BERNOULLI POLYNOMIALS AND PASCAL'S SQUARE

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In this article we discuss an interesting but not so well known matrix determinant formula for Bernoulli polynomials by considering a square version of Pascal's triangle and present an extension of this formula to a class of generalized Bernoulli polynomials.

1. Introduction

Pascal's triangle is one of the most recognized number patterns in mathematics. It is commonly arranged as the triangular array of numbers

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ & & & & \ldots & & & & \end{array} \tag{1.1}$$

However, this arrangement is not the original form of Pascal's triangle. Pascal himself presented the following right-angle form of it in 1654 in his work *Traité du triangle arithmétique*, where he called it the *arithmetical triangle* ([2]):

$$\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & \\ 1 & 3 & 6 & & \\ 1 & 4 & & & \\ 1 & & & & \\ \ldots & & & & \end{array} \tag{1.2}$$

Even earlier in 1544 Stifel from Germany had constructed the following shifted but incomplete version of the arithmetical triangle (referred to as the *figurate* or *binomial triangle* by Pascal's predecessors):

$$\begin{array}{lll} 1 & & \\ 2 & & \\ 3 & 3 & \\ 4 & 6 & \\ 5 & 10 & 10 \\ \dots & & \end{array} \tag{1.3}$$

The full version of (1.3),

$$\begin{array}{lllll} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \\ \dots & & & & \end{array} \tag{1.4}$$

was later used by Jacob Bernoulli in *Ars conjectandi*, published eight years after his death in 1713, to establish formulas for sums of powers involving certain coefficients B_n that today bear his name, the *Bernoulli numbers*:

$$\sum_{n=1}^{N} n^p = \sum_{n=0}^{p} (-1)^{\delta_{np}} \frac{p!}{n!(p+1-n)!} B_n N^{p+1-n} \tag{1.5}$$

These numbers, of which the first six are

$$\begin{array}{ll} B_0 = 1, & B_1 = -1/2, \\ B_2 = 1/6, & B_3 = 0, \\ B_4 = -1/30, & B_5 = 0, \end{array} \tag{1.6}$$

can be defined by the recursive formula

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0\,. \tag{1.7}$$

Euler later showed that Bernoulli numbers can also be defined by the series formula

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \tag{1.8}$$

an approach that effectively created a whole theory based on generating functions.

In this article we consider a square version of Pascal's triangle and demonstrate how it arises in explicit formulas for Bernoulli numbers, Bernoulli polynomials, and their generalizations. To this end, we transform (1.4) into an infinite matrix P that we refer to as *Pascal's square*:

$$
P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 1 & 0 & 0 & \ldots \\ 1 & 3 & 3 & 1 & 0 & \ldots \\ 1 & 4 & 6 & 4 & 1 & \ldots \\ \ldots & & & & & \end{pmatrix} \tag{1.9}
$$

It follows that the entries of $P = (p_{mn})$ can be defined by the binomial formula

$$
p_{mn} = \binom{m-1}{n-1}, \tag{1.10}
$$

where for nonnegative integers m and n we have

$$
\binom{m}{n} = \frac{m \cdot (m-1)(m-2)\ldots(m-n+1)}{n!}.
$$

The matrix P can in fact be found implicitly in Turnbull's classic textbook on determinants [5] where it appears (except for the first two rows and main diagonal) in an explicit determinant formula for Bernoulli numbers[1], obtained by equating series coefficients in (1.8) and solving the corresponding linear system of equations:

$$
B_n = \frac{(-1)^{n-1}}{(n+1)!} \begin{vmatrix} 1 & 2 & 0 & 0 & \ldots & 0 \\ 1 & 3 & 3 & 0 & \ldots & 0 \\ 1 & 4 & 6 & 4 & \ldots & 0 \\ 1 & 5 & 10 & 10 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \binom{n+1}{0} & \binom{n+1}{1} & \binom{n+1}{2} & \binom{n+1}{3} & \ldots & \binom{n+1}{n-1} \end{vmatrix}_n \tag{1.11}
$$

where the dimension of the matrix is $n \times n$. More generally, the Bernoulli polynomials $B_n(x)$, defined by

$$
\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \tag{1.12}
$$

and whose value at $x = 0$ equals B_n, can be similarly expressed by the formula (see [1]):

[1] The origin of this formula for the Bernoulli numbers is unclear; the authors have not been able to trace it back farther than Turnbull.

$$B_n(x) = \frac{(-1)^{(n)}}{(n-1)!}\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ x & 1/2 & 1 & 0 & 0 & 0 & \ldots & 0 \\ x^2 & 1/3 & 1 & 2 & 0 & 0 & \ldots & 0 \\ x^3 & 1/4 & 1 & 3 & 3 & 0 & \ldots & 0 \\ x^4 & 1/5 & 1 & 4 & 6 & 4 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ x^n & 1/(n+1) & \binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \ldots & \binom{n}{n-2} \end{vmatrix}_{n+1} \tag{1.13}$$

Observe that the matrix in formula (1.11) appears as a sub-matrix in formula (1.13).

The explicit formulas (1.11) and (1.13) do not seem to be very well known. We have been unable to find references that cite these formulas besides [5] and [1]. They certainly deserve more attention since they provide a beautiful connection with Pascal's triangle and illustrate a useful application of calculus and linear algebra, which is the first goal of this article. The second goal is to demonstrate how (1.13) can in fact be extended to a class of generalized Bernoulli polynomials first studied by F. Howard [4], who considered the following natural generalization of (1.12):

$$\frac{\frac{t^N}{N!}e^{xt}}{e^t - T_{N-1}(t)} = \sum_{n=0}^{\infty} B_n(N,x)\frac{t^n}{n!}. \tag{1.14}$$

Here, N is any positive integer and

$$T_{N-1}(x) = \sum_{n=0}^{N-1}\frac{t^n}{n!} \tag{1.15}$$

is the Maclaurin polynomial of e^x having degree $N-1$. Howard was able to show that the polynomials $B_n(N,x)$ defined by (1.14) and referred to as *hypergeometric Bernoulli polynomials* in [3], share many of the properties possessed by the classical Bernoulli numbers. Our contribution is a determinant formula for $B_n(N,x)$ that is analogous to (1.13):

$$B_n(N,x) = \frac{(-1)^{(n)}(N!)^n 1!2!3!\ldots(n-N-1)!}{1!2!3!\ldots(n-1)!1!2!3!\ldots N!}\left|b_{ij}\right| \tag{1.16}$$

where the matrix $\left(b_{ij}\right)$ has entries

$$b_{ij} = \begin{cases} x^{i-1} & j = 1 \\ \binom{i-j+N+1}{i-1}^{-1} & 2 \le j \le N+2 \\ \binom{i-1}{j-N-2} & j \ge N+2 \end{cases} \tag{1.17}$$

Observe that a portion of Pascal's square appears in the matrix $\left(b_{ij}\right)$. This becomes clear if we set $m = i, n = j-N-1$. Then $p_{mn} = b_{ij}$ for $i+1 \ge j, j \ge N+2$, where $P = \left(p_{mn}\right)$ is

Pascal's square defined by (1.10). Thus it can be argued that Pascal's square is a natural extension of Pascal's triangle.

Historical comment: It is possible that Bernoulli would have discovered formula (1.11) had he known about Leibniz' theory in solving linear systems of equations via matrices and determinants, which essentially evolved into our modern theory of linear algebra. Unfortunately, Leibniz never published any of his works on this topic ([6]).

2. Bernoulli Polynomials and Matrix Determinants

Let *f* and *g* be functions described by power series

$$f(t)=\sum_{n=0}^{\infty} c_n t^n,\ g(t)=\sum_{n=0}^{\infty} a_n t^n . \tag{2.1}$$

Consider their quotient:

$$\frac{f(t)}{g(t)}=\sum_{n=0}^{\infty} A_n t^n . \tag{2.2}$$

A formula for A_n can be obtained by equating coefficients in (2.2), which yields the system of equations

$$\begin{aligned} c_0 &= a_0 A_0, \\ c_1 &= a_0 A_1 + a_1 A_0, \\ c_2 &= a_0 A_2 + a_1 A_1 + a_2 A_0, \\ &\ldots \\ c_n &= a_0 A_n + a_1 A_{n-1} + \ldots + a_n A_0. \end{aligned} \tag{2.3}$$

From which, solving for A_n through a variation of Cramer's rule, we obtain (see [5])

$$A_n=(-1)^n \frac{1}{a_0^n}\begin{vmatrix} c_0 & a_0 & 0 & 0 & \ldots & 0 \\ c_1 & a_1 & a_0 & 0 & \ldots & 0 \\ c_2 & a_2 & a_1 & a_0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ c_{n-1} & a_{n-1} & a_{n-2} & a_{n-3} & \ldots & a_0 \\ c_n & a_n & a_{n-1} & a_{n-2} & \ldots & a_1 \end{vmatrix}_{n+1} , \tag{2.4}$$

where the index $n+1$ refers to the dimension of the matrix.

To apply this to the Bernoulli polynomials $B_n(x)$, we employ (1.12) and view it as the division of two series as follows:

$$\frac{e^{xt}}{\frac{e^t-1}{t}}=\frac{\sum_{n=0}^{\infty}\frac{x^n t^n}{n!}}{\sum_{n=0}^{\infty}\frac{t^n}{(n+1)!}}=\sum_{n=0}^{\infty} A_n t^n , \tag{2.5}$$

Since $B_n(x)=n!A_n$, we have from (2.4) the following determinant formula upon setting $c_n = x^n/n!$ and $a_n = 1/(n+1)!$:

$$B_n(x) = n!(-1)^{(n)} \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ \frac{x}{1!} & \frac{1}{2!} & 1 & 0 & 0 & 0 & \ldots & 0 \\ \frac{x^2}{2!} & \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & 0 & \ldots & 0 \\ \frac{x^3}{3!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{x^n}{n!} & \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \frac{1}{(n-3)!} & \ldots & 1 \end{vmatrix}_{n+1} \tag{2.6}$$

$$= n!(-1)^{(n)} \left| b_{ij}^1 \right|$$

where

$$b_{ij}^1 = \begin{cases} 0 & \text{if } i+1 < j \\ x^{i-1}/(i-1)! & j=1 \\ 1/(i-j+2)! & i+1 \geq j,\ j \neq 1 \end{cases} \tag{2.7}$$

Next, we modify the entries of the matrix (b_{ij}^1) in (2.6) by performing the following row and column operations:

1. Beginning with the first row, we factor $1/(i-1)!$ from row i to obtain

$$B_n(x) = \frac{n!(-1)^{(n)}}{1!2!3!\ldots(n-1)!(n)!} \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ x & \frac{1}{2!} & 1 & 0 & 0 & 0 & 0 \\ x^2 & \frac{2!}{3!} & 1 & 2! & 0 & 0 & 0 \\ x^3 & \frac{3!}{4!} & 1 & \frac{3!}{2!} & 3! & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ x^n & \frac{n!}{(n+1)!} & 1 & \frac{n!}{(n-1)!} & \frac{n!}{(n-2)!} & \ldots & n! \end{vmatrix}_{n+1} \tag{2.8}$$

$$= \frac{n!(-1)^{(n)}}{1!2!3!\ldots(n-1)!(n)!} \left| b_{ij}^2 \right|$$

where

$$b_{ij}^2 = \begin{cases} 0 & i+1 < j \\ x^{i-1} & j=1 \\ (i-1)/i! & j=2 \\ (i-1)!/(i-j+2)! & i+1 \geq j,\ j \neq 1,2 \end{cases} \tag{2.9}$$

2. Now, starting with the third column, we factor $(j-3)!$ from column j in (b_{ij}^2). This yields

$$B_n(x)=\frac{(-1)^{(n)}1!2!3!\ldots(n-2)!}{1!2!3!\ldots(n-1)!}\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ x & \frac{1}{2!} & 1 & 0 & 0 & 0 & 0 \\ x^2 & \frac{2!}{3!} & 1 & 2! & 0 & 0 & 0 \\ x^3 & \frac{3!}{4!} & 1 & \frac{3!}{2!} & \frac{3!}{2!} & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ x^n & \frac{n!}{(n+1)!} & 1 & \frac{n!}{(n-1)!} & \frac{n!}{(n-2)!2!} & \ldots & \frac{n!}{2!(n-2)!} \end{vmatrix}_{n+1} \tag{2.10}$$

$$=\frac{(-1)^{(n)}1!2!3!\ldots(n-2)!}{1!2!3!\ldots(n-1)!}\left|b_{ij}^3\right|$$

where

$$b_{ij}^3=\begin{cases} x^{i-1} & j=1 \\ 1/i & j=2 \\ \binom{i-1}{j-3} & j>2 \end{cases} \tag{2.11}$$

Lastly, the matrix (b_{ij}^3) in (2.10) can be simplified and written in terms of binomials from which most of Pascal's square appears (in red).

Theorem 1: (Costabile, Dell'Accio, Gualtieri)

$$B_n(x)=\frac{(-1)^{(n)}}{(n-1)!}\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ x & 1/2 & 1 & 0 & 0 & 0 & \ldots & 0 \\ x^2 & 1/3 & 1 & 2 & 0 & 0 & \ldots & 0 \\ x^3 & 1/4 & 1 & 3 & 3 & 0 & \ldots & 0 \\ x^4 & 1/5 & 1 & 4 & 6 & 4 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ x^n & 1/(n+1) & \binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \ldots & \binom{n}{n-2} \end{vmatrix}_{n+1} \tag{2.12}$$

$$=\frac{(-1)^{(n)}}{(n-1)!}\left|b_{ij}\right|$$

where

$$b_{ij}=\begin{cases} x^{i-1} & j=1 \\ 1/i & j=2 \\ \binom{i-1}{j-3} & j>2 \end{cases} \tag{2.13}$$

Note: In order to obtain formula (1.11) for Bernoulli numbers, it suffices to set $x=0$ in (2.12) and expand the determinant along the first column to obtain:

$$B_n = B_n(0) = \frac{(-1)^{n-1}}{(n+1)!}\begin{vmatrix} 1/2 & 1 & 0 & 0 & \ldots & 0 \\ 1/3 & 1 & 2 & 0 & \ldots & 0 \\ 1/4 & 1 & 3 & 3 & \ldots & 0 \\ 1/5 & 1 & 4 & 6 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{1}{n+1} & \binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \ldots & \binom{n}{n-2} \end{vmatrix}_n \tag{2.14}$$

Then perform the following row and column operations on the matrix appearing in (2.14): multiply row *i* by $i+1$ and divide column *j*, beginning with the third column, by $j-1$. From this we obtain (1.11) with Pascal's square embedded in it but with the main diagonal deleted.

3. Hypergeometric Bernoulli Polynomials

In 1977 Howard generalized Bernoulli polynomials by considering the following generating function:

$$\frac{\frac{t^N}{N!}e^{xt}}{e^t - T_{N-1}(t)} = \sum_{n=0}^{\infty} B_n(N,x)\frac{t^n}{n!} \tag{3.1}$$

where N is a positive integer and

$$T_N(t) = \sum_{n=0}^{N}\frac{t^n}{n!}. \tag{3.2}$$

We shall refer to $B_n(N,x)$ as hypergeometric Bernoulli polynomials of order *N*. Observe that for $N=1$, equation (3.1) reduces to (1.12). As before we express Howard's generating function as the division of two series as follows:

$$\frac{e^{xt}}{\frac{e^t - T_{N-1}(t)}{\frac{t^N}{N!}}} = \frac{\sum_{n=o}^{\infty} c_n t^n}{\sum_{n=0}^{\infty} a_n t^n}, \tag{3.3}$$

where $c_n = \dfrac{x^n}{n!}$ and $a_n = \dfrac{N!}{(n+N)!}$. It follows from (2.4) that

$$B_n(N,x) = n!(-1)^{(n)}\left|b_{ij}^1\right|, \tag{3.4}$$

where the $(n+1)\times(n+1)$ matrix (b_{ij}^1) has entries

$$b_{ij}^1=\begin{cases} 0 & i+1>j \\ \dfrac{x^{i-1}}{(i-1)!} & i+1\ge j,\ j=1 \\ \dfrac{N!}{(i-j+N+1)!} & i+1\ge j,\ j\ge 2 \end{cases} \tag{3.5}$$

Next, we perform the following row and column operations on the matrix (b_{ij}^1) as before:

1. Starting with the second column, we factor $N!$ from column *j* so that

$$B_n(N,x)=n!(-1)^{(n)}(N!)^n\left|b_{ij}^2\right| \tag{3.6}$$

where

$$b_{ij}^2=\begin{cases} 0 & j+1>i \\ \dfrac{x^{i-1}}{(i-1)!} & i+1\ge j,\ j=1 \\ \dfrac{1}{(i-j+N+1)!} & i+1\ge j,\ j\ge 2 \end{cases} \tag{3.7}$$

2. Beginning with the first row, we factor $1/(i-1)!$ from row *i* in (b_{ij}^2) so that

$$B_n(N,x)=n!\frac{(-1)^{(n)}(N!)^n}{1!2!3!...n!}\left|b_{ij}^3\right| \tag{3.8}$$

where

$$b_{ij}^3=\begin{cases} 0 & i+1>j \\ x^{i-1} & i+1\ge j,\ j=1 \\ \dfrac{(i-1)!}{(i-j+N+1)!} & i+1\ge j,\ j\ge 2 \end{cases} \tag{3.9}$$

3. Now, for columns 2 to $N+2$, we factor $1/(N+2-j)!$ from column *j* in (b_{ij}^3). For columns greater than $N+2$, we factor $(j-N-2)!$ from column *j*. This yields

$$B_n(N,x)=\frac{(-1)^{(n)}(N!)^n1!2!3!...(n-N-1)!}{1!2!3!...(n-1)!1!2!3!...N!}\left|b_{ij}^4\right| \tag{3.10}$$

where

$$b_{ij}^4 = \begin{cases} x^{i-1} & j=1 \\ & \end{cases} \begin{cases} 0 & i+1>j \\ x^{i-1} & i+1\geq j,\, j=1 \\ \dfrac{(i-1)!(N+2-j)!}{(i-j+N+1)!} & i+1\geq j, 2\leq j\leq N+2 \\ \dfrac{(i-1)!}{(i-j+N+1)!(j-N-2)!} & i+1\geq j,\, j\geq N+2 \end{cases} \tag{3.11}$$

The resulting matrix is then simplified and expressed in terms of binomials, which leads to our main result.

Theorem 2:

$$B_n(N,x) = \frac{(-1)^{(n)}(N!)^n 1!2!3!\ldots(n-N-1)!}{1!2!3!\ldots(n-1)!1!2!3!\ldots N!}\left|b_{ij}\right|, \tag{3.12}$$

where

$$b_{ij} = \begin{cases} x^{i-1} & j=1 \\ \dbinom{i-j+N+1}{i-1}^{-1} & 2\leq j\leq N+2 \\ \dbinom{i-1}{j-N-2} & j\geq N+2 \end{cases} \tag{3.13}$$

Discussion of Special Cases:

I. $N=0$

Below is the explicit determinant formula for hypergeometric Bernoulli polynomials when $N=0$, which reduces to the binomial polynomials:

$$B_n(0,x) = (-1)^{(n)} \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ x & 1 & 1 & 0 & 0 & 0 & \ldots & 0 \\ x^2 & 1 & 2 & 1 & 0 & 0 & \ldots & 0 \\ x^3 & 1 & 3 & 3 & 1 & 0 & \ldots & 0 \\ x^4 & 1 & 4 & 6 & 4 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ x^n & 1 & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \binom{n}{4} & \ldots & \binom{n}{n-1} \end{vmatrix}_{n+1} = (x-1)^n \tag{3.14}$$

Notice the entire version of Pascal's square (in red) is embedded in the matrix above starting with the second column.

II. $N=1$

In this case it is easy to check that formula (3.12) reduces to (1.13).

III. $N = 2$
Below is the explicit determinant formula for Bernoulli Polynomials when $N = 2$:

$$B_n(2,x) = \frac{(-1)^{(n)}2^{(n-1)}}{(n-1)!(n-2)!}\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ x & \frac{1}{3} & \frac{1}{2} & 0 & 0 & 0 & \dots & 0 \\ x^2 & \frac{1}{6} & \frac{1}{3} & 1 & 0 & 0 & \dots & 0 \\ x^3 & \frac{1}{10} & \frac{1}{4} & 1 & 3 & 0 & \dots & 0 \\ x^4 & \frac{1}{15} & \frac{1}{5} & 1 & 4 & 6 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ x^n & \frac{1}{\binom{n+2}{2}} & \frac{1}{\binom{n+1}{1}} & 1 & \binom{n}{1} & \binom{n}{2} & \dots & \binom{n}{n-3} \end{vmatrix}_{n+1} \tag{3.15}$$

Notice a smaller portion of Pascal's square appears in red in (3.15) beginning with the third row and fourth column.

Concluding Remarks: The connection between hypergeometric Bernoulli polynomials and hypergeometric functions is seen through the relation

$$\frac{\frac{t^N}{N!}e^{xt}}{e^t - T_{N-1}(t)} = \frac{e^{xt}}{{}_1F_1(1, N+1, t)}, \tag{3.16}$$

where the confluent hypergeometric function ${}_1F_1(1, N+1, t)$ is defined by

$${}_1F_1(a,b,t) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n}\frac{t^n}{n!}. \tag{3.17}$$

Since hypergeometric Bernoulli polynomials are defined by (3.1), we can employ (3.16) to further generalize $B_n(N, x)$ by using the alternate definition

$$\sum_{n=0}^{\infty} B_n(N,x)\frac{t^n}{n!} = \frac{e^{xt}}{{}_1F_1(1, N+1, t)}, \tag{3.18}$$

valid for all positive real values of *N*. An interesting open problem is to extend formula (3.12) for $B_n(N, x)$ in this situation.

References

1. F.Costabile, F. Dell'Accio, M. I. Gualtieri. *A new approach to Bernoulli polynomials*. Rendiconti di Matematica, Serie VII. Volume 26, Roma (2006), 1-12.
2. A.W.F. Edwards, Pascal's Arithmetical Triangle, Oxford University Press, New York, 1987.
3. A. Hassen and H. D. Nguyen, *Hypergeometric Bernoulli Polynomials and Appell Sequences*, to appear in Intern. J. Number Theory.

4. F. T. Howard, Numbers Generated by the Reciprocal of e^x-1-x, Math. Comput. 31 (1977) No. 138, 581-598.
5. Turnbull, H.A. The Theory of Determinants, Matrices, and Invariants. New York: Dover Publications, 1960.
6. MacTutor History of Math: Gottfried Wilhelm von Leibniz: http://www-gap.dcs.st-and.ac.uk/~history/Biographies/Leibniz.html

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