Doppler Tolerance, Complementary Code Sets and Generalized Thue-Morse Sequences

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Abstract

We generalize the construction of Doppler-tolerant Golay complementary waveforms by Pezeshki-Calderbank-Moran-Howard to complementary code sets having more than two codes. This is accomplished by exploiting number-theoretic results involving the sum-of-digits function, equal sums of like powers, and a generalization to more than two symbols of the classical two-symbol Prouhet-Thue-Morse sequence.

Keywords: Autocorrelation, sidelobe, complementary code set, Doppler tolerance, binary code, unimodular code, Thue-Morse sequence.

I. Introduction

A set of K unimodular codes of length N is complementary if corresponding sidelobes of the autocorrelations of the separate codes sum to zero. These sets find uses in waveform design for enhanced detection in radar systems [1] and in communication systems [2][3]. When the code is binary, the set is called a Golay complementary pair, after Marcel Golay who discovered these sets while solving a problem in infrared spectrometry [4]. Complementary Code Matrices (CCMs) provide a useful matrix formulation for the study of complementary code sets [5]. Given a set of K codes of length N, the corresponding $N \times K$ complementary code matrix (CCM) has the k^{th} code as its k^{th} column, $k = 1, \ldots, K$.

Complementary code sets have yet to be widely used for radar waveform designs due to certain design challenges. These include sensitivity to Doppler shift due to non-zero relative velocity of a target relative to the radar platform [1][6]. Complementary code sets may be used in a number of ways in waveform design. Two of these ways are the time-separation approach, where time-separated pulses or subpulses are phase coded using different codes in the set [7][8], and frequency-separation, where the different codes are used for phase encoding of separate components of a signal and are transmitted concurrently using pulses with different center frequencies [9][10]. The time-separation approach is especially sensitive to Doppler shift.

With time-separated pulses encoded using the codes from a complementary set, pulse returns may be match filtered separately and then added to give zero autocorrelation sidelobes, in theory, a desirable result for radar detection.

However, target relative velocity yields a phase shift pulse to pulse, and therefore a phase shift of sidelobes, thus preventing zero sidelobe sums in general.

The development in this paper builds on work by Pezeshki, Howard, Moran, Calderbank, Chi and Searle [4][7][8][9][10]. In particular, in [7], Chi, Calderbank and Pezeshki consider pulse trains in which the pulses are phase coded with binary codes in a Golay complementary pair. They show that for any given M, M^{th}-order nulls can be created about the zero-Doppler axis of the ambiguity function by mapping the codes to pulses in an order specified by the well-known (two-symbol) Prouhet-Thue-Morse sequence. We show that the result may be generalized to (N, K) complementary code sets for $K \geq 2$ by using a generalized Prouhet-Thue-Morse sequence using $m \geq 2$ symbols. The approach also makes use of results related to the Tarry-Escott problem [11][15][16], and related number-theoretic entities such as the digit-sum function [12] and equal sums of like powers [17]. Finally, it is shown that the transmission period and the total number of pulses transmitted may be reduced by using multiple antennas to transmit separate pulse trains staggered in time.

II. Notations and Terminology

Definition 2.1. A *p-phase* matrix Q is one whose entries are p-th roots of unity, i.e. roots of $z^p = 1$.

Definition 2.2. Given a unimodular code x of length N, the autocorrelation function (ACF) of x is defined as the sequence of length $2N-1$

$$\mathrm{ACF}_x = x * \overline{x}$$

where $*$ represents aperiodic convolution and $\overline{x}$ means reversal of x. The elements $\mathrm{ACF}_x(k)$ for $k = 1 - N, \ldots, -1, 0, 1, \ldots, N-1$ may be written explicitly as sums of pairwise products of the elements of x:

$$\mathrm{ACF}_x(k) = \sum_{i=1}^{N-k} x[i]\overline{x[i+k]}, \tag{1}$$

for $k = 0, 1, \ldots, N-1$, where $x[i]$ denotes the i-th component of x and $\overline{x[i]}$ represents complex conjugation. if $k = 1 - N, \ldots, -1$, then

$$\mathrm{ACF}_x(-k) = \overline{\mathrm{ACF}}_x(k).$$

- $|\mathrm{ACF}_x(N-1)| = |x[1]\overline{x[N]}| = 1.$
- When $k = 0$, $\mathrm{ACF}_x(k)$ represents the peak of the autocorrelation, which equals

$$x_[1]\overline{x[1]} + \ldots + x[N]\overline{x[N]} = ||x||^2 = N.$$

Definition 2.3. [5] A p-phase $N \times K$ matrix Q consisting of columns $(x_0, x_1, \ldots, x_{K-1})$ is said to be a *complementary code matrix* if

$$\mathrm{ACF}_{x_0}(n) + \mathrm{ACF}_{x_1}(n) + \ldots + \mathrm{ACF}_{x_{K-1}}(n) = NK\delta_n$$

for $n = -(N-1), \ldots, -1, 0, 1, \ldots, (N-1)$ where δ_n is the Kronecker delta function.

Lemma 2.4. Let $Q = (x_0, x_1, \ldots, x_{K-1})$ be a p-phase $N \times K$ CCM. Then

$$\begin{aligned}\sum_{i=0}^{K-1} X_i(z)\tilde{X}_i(z) &= |X_0(z)|^2 + \ldots + |X_{K-1}(z)|^2 \\ &= NK\end{aligned}$$

III. Doppler Shift in Radar

Let $T = (x_0, x_1, \ldots, x_{L-1})$ be a modulated pulse train whose ambiguity function is given by

$$g(k,\theta) = \sum_{n=0}^{L-1} e^{jn\theta} \mathrm{ACF}_{x_n}(k), \tag{2}$$

where k represents range or time delay and θ represents Doppler-shift-induced phase advance. We define the z-transform of a code x of length N by

$$X(z) = x[0] + x[1]z^{-1} + ... + x[N-1]z^{-N+1}$$

Following Pezeshki-Calderbank-Moran-Howard [4], the z-transform of $g(k,\theta)$ becomes

$$G(z,\theta) = \sum_{n=0}^{L-1} e^{jn\theta} |X_n(z)|^2. \tag{3}$$

where

$$\begin{aligned}|X_n(z)|^2 &= \mathrm{ACF}_{x_n}(0) + \sum_{k=1}^{N-1} \mathrm{ACF}_{x_n}(k) z^k + \\ &\sum_{k=1}^{N-1} \overline{\mathrm{ACF}_{x_n}}(k) z^{-k}.\end{aligned}$$

Next, consider the Taylor expansions of $g(k,\theta)$ and $G(z,\theta)$ about $\theta = 0$:

$$g(k,\theta) = \sum_{m=0}^{\infty} c_m(k) \frac{(j\theta)^m}{m!} \tag{4}$$

$$G(z,\theta) = \sum_{m=0}^{\infty} C_m(z) \frac{(j\theta)^m}{m!}. \tag{5}$$

Here, the Taylor coefficients $c_n(k)$ and $C_n(k)$ are given by

$$c_m(k) = \sum_{n=0}^{L-1} n^m \mathrm{ACF}_{x_n}(k) \tag{6}$$

$$C_m(z) = \sum_{n=0}^{L-1} n^m |X_n(z)|^2. \tag{7}$$

The following theorem demonstrates an equivalence in terms of the "vanishing" of the Taylor coefficients $c_m(k)$ and $C_m(z)$.

Theorem 3.1. Let m be a non-negative integer. Then $c_m(k) = 0$ for all non-zero k if and only if $C_m(z)$ is constant and independent of z.

Proof. Assume $c_m(k) = 0$ for all non-zero k. It follows from (2) that

$$\begin{aligned} C_m(z) &= \sum_{n=0}^{L-1} n^m |X_n(z)|^2 \\ &= \sum_{n=0}^{L-1} n^m (\mathrm{ACF}_{x_n}(0) + \sum_{k=1}^{N-1} \mathrm{ACF}_{x_n}(k) z^k + \sum_{k=1}^{N-1} \overline{\mathrm{ACF}_{x_n}}(k) z^{-k}) \\ &= \sum_{n=0}^{L-1} n^m \mathrm{ACF}_{x_n}(0) + \sum_{n=0}^{L-1} n^m \sum_{k=1}^{N-1} \mathrm{ACF}_{x_n}(k) z^k + \sum_{n=0}^{L-1} n^m \sum_{k=1}^{N-1} \overline{\mathrm{ACF}_{x_n}}(k) z^{-k} \\ &= \sum_{n=0}^{L-1} n^m \mathrm{ACF}_{x_n}(0) + \sum_{k=1}^{N-1} \sum_{n=0}^{L-1} n^m \mathrm{ACF}_{x_n}(k) z^k + \sum_{k=1}^{N-1} \sum_{n=0}^{L-1} n^m \overline{\mathrm{ACF}_{x_n}}(k) z^{-k} \\ &= \sum_{n=0}^{L-1} n^m \mathrm{ACF}_{x_n}(0). \end{aligned}$$

This proves that $C_m(z)$ is constant and independent of z. Conversely, assume $C_m(z)$ is constant and independent of z. Then from the previous calculation we have

$$C_m(z) = \sum_{n=0}^{L-1} n^m \mathrm{ACF}_{x_n}(0) + \sum_{k=1}^{N-1} c_m(k) z^k + \sum_{k=1}^{N-1} \overline{c_m}(k) z^{-k}.$$

It follows that $c_m(k) = 0$ for all non-zero k since $C_m(z)$ is independent of z. ■

IV. Generalized Prouhet-Thue-Morse Sequences

Denote by $S(L) = \{0, 1, \ldots, L-1\}$ to be the set consisting of the first L non-negative integers.

Definition 4.1. Let $n = n_1 n_2 \ldots n_k$ be the base-p representation of a non-negative integer n, where $n_i \in \{0, 1, \ldots, p-1\}$ for $i = 1, \ldots, k$. We define $v_p(n) \in \mathbb{Z}_p$ to be the least positive residue of the sum of the digits n_i modulo p, that is,

$$v_p(n) \equiv \left(\sum_{i=1}^{k} n_i \right) \bmod p.$$

Note that $v_p(n) = n$ if $0 \leq n < p$.

Definition 4.2 ([11]). Let p be a positive integer. We define the mod-p *Prouhet-Thue-Morse* (PTM) *sequence* $P = \{a_0, a_1, \ldots\}$ to be such that

$$a_n = v_p(n).$$

Example 4.3: Examples of P for $p = 2, 3, 4$ are given below. Observe that for $p = 2$, P reduces to the classical Prouhet-Thue-Morse sequence [13].

$p = 2$:

$P = \{0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, \ldots\}$

$p = 3$:

$P = \{0, 1, 2, 1, 2, 0, 2, 0, 1, 1, 2, 0, 2, 0, 1, 0, 1, 2, \ldots\}$

$p = 4$:

$P = \{0, 1, 2, 3, 1, 2, 3, 0, 2, 3, 0, 1, 3, 0, 1, 2, 1, 2, 3, 0, \ldots\}$

Definition 4.4. Let p and M be positive integers and set $L = p^{M+1}$. We define $\{S_0, S_1, \ldots, S_{p-1}\}$ to be a Prouhet-Thue-Morse (PTM) p-block partition of $S(L) = \{0, 1, \ldots, L-1\}$ as follows: if $v_p(n) = i$, then

$$n \in S_i.$$

Example 4.5: Examples of PTM block partitions are given below.

$p = 2,\ M = 3,\ L = 16$:

$$\begin{aligned} S_0 &= \{0, 3, 5, 6, 9, 10, 12, 15\} \\ S_1 &= \{1, 2, 4, 7, 8, 11, 13, 14\} \end{aligned}$$

$p = 3$,$M = 2,\ L = 27$:

$$\begin{aligned} S_0 &= \{0, 5, 7, 11, 13, 15, 19, 21, 26\} \\ S_1 &= \{1, 3, 8, 9, 14, 16, 20, 22, 24\} \\ S_2 &= \{2, 4, 6, 10, 12, 17, 18, 23, 25\} \end{aligned}$$

$p = 4$,$M = 2,\ L = 64$:

$$\begin{aligned} S_0 &= \{0, 7, 10, 13, 19, 22, 25, 28, 34, 37, 40, 47, 49, 52, 59, 62\} \\ S_1 &= \{1, 4, 11, 14, 16, 23, 26, 29, 35, 38, 41, 44, 50, 53, 56, 63\} \\ S_2 &= \{2, 5, 8, 15, 17, 20, 27, 30, 32, 39, 42, 45, 51, 54, 57, 60\} \\ S_3 &= \{3, 6, 9, 12, 18, 21, 24, 31, 33, 36, 43, 46, 48, 55, 58, 61\} \end{aligned}$$

Theorem 4.6 ([11][15],[16]). Let p and M be positive integers and set $L = p^{M+1}$. Define $\{S_0, S_1, \ldots, S_{p-1}\}$ to be a PTM p-block partition of $S(L) = \{0, 1, \ldots, L-1\}$. Then

$$\sum_{n \in S_0} n^m = \sum_{n \in S_1} n^m = \ldots = \sum_{n \in S_{p-1}} n^m$$

for $m = 1, \ldots, M$.

It will be convenient to define $P_m := P_m(p, M) = \sum_{n \in S_0} n^m$ to be the m^{th} Prouhet sum corresponding to p and M.

Let $(A_0, A_1, \ldots)$ be a sequence of elements satisfying the aperiodic property

$$A_n = A_{v_p(n)}.$$

We shall define an orthogonal set of sequences $w_i(n)$ whose values are given by the Rademacher functions [14]. These sequences will be used to define a transformation of the elements $(A_0, A_1, \ldots, A_{p-1})$ whose invertibility provides a useful decomposition for isolating sidelobes in the total autocorrelation of a train of coded pulses.

Definition 4.7. Let

$$i = d_{p-1}^{(i)} 2^{p-1} + d_{p-2}^{(i)} 2^{p-2} + \ldots + d_1^{(i)} 2^1 + d_0^{(i)} 2^0$$

be the binary expansion of i, where i is a non-negative integer with $0 \le i \le 2^p - 1$. Define $w_0(n), w_1(n), \ldots, w_{2^p-1}(n)$ to be binary ± 1-sequences

$$w_i(n) = (-1)^{d_{p-1-v_p(n)}^{(i)}}$$

for $n = 0, 1, \ldots$.

Theorem 4.8 ([14]). Define

$$B_i = \sum_{n=0}^{p-1} w_i(n) A_n$$

for $i = 0, 1, \ldots, 2^p - 1$. Then

$$A_n = \left(1/2^{p-1}\right) \sum_{i=0}^{2^{p-1}-1} w_i(n) B_i$$

for $n = 0, 1, \ldots$.

Because of Theorem 4.8, we shall call $w_0(n), w_1(n), \ldots, w_{2^{p-1}-1}(n)$ the PTM weights of A_n with respect to $(B_0, B_1, \ldots, B_{2^{p-1}-1})$.

Example 4.9: Examples illustrating Theorem 4.8 are given below.

(1) $p = 2$:

$$B_0 = A_0 + A_1, \quad A_0 = \tfrac{1}{2}(B_0 + B_1)$$
$$B_1 = A_0 - A_1, \quad A_1 = \tfrac{1}{2}(B_0 - B_1)$$

(2) $p = 3$:

$$B_0 = A_0 + A_1 + A_2, \quad A_0 = \tfrac{1}{4}(B_0 + B_1 + B_2 + B_3)$$
$$B_1 = A_0 + A_1 - A_2, \quad A_1 = \tfrac{1}{4}(B_0 + B_1 - B_2 - B_3)$$
$$B_2 = A_0 - A_1 + A_2, \quad A_2 = \tfrac{1}{4}(B_0 - B_1 + B_2 - B_3)$$
$$B_3 = A_0 - A_1 - A_2$$

Theorem 4.10 ([14]). Suppose $L = p^{M+1}$ where M is a non-negative integer. Write

$$A_n = (1/2^{p-1})w_0(n)B_0 + (1/2^{p-1})S_p(n) \tag{8}$$

where

$$S_p(n) = \sum_{i=1}^{2^{p-1}-1} w_i(n)B_i.$$

Then

$$\sum_{n=0}^{L-1} n^m S_p(n) = N_m B_0 \tag{9}$$

for $m = 1, \ldots, M$ where

$$N_m = 2^{p-1}P_m - \sum_{n=0}^{L-1} n^m.$$

V. Doppler-Tolerant CCM Waveforms

In this section we generalize the results in [7] and [10] by constructing Doppler-tolerant CCM waveforms.

Definition 5.1. We define a mod-p Prouhet-Thue-Morse (PTM) pulse train $T = (x_0, x_1, \ldots, x_{L-1})$ to be a sequence satisfying

$$x_n = x_{v_p(n)}.$$

Let $A_n(k)$ represent sidelobe k for the autocorrelation ACF_{x_n} of code x_n. It follows that $A_n(k) = A_{v_p(n)}(k)$. At times, the sidelobe index k will be suppressed, when the property being discussed applies regardless of the particular sidelobe.

We now use the results from the previous section to isolate the sidelobe term given by (9) in the ambiguity function $g(k, \theta)$. Suppose $L = p^{M+1}$ where M is a non-negative integer. It follows from (2) and (8) that

$$\begin{aligned}
g_p(\theta) &:= g(k, \theta) \\
&= \sum_{n=0}^{L-1} A_{v_p(n)} e^{jn\theta} \\
&= \sum_{n=0}^{L-1} ((1/2^{p-1})w_0(n)B_0 + (1/2^{p-1})S_p(n))e^{jn\theta} \\
&= (1/2^{p-1})B_0 \sum_{n=0}^{L-1} e^{jn\theta} + (1/2^{p-1}) \sum_{n=0}^{L-1} S_p(n)e^{jn\theta}.
\end{aligned}$$

The argument uses the fact that $w_0(n) = 1$.

Example 5.2. Let $p = 2$. Then $g_2(\theta)$ reduces to equation (11) in [7]:

$$\begin{aligned} g_2(\theta) &= (1/2)B_0 \sum_{n=0}^{L-1} e^{jn\theta} + (1/2) \sum_{n=0}^{L-1} S_2(n)e^{jn\theta} \\ &= (1/2)(A_0 + A_1) \sum_{n=0}^{L-1} e^{jn\theta} + (1/2)(A_0 - A_1) \sum_{n=0}^{L-1} w_1(n)e^{jn\theta}, \end{aligned}$$

where $w_1(n) = p_n$ is the classical Prouhet-Thue-Morse sequence defined by the recurrence $p_0 = 1$, $p(2n) = p(n)$, and $p(2n + 1) = -p(n)$.

Define

$$h_p(\theta) = (1/2^{p-1}) \sum_{n=0}^{L-1} S_p(n)e^{jn\theta}$$

so that

$$g_p(\theta) = (1/2)B_0 \sum_{n=0}^{L-1} e^{jn\theta} + h_p(\theta).$$

If $Q = (x_0, x_1, \ldots, x_{K-1})$ is a unimodular $N \times K$ CCM, then $h_p(\theta)$ represents the sidelobes of $g_p(\theta)$ since $B_0 = A_0 + A_1 + \ldots + A_{K-1}$ vanishes for all non-zero k, being the sum of the autocorrelation functions of $x_0, x_1, \ldots, x_{K-1}$. Expanding $h_p(\theta)$ in a Taylor series about $\theta = 0$:

$$h_p(\theta) = (1/2^{p-1}) \sum_{m=0}^{\infty} s_m \left((j\theta)^m/m!\right)$$

where

$$s_m = \sum_{n=0}^{L-1} n^m S_p(n).$$

The following result generalizes Theorem 2 in [7].

Theorem 5.3. Let Q be a unimodular $N \times K$ CCM consisting of columns $(x_0, x_1, \ldots, x_{K-1})$ and M a positive integer. Set $L = K^{M+1}$ and extend Q to a pulse train $T = (x_0, x_1, \ldots, x_{K-1}, x_K, \ldots, x_{L-1})$ where

$$x_n = x_{v_K(n)}$$

for all $n = 0, 1, \ldots, L - 1$. Then the Taylor coefficients s_m of $h_K(\theta)$ vanish up to order M, namely

$$s_m = 0$$

for $m = 1, \ldots, M$.

Proof. Set $p = k$. It follows from (9) that

$$\begin{aligned} s_m &= N_m B_0 \\ &= N_m(A_0 + A_1 + \ldots + A_{K-1}) \\ &= N_m(\mathrm{ACF}_{x_0}(k) + \mathrm{ACF}_{x_1}(k) + \ldots + \mathrm{ACF}_{x_{K-1}}(k) \\ &= 0 \end{aligned}$$

for all non-zero k. ∎

Next, we move to the z-domain and prove an equivalent version of Theorem 5.3 by generalizing Theorem 2 in [8], which constructs Doppler-tolerant pulse trains in the z-domain.

Theorem 5.4. Let Q be a unimodular $N \times K$ CCM consisting of columns $(x_0, x_1, \ldots, x_{K-1})$ and M a positive integer. Set $L = K^{M+1}$ and extend Q to a pulse train $T = (x_0, x_1, \ldots, x_{K-1}, x_K, \ldots, x_{L-1})$ where

$$x_n = x_{v_K(n)}$$

for all $n = 0, 1, \ldots, L-1$. Then the Taylor coefficients $C_m(z)$ are independent of z up to order M, namely

$$C_m(z) = NKP_m$$

for $m = 1, \ldots, M$ where P_m is the m^{th} Prouhet sum corresponding to K and M.

As in [4], we call T a mod-K Prouhet-Thue-Morse (PTM) pulse train of length L.

Proof. Let $\{S_0, S_1, \ldots, S_{K-1}\}$ be a PTM K-block partition of $S = \{0, 1, \ldots, L-1\}$. It follows from Theorem 4.6 and Lemma 2.4 that

$$\begin{aligned} C_m(z) &= \sum_{n=0}^{L-1} n^m |X_n(z)|^2 \\ &= \sum_{n\in S_0} n^m |X_{v_K(n)}(z)|^2 + \sum_{n\in S_1} n^m |X_{v_K(n)}(z)|^2 \\ &\quad + \ldots + \sum_{n\in S_{K-1}} n^m |X_{v_K(n)}(z)|^2 \\ &= |X_0(z)|^2 \sum_{n\in S_0} n^m + |X_1(z)|^2 \sum_{n\in S_1} n^m \\ &\quad + \ldots + |X_{K-1}(z)|^2 \sum_{n\in S_{K-1}} n^m \\ &= (|X_0(z)|^2 + |X_1(z)|^2 + \ldots + |X_{K-1}(z)|^2) P_m \\ &= NKP_m \end{aligned}$$

for $m = 1, 2, \ldots, M$. ∎

Example 5.5: Examples of PTM pulse trains are given below.

1. Let $K = 2$, $M = 3$, and (x_0, x_1) be a binary $N \times 2$ CCM (Golay pair). Then the following is a mod-2 PTM pulse train of length $L = 2^4 = 16$:

$$T = (x_0, x_1, x_1, x_0, x_1, x_0, x_0, x_1, x_1, x_0, x_0, x_1, x_0, x_1, x_1, x_0)$$

2. Let $K = 3$, $M = 2$, and (x_0, x_1, x_2) be a tri-phase $N \times 3$ CCM. Then the following is a mod-3 PTM pulse train of length $L = 3^3 = 27$:

$$\begin{aligned} T \quad = \quad & (x_0, x_1, x_2, x_1, x_2, x_0, x_2, x_0, x_1, x_1, x_2, x_0, x_2, x_0, \\ & x_1, x_0, x_1, x_2, x_2, x_0, x_1, x_0, x_1, x_2, x_1, x_2, x_0) \end{aligned}$$

3. Let $K = 4$, $M = 2$, and (x_0, x_1, x_2, x_3) be a unimodular $N \times 4$ CCM. Then the following is a mod-4 PTM pulse train of length $L = 4^3 = 64$:

$$\begin{aligned} T \quad = \quad & (x_0, x_1, x_2, x_3, x_1, x_2, x_3, x_0, x_2, x_3, x_0, x_1, \\ & x_3, x_0, x_1, x_2, x_1, x_2, x_3, x_0, x_2, x_3, x_0, x_1, \\ & x_3, x_0, x_1, x_2, x_0, x_1, x_2, x_3, x_2, x_3, x_0, x_1, \\ & x_3, x_0, x_1, x_2, x_0, x_1, x_2, x_3, x_1, x_2, x_3, x_0, \\ & x_3, x_0, x_1, x_2, x_0, x_1, x_2, x_3, x_1, x_2, x_3, x_0, \\ & x_2, x_3, x_0, x_1) \end{aligned}$$

VI. ESP Staggered Pulse Trains

In this section we introduce pulse trains, called ESP staggered pulse trains, that provide the same Doppler tolerance as PTM pulse trains but are generally shorter in length, by using multiple antennas to transmit separate pulse trains staggered in time. We begin with definitions of delayed pulse trains and partitions of arbitrary sets of non-negative integers (not necessarily consecutive as with PTM partitions) having equal sums of powers.

Definition 6.1. We define a *delayed* pulse train

$$T(d) = (x_0, x_1, ..., x_{L-1})$$

of length L as one having a delay of d pulses in the sense that its ambiguity function has the form

$$g_T(k, \theta, d) = \sum_{n=0}^{L-1} \mathrm{ACF}_{x_n} e^{i(n+d)\theta}$$

Definition 6.2. Let S be a set of non-negative integers and $P = \{S_0, S_1, \ldots, S_{p-1}\}$ be a p-block partition of S.

We say that P has *equal sums of (like) powers (ESP)* of degree M if

$$\sum_{n\in S_0} n^m = \sum_{n\in S_1} n^m = \ldots = \sum_{n\in S_{p-1}} n^m$$

for $m = 1, \ldots, M$. In that case, we define

$$P_m := P_m(C) = \sum_{n\in S_0} n^m.$$

The following examples demonstrates our concept of using MIMO (multiple-input multiple-output) radar to transmit ESP pulse trains whose overall transmission period is shorter than PTM pulse trains.

Example 6.3: (Second-order nulls) Let $S = \{0, 1, 2, 4, 5, 6\}$ and consider the 2-block partition $\mathcal{P} = (S_0, S_1)$ of S, where $S_0 = (0, 4, 5)$ and $S_1 = (1, 2, 6)$. Then $\mathcal{P}$ has ESP of degree 2 since

$$0 + 4 + 5 = 1 + 2 + 6$$
$$0^2 + 4^2 + 5^2 = 1^2 + 2^2 + 6^2$$

Observe that this partition consists of only six values (skipping the value 3) and is smaller in size than the 2-block PTM partition of $\{0, 1, ..., 7\}$. Then given a Golay pair of codes (x_0, x_1), we can of course construct a single pulse train based on the partition above by inserting a gap or fill pulse for the value at position 3:

$$T = (x_0, x_1, x_1, \_\_, x_0, x_0, x_1)$$

This approach however is impractical in terms of transmission. On the other hand, we can modify the partition $\mathcal{P}$ so that it includes the value 3 in both sets:

$$S_0 = (0, 3, 4, 5)$$
$$S_1 = (1, 2, 3, 6)$$

Note that $\mathcal{P}$ is no longer a collection of mutually disjoint sets but continues to have ESP of degree 2. Suppose we then transmit two separate pulse trains of length 4, T_0 and T_1 (each from a separate antenna), but staggered in the sense that we delay the transmission of T_1 by 3 pulses as follows:

$$T_0 = (x_0, x_1, x_1, x_0)$$
$$T_1(3) = \qquad\qquad (x_1, x_0, x_0, x_1)$$

Here, T_0 transmits pulses corresponding to the first two values of S_0 (positions 0 and 3) and the first two values of S_1 (positions 1 and 2). Similarly for $T_1(3)$, but corresponding to the last two values of S_0 and S_1. If we sum the composite ACFs of both pulse trains, then we obtain

$$\begin{aligned} g(k,\theta) &= g_{T_0}(k,\theta) + g_{T_1}(k,\theta,3) \\ &= \mathrm{ACF}_{x_0}(k) + \mathrm{ACF}_{x_1}(k)e^{i\theta} + \mathrm{ACF}_{x_1}(k)e^{2i\theta} \\ &\quad + (\mathrm{ACF}_{x_0}(k) + \mathrm{ACF}_{x_1}(k))e^{3i\theta} + \mathrm{ACF}_{x_0}(k)e^{4i\theta} \\ &\quad + \mathrm{ACF}_{x_0}(k)e^{5i\theta} + \mathrm{ACF}_{x_1}(k)e^{6i\theta} \end{aligned}$$

To show that $g(k,\theta)$ has Doppler nulls of order 2 at $\theta = 0$, we compute its Doppler (Taylor) coefficients:

$$\begin{aligned} c_m(k) &= g^{(m)}(k,0) \\ &= (0^m + 3^m + 4^m + 5^m)\mathrm{ACF}_{x_0}(k) \\ &\quad + (1^m + 2^m + 3^m + 6^m)\mathrm{ACF}_{x_1}(k) \\ &= P_m(\mathrm{ACF}_{x_0}(k) + \mathrm{ACF}_{x_1}(k)) \\ &= 2NP_m\delta_k \end{aligned}$$

for $m = 0, 1, 2$. This demonstrates that we can achieve the same Doppler tolerance as with a single PTM pulse train of length 8 by using instead two staggered (but overlapping) pulse trains of length 4 to reduce the total transmission time from 8 pulses down to 7 pulses. Note however that the total number of pulses transmitted is the same, namely 8, in both cases.

Example 6.4: (Third-order nulls) Consider the following 2-block partition $\mathcal{P} = (S_0, S_1)$, where

$$\begin{aligned} S_0 &= (0, 4, 7, 11) \\ S_1 &= (1, 2, 9, 10) \end{aligned}$$

which has ESP of degree 3, namely

$$0^m + 4^m + 7^m + 11^m = 1^m + 2^m + 9^m + 10^m$$

for $m = 0, 1, 2, 3$. As in the previous example, we modify this partition so that both sets S_0 and S_1 contain each of the values 3, 5, 6, and 8:

$$\begin{aligned} S_0 &= (0, 3, 4, 5, 6, 7, 8, 11) \\ S_1 &= (1, 2, 3, 5, 6, 8, 9, 10) \end{aligned}$$

We now transmit four pulse trains T_0, $T_1(3)$, $T_2(5)$, $T_3(8)$ on separate antennas having delays $0, 3, 5, 8$, respectively:

$$\begin{aligned} T_0 &= (x_0, x_1, x_1, x_0) \\ T_1(3) &= (x_1, x_0, x_0, x_1) \\ T_2(5) &= (x_1, x_0, x_0, x_1) \\ T_3(8) &= (x_0, x_1, x_1, x_0) \end{aligned}$$

Then it can be shown that the Doppler coefficients of the composite ambiguity function $g(k, \theta)$ has Doppler nulls of order 3:

$$\begin{aligned} c_m(k) &= (0^m + 3^m + 4^m + 5^m + 6^m + 7^m + 8^m \\ &\qquad + 11^m)\mathrm{ACF}_{x_0}(k) + (1^m + 2^m + 3^m + 5^m \\ &\qquad + 6^m + 8^m + 9^m + 10^m)\mathrm{ACF}_{x_1}(k) \\ &= P_m(\mathrm{ACF}_{x_0}(k) + \mathrm{ACF}_{x_1}(k)) \\ &= 2NP_m\delta_k \end{aligned}$$

for $m = 0, 1, 2, 3$. Thus, we have reduced the total transmission time from 16 pulses (for a single PTM pulse train of length 16 having the same Doppler tolerance) down to 12 by using instead four pulse trains transmitted separately. Again, note that the total number of pulses transmitted is the same (16) in both cases.

Example 6.5: (Fifth-order nulls) Consider the following 2-block partition $\mathcal{P} = (S_0, S_1)$ which has ESP of degree 5:

$$\begin{aligned} S_0 &= (0, 5, 6, 16, 17, 22) \\ S_1 &= (1, 2, 10, 12, 20, 21) \end{aligned}$$

We again modify this partition to include the values $\{3, 4, 7, 8, 9, 11, 13, 14, 15, 18, 19\}$ without changing its degree:

$$\begin{aligned} S_0 &= (0, 3, 4, 5, 6, 7, 8, 9, 11, 13, 14, 15, 16, 17, 18, 19, 22) \\ S_1 &= (1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 13, 14, 15, 18, 19, 20, 21) \end{aligned}$$

We then transmit seven pulse trains T_0, $T_1(3)$, $T_2(7)$, $T_3(8)$, $T_4(10)$, and $T_5(13)$, and $T_6(18)$ having delays 0, 3, 7, 8, 10, 13, and 18, respectively:

$$
\begin{aligned}
T_0 &= (x_0, x_1, x_1, x_1, x_1) \\
T_1(3) &= (x_0, x_0, x_0, x_0, x_1) \\
T_2(7) &= (x_0, x_0, x_0) \\
T_3(8) &= (x_1, x_1, x_1, x_1, x_1) \\
T_4(10) &= (x_0, x_1, x_1, x_1, x_1) \\
T_5(13) &= (x_0, x_0, x_0, x_0, x_0, x_0, x_0) \\
T_6(18) &= (x_1, x_1, x_1, x_1, x_0)
\end{aligned}
$$

Again it can be shown that the Doppler coefficients of the composite ambiguity function $g(k, \theta)$ has Doppler nulls of order 5. Thus, we have reduced the total transmission time from 64 pulses (for a single PTM pulse train of length 64 having the same Doppler tolerance) down to 23 by using instead seven pulse trains transmitted by separate antennas. Unlike Examples 6.3 and 6.4, the total number of pulses transmitted for all seven staggered pulse trains is only 40 in comparison to 64 for a single PTM pulse train. We observe that the three pulse trains $T_2(7)$, $T_3(8)$, and $T_5(13)$ are constant in value.

VII. Conclusions

Pezeshki, Calderbank, Howard, and Moran have shown that Doppler tolerance can be achieved in match-filtered trains of time-separated pulses encoded with Golay complementary pairs. The key is to map the two codes to the pulses in the train using the well-known Thue-Morse sequence. Depending on the number of pulses that can be supported for a particular application, the Doppler tolerance can be achieved to any desired order. This paper has shown that the same is possible with complementary code sets containing more than two codes. Generalization is achieved by exploiting several number-theoretic concepts, including equal sums of like powers, the digit sum function, and the generalization to $m \geq 2$ symbols of the classical two-symbol Thue-Morse sequence. In addition, it is shown that certain ESP pulse trains having shorter lengths than PTM pulse trains can be used to obtain the same Doppler tolerance by employing multiple antennas to transmit these pulse trains staggered in time.

VIII. References

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