformulate the imbalance of all columns as the vector sum of all rows, which we denote by R.

Lemma 4.1. Let Q be an $N \times K$ CCM consisting of rows $\{r_1, r_2, \ldots, r_N\}$. Define

$$R = r_1 + r_2 + \ldots + r_N.$$

Then

$$|R|^2 = NK.$$

Proof. We have

$$\begin{aligned} |R|^2 &= (r_1 + \ldots + r_N) \cdot (\overline{r_1 + \ldots + r_N}) \\ &= \sum_{j=1}^{N} r_j \cdot \overline{r}_j + \sum_{k=1}^{N-1} \sum_{j=1}^{N-k} r_j \cdot \overline{r}_{j+k} + \\ &\quad \sum_{k=1}^{N-1} \sum_{j=1}^{N-k} r_{j+k} \cdot \overline{r}_j \\ &= \sum_{j=1}^{N} K + \sum_{k=1}^{N-1} \mathrm{RCF}_Q(k) + \sum_{k=1}^{N-1} \mathrm{RCF}_Q(-k) \\ &= NK \end{aligned}$$

because of (3). ■

Note if the components of R are given by $(\lambda_1, ..., \lambda_K)$, then Lemma 4.1 shows that NK can be written as a sum of K non-negative squares:

$$\lambda_1^2 + ... + \lambda_K^2 = NK$$

For binary CCMs, this result was first proven by Feng, Shiue, and Xiang in [18] (Lemma 2.2) by a different method. It is also equivalent to Corollary 3.2 in [8] where λ_k represents the imbalance of the k-th column of Q. The next theorem generalizes Theorem 5.1 in [18] where an existence test was established via the imbalance of the even and odd numbered rows.

Theorem 4.2. Let Q be an $N \times K$ CCM consisting of rows $\{r_1, r_2, \ldots, r_N\}$. Define $R_o = r_1 + r_3 + \ldots + r_{2\lfloor (N-1)/2 \rfloor + 1}$ and $R_e = r_2 + r_4 + \ldots + r_{2\lfloor N/2 \rfloor}$. Then

$$|R_o|^2 + |R_e|^2 = NK.$$

Moreover, if $\mathrm{Im}(R_o \cdot \overline{R}_e) = 0$, which holds if Q is binary, then

$$R_o \cdot \overline{R}_e = 0.$$

Proof. We have

$$\begin{aligned}
|R_o|^2 &= \left(\sum_{j=1}^{\lfloor (N+1)/2 \rfloor} r_{2j-1}\right)\left(\sum_{j=1}^{\lfloor (N+1)/2 \rfloor} \overline{r}_{2j-1}\right) \\
&= \sum_{j=1}^{\lfloor (N+1)/2 \rfloor} r_{2j-1} \cdot \overline{r}_{2j-1} + \\
&\quad \sum_{k=1}^{\lfloor (N-1)/2 \rfloor} \sum_{j=1}^{\lfloor (N+1)/2 \rfloor - k} r_{2j-1} \cdot \overline{r}_{2j-1+2k} + \\
&\quad \sum_{k=1}^{\lfloor (N-1)/2 \rfloor} \sum_{j=1}^{\lfloor (N+1)/2 \rfloor - k} r_{2j-1+2k} \cdot \overline{r}_{2j-1}
\end{aligned}$$

and

$$\begin{aligned}
|R_e|^2 &= \left(\sum_{j=1}^{\lfloor N/2 \rfloor} r_{2j}\right)\left(\sum_{j=1}^{\lfloor N/2 \rfloor} \overline{r}_{2j}\right) \\
&= \sum_{j=1}^{\lfloor N/2 \rfloor} r_{2j} \cdot \overline{r}_{2j} + \\
&\quad \sum_{k=1}^{\lfloor (N-2)/2 \rfloor} \sum_{j=1}^{\lfloor N/2 \rfloor - k} r_{2j} \cdot \overline{r}_{2j+2k} + \\
&\quad \sum_{k=1}^{\lfloor (N-2)/2 \rfloor} \sum_{j=1}^{\lfloor N/2 \rfloor - k} r_{2j+2k} \cdot \overline{r}_{2j}
\end{aligned}$$

It follows that

$$\begin{aligned}
|R_o|^2 + |R_e|^2 &= \sum_{j=1}^{N} r_j \cdot \overline{r}_j + \sum_{k=1}^{\lfloor (N-1)/2 \rfloor} \sum_{j=1}^{N-2k} r_j \cdot \overline{r}_{j+2k} + \\
&\quad \sum_{k=1}^{\lfloor (N-1)/2 \rfloor} \sum_{j=1}^{N-2k} r_{j+2k} \cdot \overline{r}_j \\
&= \sum_{j=1}^{N} K + \sum_{k=1}^{\lfloor (N-1)/2 \rfloor} \mathrm{RCF}_Q(2k) + \\
&\quad \sum_{k=1}^{\lfloor (N-1)/2 \rfloor} \mathrm{RCF}_Q(-2k) \\
&= NK
\end{aligned}$$

Define $R = R_o + R_e$. Then from Lemma 4.1, we have

$$\begin{aligned}
NK &= |R|^2 \\
&= (R_o + R_e) \cdot (\overline{R_o + R_e}) \\
&= |R_o|^2 + |R_e|^2 + R_o \cdot \overline{R}_e + R_e \cdot \overline{R}_o. \\
&= NK + 2\mathrm{Re}(R_o \cdot \overline{R}_e)
\end{aligned}$$

It follows that $\mathrm{Re}(R_o \cdot \overline{R}_e) = 0$. Since $\mathrm{Im}(R_o \cdot \overline{R}_e) = 0$ by assumption, we conclude that $R_o \cdot \overline{R}_e = 0$. ■

Observe that Lemma 4.1 and Theorem 4.2 imply the following Pythagorean relationship for a CCM:

$$|R_o|^2 + |R_e|^2 = |R|^2$$

We now turn to developing divisibility tests for the existence of CCMs.

Lemma 4.3. Let p be a prime integer and Q a p-phase $N \times K$ CCM. Then p divides K.

Proof. We transform $Q = (q_{j,k})$ into dephased form where its entries in the first row consists of all 1s. Since $\mathrm{RCF}_Q(N-1) = r_1 \cdot \overline{r}_N = 0$ for a CCM, it follows that the entries in the last row must sum to zero, that is,

$$\sum_{k=1}^{K} q_{N,k} = 0.$$

Recall that the values $q_{N,1}, \ldots, q_{N,K}$ are roots of $z^p = 1$. But there is only one way for these K values to sum to zero when p is prime, namely as the sum of all p distinct roots or a positive integer multiple of it (recall that $\sum_{n=0}^{p-1} e^{2\pi i n/p} = 0$) [11]. Hence, K is divisible by p. ■

Lemma 4.3 can be extended to all positive integers p if we can determine when a sum of roots of unity vanishes. Towards this end, let $A = \{\alpha_1, \ldots, \alpha_n\}$ be an n-element set consisting of p-th roots of unity. We say that A represents a *vanishing sum of roots of unity* of weight n if

$$\alpha_1 + \ldots + \alpha_n = 0.$$

The following theorem of Lam and Leung [11] completely characterizes vanishing sums of roots of unity.

Theorem 4.4 (Lam-Leung). Let p be a positive integer with prime factorization $p = p_1^{n_1} p_2^{n_2} \dots p_s^{n_s}$ where $p_1, \dots, p_s$ are primes. Then there exists a vanishing sum of p-th roots of unity of weight n if and only if n is a non-negative integer linear combination of $p_1 + \dots, p_s$, that is,

$$n = c_1 p_1 + c_2 p_2 + \dots + c_s p_s$$

for some set of nonnegative integers $c_1, c_2, \dots, c_s$.

Moreover, Lam and Leung note that if $6|p$, then the weight n takes on every positive integer greater than 1. As a corollary, we obtain a necessary condition for the existence of p-phase CCMs.

Corollary 4.5. Let Q be a p-phase $N \times K$ CCM where p has prime factorization $p = p_1^{n_1} p_2^{n_2} \dots p_s^{n_s}$. Then K is a non-trivial non-negative integer linear combination of $p_1, p_2, \dots, p_s$, that is,

$$K = c_1 p_1 + c_2 p_2 + \dots + c_s p_s$$

for some set of nonnegative integers $c_1, c_2, \dots, c_s$.

Note that if $6|p$, then Corollary 4.5 allows for the possibility of a p-phase CCM to exist having an arbitrary number of columns K greater than 1.

V. Quad-Phase Golay Pairs

In this section we present results on quad-phase Golay pairs and CCMs by investigating congruences satisfied by their row sums.

Theorem 5.1. Let Q be a quad-phase $N \times K$ CCM. Then K must be even.

Proof. We assume $Q = \{r_1, \dots, r_N\}$ is de-phased so that the first row r_1 consists of all 1s. Again, it follows that

$$\mathrm{RCF}_Q(N-1) = r_1 \cdot \overline{r}_N = 0 \Rightarrow q_{N,1} + \dots + q_{N,K} = 0. \quad (4)$$

The second equation in (4) expands as

$$(x_{N,1} + \dots + x_{N,K}) + i(y_{N,1} + \dots + y_{N,K}) = 0,$$

yielding

$$(x_{N,1} + \dots + x_{N,K}) = (y_{N,1} + \dots + y_{N,K}) = 0 \bmod 2. \quad (5)$$

Suppose on the contrary that K is odd. Assume $x_{N,k} = 1$ for an odd number of k values. Then

$$x_{N,1} + \dots + x_{N,K} = 1 \bmod 2,$$

which contradicts (5). On the other hand, if $x_{N,k} = 1$ for an even number of k values, then $x_{N,k} = 0$ for an even number of such k values, since K is odd. It follows that $y_{N,k} = 1$ for an odd number of k values since all entries of Q satisfy $|x_{j,k}| + |y_{j,k}| = 1$. But then

$$y_{N,1} + \dots + y_{N,K} = 1 \bmod 2$$

again contradicting (3). This proves that K must be even. ■

Lemma 5.2 (Golay). Let $Q = (A, B)$ be an Golay pair, that is, a binary $N \times 2$ CCM, whose two columns have entries $A = \{a_1, a_2, \dots, a_N\}$ and $B = \{b_1, b_2, \dots, b_N\}$. Then

$$a_n + b_n + a_{N-n+1} + b_{N-n+1} = 2 \bmod 4.$$

The proof of Lemma 5.2 relies on the following fact.

Lemma 5.3. Let a and b be two values such that $|a| = |b| = 1$. Then

$$ab = a + b - 1 \bmod 4.$$

We now generalize Lemma 5.2 to binary $N \times K$ CCMs.

Theorem 5.4. Let $Q = (Q_1, Q_2, \dots, Q_K)$ be a binary $N \times K$ CCM whose k^{th} column Q_k has entries $\{q_{1,k}, \dots, q_{N,k}\}$ for $k = 1, \dots, K$. Then

$$q_{N-n+1,1} + \dots + q_{N-n+1,K} + q_{n,1} + \dots + q_{n,K} = K \bmod 4$$

for $n = 1, \dots, N$.

Proof. Since Q is a CCM, it follows that $\mathrm{RCF}_Q(n) = 0$, or equivalently,

$$\sum_{j=1}^{N-n} (q_{j,1}q_{j+n,1} + \ldots + q_{j,K}q_{j+n,K}) = 0. \tag{6}$$

As all the entries of Q satisfy $|q_{j,k}| = 1$, it follows from Lemma 5.3 that

$$\sum_{j=1}^{N-n} [(q_{j,1} + q_{j+n,1} - 1) + \ldots + (q_{j,K} + q_{j+n,K} - 1)] = 0 \bmod 4$$

which implies

$$\sum_{j=1}^{N-n} [(q_{j,1} + q_{j+n,1}) + \ldots + (q_{j,K} + q_{j+n,K})] - \sum_{j=1}^{N-n} K = 0 \bmod 4$$

which implies

$$\sum_{j=1}^{N-n} (q_{j,1} + \ldots + q_{j,K}) + \sum_{j=n+1}^{N} (q_{j,1} + \ldots + q_{j,K}) = (N-n)K \bmod 4. \tag{7}$$

Subtracting case n of (5) from case $n-1$ yields

$$q_{N-n+1,1} + \ldots + q_{N-n+1,K} + q_{n,1} + \ldots + q_{n,K} = K \bmod 4$$

for $n = 2, \ldots, N-1$. But this result holds for $n = 1$ and $n = N$ as well since

$$q_{1,1}q_{N,1} + \ldots + q_{1,K}q_{N,K} = 0$$

which follows by setting $n = N-1$ in (5). ■

Corollary 5.5 Let $Q = (Q_1, Q_2)$ be a binary $N \times 2$ CCM consisting of rows $r_1, r_2, ..., r_N$ and whose k^{th} column Q_k has entries $\{q_{1,k}, \ldots, q_{N,k}\}$ for $k = 1, 2$. Then $r_n \cdot r_{N+1-n} = 0$ for $n = 1, \ldots, N$.

Proof. Given any row index $1 \leq n \leq N$,

$$q_{N-n+1,1} + \ldots + q_{N-n+1,K} + q_{n,1} + \ldots + q_{n,K} = 2 \bmod 4,$$

by Theorem 5.4. Since the sum involves four ± 1 matrix elements in rows r_n and r_{N-n+1}, it must be either $+2$ or -2. But then the four summed elements are either three 1s and one -1s or three -1s and one 1. For either case, $r_n \cdot r_{N+1-n} = 0$. ■

We now generalize Theorem 5.4, extending the result for binary CCMs to quad-phase CCMs. First, we prove the following fact.

Lemma 5.6. Let $q_1 = x_1 + iy_1$ and $q_2 = x_2 + iy_2$ be such that $|x_1| + |y_1| = 1$ and $|x_2| + |y_2| = 1$. Then

$$\mathrm{Re}(q_1\overline{q}_2) = x_1x_2 + y_1y_2 = x_1 + y_2 = (y_1 + x_2) \bmod 2$$

and

$$\mathrm{Im}(q_1\overline{q}_2) = -x_1y_2 + y_1y_2 = x_1 + y_2 - 1 = (y_1 + x_2 - 1) \bmod 2.$$

We now present our result for quad-phase CCMs.

Theorem 5.7. Let $Q = (Q_1, Q_2, \ldots, Q_K)$ be a quad-phase $N \times K$ CCM whose k^{th} column Q_k has entries $\{q_{1,k}, \ldots, q_{N,k}\}$ for $k = 1, \ldots, K$. Then

$$x_{N-n+1,1} + \ldots + x_{N-n+1,K} + y_{n,1} + \ldots y_{n,K} = 0 \bmod 2$$

for $n = 1, \ldots, N$.

Proof. Write $q_{j,k} = x_{j,k} + iy_{j,k}$. Then

$$\mathrm{RCF}_Q(n) = \sum_{j=1}^{N-n} (q_{j,1}\overline{q}_{j+n,1} + \ldots + q_{j,K}\overline{q}_{j+k,K}) = 0$$

implies

$$\mathrm{Re}\left(\sum_{j=1}^{N-n} (q_{j,1}\overline{q}_{j+n,1} + \ldots + q_{j,K}\overline{q}_{j+k,K})\right) = 0$$

which implies

$$\sum_{j=1}^{N-n} [(x_{j,1}x_{j+n,1} + y_{j,1}y_{j+n,1}) + \ldots + (x_{j,K}x_{j+n,K} + y_{j,K}y_{j+n,K})] = 0 \tag{8}$$

which implies

$$\sum_{j=1}^{N-n} [(x_{j,1} + y_{j+n,1}) + \ldots + (x_{j,K} + y_{j+n,K})] = 0 \bmod 2$$

which implies

$$\sum_{j=1}^{N-n} (x_{j,1} + \ldots + x_{j,K}) + \sum_{j=n+1}^{N} (y_{j,1} + \ldots + y_{j,K}) = 0 \bmod 2.$$

Subtracting case n of (7) from case $n-1$ yields

$$x_{N-n+1,1} + \ldots + x_{N-n+1,K} + y_{n,K} + \ldots + y_{n,K} = 0 \bmod 2$$

for $n = 2, \ldots, N-1$. But this result holds for $n = 1$ and $n = N$ as well, since

$$(x_{1,1}x_{N,1} + y_{1,1}y_{N,1}) + \ldots + (x_{1,K}x_{N,K} + y_{1,K}y_{N,K}) = 0$$

which follows by setting $n = N-1$ in (7). ■

Corollary 5.8 Let Q be a quad-phase $N \times 2$ CCM (complex Golay pair) consisting of rows $r_1, \ldots, r_N$. Then $r_n \cdot \overline{r}_{N-n+1} = 0, \pm 2$, or $\pm 2i$ for $n = 1, \ldots, N$.

Proof. Let $r_n = (q_{n,1}, q_{n,2})$ denote the entries in row n and consider the four entries $q_{n,1}, q_{n,2}, q_{N-n+1,1}, q_{N-n+1,2}$ where each is a fourth root of unity. We claim that the number of entries that are real, i.e. ± 1, and the number of entries that are imaginary, i.e. $\pm i$, must both have even parity. To prove this, assume on the contrary so that the four entries are say $1, 1, 1, i$ (three real and one imaginary). Then $r_n \cdot \overline{r}_{N-n+1} = 1 - i$, which contradicts Lemma 5.6. All other odd cases can be eliminated in the same manner. It follows that $r_n \cdot \overline{r}_{N-n+1} = 0$, ± 2, or $\pm 2i$, which can be verified case by case, e.g. when the four entries are $1, 1, i, i$ (two real and two imaginary). ■

VI. Conclusions

A new approach to characterizing complementary code matrices (CCMs) in terms of the row-correlation function was presented. Existence tests were derived for p-phase CCMs in terms of the factorization of p via a theorem of Lam and Leung on vanishing sums of roots of unity. Other tests were derived by considering dot product and congruence relations obtained from the row-correlation function.

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