COMPACT WEAKLY SYMMETRIC SPACES AND SPHERICAL PAIRS

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ABSTRACT. Let (G, H) be a spherical pair and assume that G is a connected compact simple Lie group and H a closed subgroup of G. We prove in this paper that the homogeneous manifold G/H is weakly symmetric with respect to G and possibly an additional fixed isometry μ. It follows that M. Krämer's classification list of such spherical pairs also becomes a classification list of compact weakly symmetric spaces. In fact, our proof involves a case-by-case study of all the spherical pairs on Krämer's list.

CONTENTS

1. INTRODUCTION

Let $M = G/H$ be a weakly symmetric space with respect to an isometry group G and possibly an additional fixed isometry μ. A. Selberg was able to prove in [S] that the space of G-invariant differential operators on M is commutative. If G is connected, then this is equivalent to the property of the function space $L^1(H\backslash G/H)$ being commutative, i.e. (G, H) is a Gelfand pair (cf. [Th] and [N2]), or the property that every unitary irreducible representations of G contain at most one H-fixed vector, i.e. (G, H) is a spherical pair and H is a called a spherical subgroup of G (cf. [Kr] and [H2], Ch. IV, Theorem 3.4).

It was proven by J. Lauret in [L] that the converse to the above statement is false, namely that not all Gelfand pairs (G, H) are weakly symmetric. Lauret's result involved constructing nilmanifolds $M = G/H$ with a modified Heisenberg-type metric and showing that M cannot be weakly symmetric with respect to the full isometry group $I(M)$. However, we prove in this paper that the converse does hold (in the context of spherical pairs) if we assume G to be a connected compact simple Lie group. Therefore, even in light of Lauret's counterexamples, our result demonstrates that for many of the compact spherical pairs (G, H), proving weak symmetry of M becomes a very easy task.

We now give a precise statement of our main result.

Theorem 1.1. *Let (G, H) be a spherical pair with G a connected compact simple Lie group and H a closed subgroup of G. Then the homogeneous manifold $M = G/H$ is weakly symmetric with respect to G and possibly an additional fixed isometry μ.*

Our strategy for proving this theorem is as follows: if (G, H) is a spherical pair, then Krämer in [Kr] has proven that (G, H_0) is also a spherical pair, where H_0 is the identity component of H. Furthermore, (G, H_0) must appear on his classification list given in the same paper. This allows us then to go through a

case-by-case study of each pair appearing on his list and prove that the corresponding homogeneous manifold $M_0 = G/H_0$ is weakly symmetric with respect to G and possibly an additional fixed isometry μ. This is done by understanding how the isotropy representation reverses tangent vectors in each case. It follows that $M = G/H$ is also weakly symmetric with respect to G and μ as it has an even larger isotropy subgroup than M_0.

Acknowledgements. The author wishes to thank W. Ziller for his generous help and guidance. Many of the arguments for the spherical pairs of constant curvature and of Cayley-type actually resulted from joint work between us. Furthermore, he has recently informed the author that D.N. Akhiezer and E.B. Vinberg have announced a similar result in [AV] but within the context of spherical varieties. In fact, they prove that Theorem 1.1 holds more generally for reductive algebraic groups G defined over $\mathbb{R}$.

2. PRELIMINARIES

Let $M = G/H$ be a homogeneous Riemannian manifold, G a transitive group of isometries and μ a fixed isometry of M (not necessarily in G) satisfying $\mu G \mu^{-1} = G$ and $\mu^2 \in G$.

Definition 2.1. M is *weakly symmetric* with respect to G and μ if given any two points x and y in M, there exists an element $g \in G$ such that $gx = \mu y$ and $gy = \mu x$.

We now present some results that will make it easier for us to identify weakly symmetric spaces in terms of their linear isotropy representations. The first is a lemma that characterizes weak symmetry as the reversal of tangent vectors, first observed by W. Ziller [Z].

Lemma 2.2. *(cf. [N1], Lemma 2.2) Assume that M is connected and μ fixes the point z_o. Define H to be the isotropy subgroup of G at z_o and $T_{z_o}(M)$ to be the tangent space of M at z_o. Then M is weakly symmetric with respect to G and μ if and only if given any tangent vector $v \in T_{z_o}(M)$, there exists an element $h \in H$ such that $d(h \circ \mu)_{z_o}(v) = -v$.*

As weakly symmetric spaces are inherently homogeneous manifolds, Lemma 2.2 now allows us to also view weak symmetry of M in terms of the adjoint representation of H. We wish to develop this relationship as a next step. Let G be a connected Lie group and θ an involutive automorphism of G. If H be a compact θ-invariant subgroup of G, then there exists a reductive composition $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ such that $\mathfrak{q}$ is $Ad_{G/H}(H)$-invariant. Here, $Ad(H) = Ad_{G/H}(H)$ is the adjoint representation of H on $\mathfrak{q} \cong \mathfrak{g}/\mathfrak{h}$.

Definition 2.3. (G, H, θ) is called a *weakly symmetric triple* if given any element $X \in \mathfrak{q}$, there exists an element $h \in H$ such that $(Ad(h) \circ d\theta)(X) = -X$.

Lemma 2.4. *If (G, H, θ) is a weakly symmetric triple, then the homogeneous manifold $M = G/H$, equipped with any G-invariant Riemannian metric, is weakly symmetric with respect to G and the corresponding isometry μ induced from θ.*

Proof. Consider the natural projection map $\pi : G \to G/H$. The automorphism θ induces an involutive diffeomorphism μ of the homogeneous manifold G/H:

$$\mu(gH) = \theta(g)H, \quad gH \in G/H.$$

Endow $M = G/H$ with any G-invariant Riemannian metric. Then G becomes a transitive group of isometries acting by left translations on right cosets of H. It follows that μ is an involutive isometry satisfying $\mu G \mu^{-1} = G$ and $\mu^2 \in G$.

Next, we identify $T_o(M)$, the tangent space of $M = G/H$ at the origin $o = eK$, with $\mathfrak{q}$ via π. It follows that the isotropy action of H on $T_o(M)$ is precisely the adjoint action of H on $\mathfrak{q}$. It becomes clear that M is weakly symmetric with respect to G and μ as a result of Lemma 2.2. $\square$

Remark 2.5. It is trivial to see that if G/K is a symmetric space, where K is an open subgroup of G^θ (the fixed point set of the corresponding involution θ), then (G, K, θ) is a weakly symmetric triple. In fact, the adjoint representation of K is not necessary here to reverse tangent vectors.

Lemma 2.6. *Let $\rho : \tilde{G} \to G$ be a covering homomorphism between connected Lie groups, $\tilde{H} \subset G$ a connected subgroup and $H = \rho(\tilde{H})$. Then $\tilde{M} = \tilde{G}/\tilde{H}$ is weakly symmetric with respect to $\tilde{G}$ if and only if $M = G/H$ is weakly symmetric with respect to G.*

Proof. Both $\tilde{G}$ and G have isomorphic Lie algebras $\tilde{\mathfrak{g}}$ and $\mathfrak{g}$, respectively, under $d\rho$. Similarly, both $\tilde{H}$ and H have isomorphic Lie algebras $\tilde{\mathfrak{h}}$ and $\mathfrak{h}$, respectively. It follows that $T_{\tilde{o}}(\tilde{M}) \cong \tilde{\mathfrak{p}} \cong \mathfrak{p} \cong T_o(M)$, where $\tilde{\mathfrak{p}}$ and $\mathfrak{p}$ come from the $Ad_{\tilde{G}/\tilde{H}}(\tilde{H})$- and $Ad_{G/H}(H)$-invariant decompositions $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ and $\tilde{\mathfrak{g}} = \tilde{\mathfrak{h}} + \tilde{\mathfrak{p}}$, respectively. Furthermore, given any $\tilde{h} \in \tilde{H}$ with $h = \rho(\tilde{h})$, the following diagram commutes:

$$
\begin{array}{ccc}
\tilde{\mathfrak{g}} & \xrightarrow{d\rho} & \mathfrak{g} \\
{\scriptstyle Ad_{\tilde{G}/\tilde{H}}(\tilde{h})} \downarrow & & \downarrow {\scriptstyle Ad_{G/H}(h)} \\
\tilde{\mathfrak{g}} & \xrightarrow{d\rho} & \mathfrak{g}
\end{array}
$$

Now, fix $\tilde{v} \in \tilde{\mathfrak{p}}$ and set $v = d\rho(\tilde{v}) \in \mathfrak{p}$. Then $Ad_{\tilde{G}/\tilde{H}}(\tilde{h})(\tilde{v}) = -\tilde{v}$ for some $\tilde{h} \in \tilde{H}$ if and only if $Ad_{G/H}(h)(v) = -v$ for some $h \in H$ with $h = \rho(\tilde{h})$. $\square$

3. Proof of Theorem

As described earlier in the introduction, we shall prove Theorem 1.1 by decomposing the spherical pairs on Kramer's classification list into six families (categorized below) and showing weak symmetry for each family:

I. Symmetric spaces, including $SO(8)/(SU(2) \cdot Sp(2))$.
II. S^1-bundles over hermitian symmetric spaces of nontube type.
III. $SU(2n + 1)/Sp(n)$ and $SU(2n + 1)/(Sp(n) \cdot U(1))$.
IV. Spaces of constant curvature: G_2/A_2, $SO(7)/G_2$ and $SO(8)/Spin(7)$.
V. Spaces of Cayley-type: $SO(10)/(SO(2) \times Spin(7))$, $SO(9)/Spin(7)$ and $SO(8)/G_2$.
VI. $SO(2n + 1)/U(n)$ and $Sp(n)/(Sp(n - 1) \times U(1))$.

3.1. I. If G/K is a symmetric space, then it is obviously weakly symmetric with respect to the corresponding involutive isometry (the group G is not needed here). We note that the homogeneous manifold $SO(8)/(SU(2) \cdot Sp(2))$ appearing on Kramer's list is also a symmetric space (cf. [WZ], Table 3, p. 325).

3.2. II. Let G be a connected semisimple matrix Lie group with finite center and K a maximal compact subgroup of G such that $D = G/K$ is a hermitian symmetric space. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to a Cartan involution σ, where $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebras of G and K, respectively. Then $\mathfrak{k} = \mathfrak{k}_s + \mathfrak{z}_\mathfrak{k}$, where $\mathfrak{k}_s$ is the semisimple part of $\mathfrak{k}$ and $\mathfrak{z}_\mathfrak{k}$ the one-dimensional center of $\mathfrak{k}$. Let K_s be the subgroup of K with Lie algebra $\mathfrak{k}_s$ and Z_K the center of K. Then K_s is connected and $K = K_s Z_K^0$, where Z_K^0 is the connected component of Z_K.

Fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and let A and P be subgroups of G with Lie algebras $\mathfrak{a}$ and $\mathfrak{p}$, respectively. According to Flensted-Jensen [F], there exists an involutive automorphism θ satisfying:

Lemma 3.3. *(i) $\sigma\theta = \theta\sigma$,*
(ii) $\theta(a) = a^{-1}$ for all $a \in A$,
(iii) $\theta(K_s) = K_s$ and $\theta(c) = c^{-1}$ for all $c \in Z_K^0$.

At the Lie algebra level, these properties translate to

(i′) $d\sigma d\theta = d\theta d\sigma$,
(ii′) $d\theta(X) = -X$ for all $X \in \mathfrak{a}$,
(iii′) $\theta(K_s) = K_s$ and $d\theta(X) = -X$ for all $X \in \mathfrak{z}_K$.

Proposition 3.4. *If G/K is not of tube type, then (G, K_s, θ) is a weakly symmetric triple.*

In order to prove this proposition, we shall need some finer structure theory about hermitian symmetric spaces due to Wolf and Korányi [KW]. With their permission, we also present some unpublished results of theirs obtained in the early 1980's [WK].

Extend $\mathfrak{a}$ to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let $Z^J \in \mathfrak{z}_\mathfrak{k}$ be the element which gives $\mathfrak{p}$ a complex structure J corresponding to the root system for $(\mathfrak{g}, \mathfrak{h})$ as described in [KW]. We then split $Z^J = Z^0 + Z'$, where Z^0 defines the complex structure on the polydisc or the polysphere inside G/K and Z' is an element inside $\mathfrak{k}$ which centralizes $\mathfrak{a}$.

Lemma 3.5. *([KW]) G/K is not of tube type if and only if $Z' \neq 0$.*

Lemma 3.6. *([WK]) Let $S' = \{\exp t Z' : t \in \mathbb{R}\}$. If G/K is not of tube type, then*
(i) $Z' \notin \mathfrak{k}_s$,
(ii) $K_s S' = K = S' K_s$.

Proof. (i) We first note that $Z^J \perp \mathfrak{k}_s$. Then $Z^0 \perp Z'$ because Z' centralizes $\mathfrak{a}$ and $Z^0 \in [\mathfrak{a}, J\mathfrak{a}] \subset ad(\mathfrak{a})\mathfrak{g}$. Were $Z' \in \mathfrak{k}_s$, then $0 = < Z^J, Z' > = < Z', Z' >$ and forces $Z' = 0$ (here $< \cdot, \cdot >$ is the Killing form on $\mathfrak{g}$).
(ii) As $Z' \notin \mathfrak{k}_s$, the circle group S' acts nontrivially on the circle K/K_s. Therefore, S' is transitive there and hence $S' K_s = K$. Use the map $k \mapsto k^{-1}$ to obtain $K_s S' = K$. $\square$

Lemma 3.7. *If G/K is not of tube type, then $Ad(K_s)(\mathfrak{a}) = \mathfrak{p}$.*

Proof. It is a standard result that $Ad(K)(\mathfrak{a}) = \mathfrak{p}$. Now write $K = K_s S'$ and use the fact that S' centralizes $\mathfrak{a}$ to see the our lemma immediately follows. $\square$

Proof. (second part of Prop. 3.4) Decompose $\mathfrak{g} = \mathfrak{k}_s + \mathfrak{z}_\mathfrak{k} + \mathfrak{p}$. Write any element $X \in \mathfrak{z}_\mathfrak{k} + \mathfrak{p}$ as $X = Z + Y$ with $Z \in \mathfrak{z}_\mathfrak{k}$ and $Y \in \mathfrak{p}$. By Lemma 3.7, we can write $Y = Ad(h)(W)$ with $h \in K_s$ and $W \in \mathfrak{a}$. Then

$$d\theta(Y) = Ad(\theta(h))(-W).$$

Set $k = h\theta(h)$. Now use the fact that the adjoint action of K commutes with J and is trivial on $\mathfrak{z}_\mathfrak{z}$ to check

$$(Ad(k) \circ d\theta)(X) = -X.$$

This proves that (G, K_s, θ) is a weakly symmetric triple. $\square$

3.8. III. We fix $G = SU(2n+1)$, $K = S(U(2n) \times U(1))$ and $K_s = SU(2n)$ so that G/K is complex projective space $\mathbb{C}P^{2n}$. We choose a Cartan involution σ so that G and K have the following form as matrices (see Helgason [H1]):

$$G = \{g \in SL(2n+1, \mathbb{C}) : {}^t g I_{p,q} \bar{g} = I_{p,q} \text{ and } \det g = 1.\}, \quad I_{p,q} = \begin{pmatrix} I_{2n} & 0 \\ 0 & -1 \end{pmatrix}$$

$$K = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in G : A \in U(2n), D \in U(1) \text{ and } \det A \det D = 1. \right\}$$

$$K_s = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in G : A \in SU(2n). \right\}$$

$$Z_K^0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & e^{-i2n\phi} \end{pmatrix} \in K : A = \text{Diag}(e^{i\phi}, ..., e^{i\phi}), \quad \phi \in [0, 2\pi). \right\}$$

Let $\mathfrak{a} \subset \mathfrak{p}$ be the following:

$$\mathfrak{a} = \left\{ \begin{pmatrix} 0 & iw \\ -iw & 0 \end{pmatrix} : w \in \mathbb{R} \right\}, \text{ where } \mathfrak{p} = \left\{ \begin{pmatrix} 0 & W \\ \bar{W}^t & 0 \end{pmatrix} : W \text{ complex } 2n \times 1 \text{ matrix} \right\}.$$

Then the corresponding involutive automorphism θ of G described in Lemma 3.3 becomes such that $\theta(g)$ is complex conjugation of all the entries of g as a matrix in $SL(n, \mathbb{C})$. As G/K is not of tube type, $(SU(2n+1), SU(2n), \theta)$ is a weakly symmetric triple by Prop. 3.4.

Let $H = Sp(n)$ be a maximal subgroup of $K_s = SU(2n)$ embedded in G as follows so that K_s/H is a symmetric space:

$$H = \left\{ \begin{pmatrix} Sp(n) & 0 \\ 0 & 1 \end{pmatrix} \right\}, \text{ where } Sp(n) = \left\{ \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} : AA^* + BB* = I_n, \ AB^t = BA^t. \right\}$$

Proposition 3.9. $(SU(2n+1), Sp(n), \theta)$ *is a weakly symmetric triple.*

Proof. We first write $\mathfrak{su}(2n+1) = \mathfrak{su}(2n) + \mathfrak{z}_{\mathfrak{k}} + \mathfrak{p}$, where $\mathfrak{su}(2n) = \mathfrak{k}_s$. Then by writing $\mathfrak{su}(2n) = \mathfrak{sp}(n) + \mathfrak{q}$, we can decompose it further as $\mathfrak{su}(2n+1) = \mathfrak{sp}(n) + \mathfrak{q} + \mathfrak{z}_{\mathfrak{k}} + \mathfrak{p}$. It now comes down to proving that given any element $X = V + Z + W \in \mathfrak{q} + \mathfrak{z}_{\mathfrak{k}} + \mathfrak{p}$ with V, Z and W in $\mathfrak{q}$, $\mathfrak{z}_{\mathfrak{k}}$ and $\mathfrak{p}$, respectively, there exists an element $k \in Sp(n)$ such that $(Ad(k) \circ d\theta)(X) = -X$.

The Lie subalgebras $\mathfrak{sp}(n)$ and $\mathfrak{q}$ of $\mathfrak{su}(2n)$ are defined as follows:

$$\mathfrak{sp}(n) = \left\{ \begin{pmatrix} V_1 & V_2 & 0 \\ -\bar{V}_2 & \bar{V}_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} : V_1 \in \mathfrak{u}(n), V_2 \ n \times n \text{ complex matrix.} \right\}$$

$$\mathfrak{q} = \left\{ \begin{pmatrix} V_1 & V_2 & 0 \\ \bar{V}_2 & -\bar{V}_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} : V_1 \in \mathfrak{su}(n), V_2 \in \mathfrak{so}(n, \mathbb{C}). \right\}$$

We define a maximal abelian subalgebra $\mathfrak{b}$ of $\mathfrak{q}$ as follows:

$$\mathfrak{b} = \left\{ \begin{pmatrix} D & 0 & 0 \\ 0 & -\bar{D} & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{q} : D = \text{Diag}(id_1, ..., id_n) \text{ with } d_1, ..., d_n \in \mathbb{R} \text{ and } \sum_{j=1}^{n} d_j = 0. \right\}$$

Notice that $(SU(2n), Sp(n), \theta|_{SU(2n)})$ is a weakly symmetric triple since $\theta|_{SU(2n)}$, the restriction of θ to $SU(2n)$, is also an involutive automorphism with $Sp(n)$ a θ-invariant subgroup and satisfies all the conditions of Lemma 3.3.

As $SU(2n)/Sp(n)$ is a symmetric space, we can write $V = Ad(h)(U)$, where $h \in Sp(n)$ and $U \in \mathfrak{b}$. Then as the adjoint action of $Sp(n)$ is trivial on $\mathfrak{z}_{\mathfrak{k}}$,

$$Ad(h^{-1})(X) = U + Z + W', \quad W' \in \mathfrak{p}.$$

Now, let $N = N_{Sp(n)}(\mathfrak{b})$ be the normalizer of $\mathfrak{b}$ in $Sp(n)$, i.e.

$$N(\mathfrak{b}) = \{l \in Sp(n) : Ad(l)(\mathfrak{b}) \subseteq \mathfrak{b}\}.$$

According to Lemma 3.11 below, there exists an element $l \in N$ such that $Ad(l)(W') = Y$ where $Y \in \mathfrak{a}$. Hence, $Ad(lh^{-1})(X) = U' + Z + Y$ with $U' \in \mathfrak{b}$. In other words,

$$X = Ad(hl^{-1})(U' + Z + Y).$$

As $d\theta = -\text{Id}$ on $\mathfrak{b} + \mathfrak{z}_{\mathfrak{k}} + \mathfrak{a}$, it follows that

$$d\theta(X) = Ad(\theta(hl^{-1}))(-U' - Z - Y).$$

Setting $k = hl^{-1}\theta(hl^{-1}) \in Sp(n)$, we get

$$(Ad(k) \circ d\theta)(X) = -X.$$

$\square$

We also obtain as an obvious result the following corollary:

Corollary 3.10. $(SU(2n+1), Sp(n) \cdot U(1), \theta)$ *is also a weakly symmetric triple.*

Lemma 3.11. $Ad_G(N)(\mathfrak{a}) = \mathfrak{p}$.

Proof. It is clear that $Ad_G(N)(\mathfrak{a}) \subset \mathfrak{p}$. We introduce some notation. Let $E_{i,j}$ denote the $n \times n$ matrix with 1 in the ij entry and 0's everywhere else, $F_{i,j}$ the $n \times n$ matrix obtained from the identity matrix by switching the i-th row with the j-th row, and W_j the $(2n+1) \times 1$ matrix with 1 in the j-th row and 0's everywhere else. Now, decompose $\mathfrak{p} = \sum_{j=1}^{2n} \mathfrak{a}_j$, where $\mathfrak{a}_j$ is the complex subspace spanned by the matrix

$$\begin{pmatrix} 0 & W_j \\ W_j^t & 0 \end{pmatrix}.$$

The following assertions can be easily verified:

(i) Since N contains the diagonal matrices in $Sp(n)$, it follows that $Ad(N)(\mathfrak{a}) \supset \mathfrak{a}_1$.

(ii) Since N contains matrices of the form

$$L_j = \begin{pmatrix} F_{j,j+1} & 0 & 0 \\ 0 & F_{j,j+1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad j = 1, ..., n,$$

it follows that $Ad(L_j)(\mathfrak{a}_j) = \mathfrak{a}_{j+1}$ and $Ad(L_j)(\mathfrak{a}_{n+j}) = \mathfrak{a}_{n+j+1}$.

(iii) Since N contains matrices of the form

$$\tilde{L}_j = \begin{pmatrix} I_n - E_{j,j} & E_{j,j} & 0 \\ -E_{j,j} & I_n - E_{j,j} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad j = 1, ..., n,$$

it follows that $Ad(\tilde{L}_j)(\mathfrak{a}_j) = \mathfrak{a}_{n+j}$.

From these assertions, we conclude that $Ad(N)(\mathfrak{a}) = \mathfrak{p}$. This completes the proof of the lemma and Proposition 3.9. $\square$

3.12. **IV.** Spaces of constant curvature:

(i) $SO(8)/Spin(7)$: We have the double cover $Spin(8)/Spin(7) \to SO(8)/Spin(7)$. Since we have that $Spin(8)/Spin(7) \cong S^7$ is a space of constant curvature and the isotropy action is transitive on the unit tangent sphere, it follows that $Spin(8)/Spin(7)$ is weakly symmetric with respect $Spin(8)$. Hence, $SO(8)/Spin(7)$ is weakly symmetric with respect to $SO(8)$ by Lemma 2.6.

(ii) $SO(7)/G_2$: Again, we have the double cover $Spin(7)/G_2 \to SO(7)/G_2$. Now, $Spin(7)/G_2 \cong S^7$ is a space of constant curvature and therefore weakly symmetric with respect to G_2. Hence, $SO(7)/G_2$ is weakly symmetric with respect to $SO(7)$.

(iii) G_2/A_2: From [Wo], Prop. 8.12.7, it is known that $G_2/A_2 = S^6$ is a space of constant curvature and its isotropy action is transitive on the unit tangent sphere. Hence, G_2/A_2 is weakly symmetric with respect to G_2.

3.13. **V.** Spaces of Cayley-type:

(i) $SO(10)/(SO(2) \times Spin_\pm(7))$: Following [Ta], Theorem 5.4, we have the decomposition

$$\mathfrak{so}(10)/(\mathfrak{spin}(7) \oplus \mathfrak{so}(2)) = \mathbb{R}^7 \oplus (\mathbb{R}^8 \otimes \mathbb{R}^2).$$

Here, the adjoint representation of $Spin_\pm(7) \times SO(2)$ on the factor $\mathbb{R}^7$ acts through the standard representation of $Spin(7)$ on $\mathbb{R}^7$ (the action of $SO(2)$ is trivial) and on the factor $\mathbb{R}^8 \otimes \mathbb{R}^2$ through the spin representation of $Spin(7)$ on $\mathbb{R}^8$ and by rotation of $SO(2)$ on $\mathbb{R}^2$. Furthermore, if $v \in \mathbb{R}^7$, then the isotropy subgroup $(Spin(7) \times SO(2))_v = SU(4) \times SO(2)$, where $SU(4)$ acts on $\mathbb{R}^8 \cong \mathbb{C}^4$ by the standard action.

Table 3.14 - Weakly symmetric homogeneous manifolds of a connected compact simple Lie group

Family	\multicolumn{2}{c}{$M = G/H$ weakly symmetric}		with respect to
	G	H	for	
I	Symmetric spaces with involutive isometry μ			μ
II	S^1-bundles over hermitian symmetric spaces:			
	$SU(n+m)$	$SU(m) \times SU(m)$	$n > m \geq 1$	G and μ
	$SO(2n)$	$SU(n)$	$n \geq 3$, n odd	G and μ
	E_6	D_5		G and μ
III	$SU(2n+1)$	$Sp(n)$	$n \geq 1$	G and μ
	$SU(2n+1)$	$Sp(n) \cdot U(1)$	$n \geq 1$	G and μ
IV	Spaces of constant curvature:			
	$SO(8)$	$Spin(7)$		G
	$SO(7)$	G_2		G
	G_2	A_2		G
V	Spaces of Cayley-type:			
	$SO(10)$	$SO(2) \times Spin(7)$		G
	$SO(9)$	$Spin(7)$		G
	$SO(8)$	G_2		G
VI	$SO(2n+1)$	$U(n)$	$n \geq 2$	G
	$Sp(n)$	$Sp(n-1) \times U(1)$	$n \geq 1$	G

Therefore, given any vector $v + w \in \mathbb{R}^7 \oplus (\mathbb{R}^8 \otimes \mathbb{R}^2)$, there exists an element $k \in Spin(7)$ which sends v to its negative. Next, we write $w = \sum_{i=1}^{p} e_i \otimes f_j$, where $e_i \in \mathbb{R}^8$ and $f_j \in \mathbb{R}^2$. As $SO(2)$ acts by rotations, we use the rotation $e^{i\pi}$ to reverse each f_j and therefore send w to $-w$. Together, the element $(k, e^{i\pi})$ in $Spin(7) \times SO(2)$ will then send $v+w$ to $-(v+w)$. This proves weak symmetry for $SO(10)/(SO(2) \times Spin(7))$.

(ii) $SO(9)/Spin(7)$ and $SO(8)/G_2$: According to [Mu], there exists double covers

$$Spin(9)/Spin(7) \to SO(9)/Spin(7) \quad \text{and} \quad Spin(8)/G_2 \to SO(8)/G_2.$$

But $Spin(9)/Spin(7)$ and $Spin(8)/G_2$ are already known to be weakly symmetric with respect to $Spin(9)$ and $Spin(8)$, respectively, from [Z], pp. 357 and 361.

3.14. **VI.** $SO(2n+1)/SU(n)$ and $Sp(n)/(U(1) \times Sp(n-1))$: Again, these spaces have already been proven to be weakly symmetric in [Z], pp. 359-360.

This concludes the proof of our main theorem. Table 3.14 is a summary of our results with appropriate descriptions of weak symmetry for each family.

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