MOMENTS OF HYPERGEOMETRIC HURWITZ ZETA FUNCTIONS

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Abstract. This paper investigates a generalization the classical Hurwitz zeta function. It is shown that many of the properties exhibited by this special function extends to class of functions called hypergeometric Hurwitz zeta functions, including their analytic continuation to the complex plane and a pre-functional equation satisfied by them. As an application, a formula for moments of hypergeometric Hurwitz zeta functions is developed, extending the formula by Espinosa and Moll for moments of the classical Hurwitz zeta function.

1. INTRODUCTION

In our paper [2], we investigated a family of special functions called hypergeometric zeta functions as a generalization of the Riemann zeta function. In this paper, we investigate a generalization of the Hurwitz zeta function, which we refer to as *hypergeometric Hurwitz zeta functions*, defined by

$$\zeta_N(s,a) = \frac{1}{\Gamma(s+N-1)} \int_0^\infty \frac{x^{s+N-2} e^{(1-a)x}}{e^x - T_{N-1}(x)}\, dx \qquad (0 < a \leq 1). \tag{1.1}$$

Here, N is a positive integer and $T_{N-1}(x)$ is the $(N-1)$-order Taylor polynomial of e^x. Observe that for $N=1$, $\zeta_1(s,a) = \zeta(s,a)$, where

$$\zeta(s,a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1} e^{(1-a)x}}{e^x - 1}\, dx \tag{1.2}$$

is the classical Hurwitz zeta function. Following Riemann, we develop the analytic continuation of $\zeta_N(s,a)$ to the entire complex plane, except for N simple poles at $s = 1, 0, -1, \cdots, 2-N$, and establish many properties analogous to those satisfied by the classical Hurwitz zeta function as was done in [2] for hypergeometric zeta functions.

In section 2, we formally define hypergeometric Hurwitz zeta functions, establish their convergence on a right half-plane, and develop their series representations. In section 3, we reveal their analytic continuation to the entire complex plane, except at a finite number of poles, and calculate their residues in terms of generalized Bernoulli numbers. In section 4, we establish a series formula (called a pre-functional equation) for hypergeometric Hurwitz zeta functions that is valid on a left half-plane and use it to establish a formula involving their moments, extending a formula for moments of the classical Hurwitz zeta function given by Espinosa and Moll in [1].

2. PRELIMINARIES

In this section we formally define hypergeometric zeta functions, establish a domain of convergence, and demonstrate their series representations.

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Definition 2.1. Denote the Maclaurin (Taylor) polynomial of the exponential function e^x by

$$T_N(x) = \sum_{k=0}^{N} \frac{x^k}{k!}.$$

We define the N^{th}*-order hypergeometric Hurwitz zeta function* (or just *hypergeometric Hurwitz zeta function* for short) to be

$$\zeta_N(s,a) = \frac{1}{\Gamma(s+N-1)} \int_0^\infty \frac{x^{s+N-2} e^{(1-a)x}}{e^x - T_{N-1}(x)}\, dx \qquad (N \geq 1, \ \ 0 < a \leq 1). \tag{2.1}$$

Lemma 2.1. *$\zeta_N(s,a)$ converges absolutely for $\sigma = \Re(s) > 1$.*

Proof. Choose $0 < \alpha < 1$ such that $a + \alpha > 1$. Let $K > 0$ be such that $e^x \geq e^{\alpha x} + T_{N-1}(x)$ for all $x \geq K$. This is equivalent to $e^x - T_{N-1}(x) \geq e^{\alpha x}$. For $\sigma > 1$, we have

$$\begin{aligned} |\zeta_N(s,a)| &\leq \frac{1}{|\Gamma(s+N-1)|} \left[\int_0^K \left| \frac{e^{(1-a)x} x^{s+N-2}}{e^x - T_{N-1}(x)} \right| dx + \int_K^\infty \left| \frac{e^{(1-a)x} x^{s+N-2}}{e^x - T_{N-1}(x)} \right| dx \right] \\ &\leq \frac{1}{|\Gamma(s+N-1)|} \left[\int_0^K \frac{e^{(1-a)x} x^{\sigma+N-2}}{x^N/N!} dx + \int_K^\infty x^{\sigma+N-2} e^{(1-a-\alpha)x} dx \right]. \end{aligned}$$

The first integral is finite and since $1 - a - \alpha < 0$, the second integral is convergent. This proves our lemma. ☐

The next lemma establishes a series representation for $\zeta_N(s,a)$.

Lemma 2.2. *For $\sigma > 1$, we have*

$$\zeta_N(s,a) = \sum_{n=0}^{\infty} f_n(N,s,a), \tag{2.2}$$

where

$$f_n(N,s,a) = \frac{1}{\Gamma(s+N-1)} \int_0^\infty x^{s+N-2} T_{N-1}^n(x) e^{-(n+a)x}\, dx. \tag{2.3}$$

Proof. Since $|T_{N-1}(x)e^{-x}| < 1$ for all $x > 0$, we can rewrite the integrand in (2.1) as a geometric series:

$$\frac{x^{s+N-2} e^{(1-a)x}}{e^x - T_{N-1}(x)} = \frac{e^{-ax} x^{s+N-2}}{1 - T_{N-1}(x)e^{-x}} = e^{-ax} x^{s+N-2} \sum_{n=0}^{\infty} \left[T_{N-1}(x) e^{-x} \right]^n = x^{s+N-2} \sum_{n=0}^{\infty} T_{N-1}^n(x) e^{-(n+a)x}.$$

The lemma now follows by reversing the order of integration and summation because of Dominated Convergence Theorem:

$$\begin{aligned} \zeta_N(s,a) &= \frac{1}{\Gamma(s+N-1)} \int_0^\infty x^{s+N-2} \sum_{n=0}^{\infty} T_{N-1}^n(x) e^{-(n+a)x}\, dx \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{\Gamma(s+N-1)} \int_0^\infty x^{s+N-2} T_{N-1}^n(x) e^{-(n+a)x}\, dx \right]. \end{aligned}$$

☐

Unfortunately, the coefficients $f_n(N,s,a)$ given by (2.3) have no closed form in general. However, in the special case where $s = 1$, we find that $f_n(N,1,a)$ is harmonic so that $\zeta_N(s,a)$ has the same formal harmonic series representation as the classical Hurwitz zeta function at $s = 1$. This is the content of the next lemma.

Lemma 2.3. *For $f_n(N,s,a)$ given by (2.3), we have*

$$f_n(N,1,a)=\frac{1}{n+a}. \tag{2.4}$$

Proof. Since $x^{N-1}=(N-1)!\,[T_{N-1}(x)-T_{N-2}(x)]$, it follows that

$$\begin{aligned} f_n(N,1,a) &= \frac{1}{(N-1)!}\int_0^\infty x^{N-1}T_{N-1}^n(x)e^{-(n+a)x}\,dx \\ &= \int_0^\infty T_{N-1}^{n+1}(x)e^{-(n+a)x}\,dx-\int_0^\infty T_{N-2}(x)T_{N-1}^n(x)e^{-(n+a)x}\,dx. \end{aligned}$$

But the two integrals above merely differ by $1/(n+a)$, which results from integrating by parts:

$$\int_0^\infty T_{N-1}^{n+1}(x)e^{-(n+a)x}\,dx=\frac{1}{n+a}+\int_0^\infty T_{N-2}(x)T_{N-1}^n(x)e^{-(n+a)x}\,dx.$$

This establishes the lemma. □

We are now ready to describe a 'Dirichlet series' representation of hypergeometric Hurwitz zeta functions.

Lemma 2.4. *For $\Re(s)=\sigma>1$, we have*

$$\zeta_N(s,a)=\sum_{n=0}^\infty\frac{\mu_N(n,s,a)}{(n+a)^{s+N-1}}, \tag{2.5}$$

where

$$\mu_N(n,s,a)=\sum_{k=0}^{n(N-1)}\frac{a_k(N,n)}{(n+a)^k}(s+N-1)_k. \tag{2.6}$$

Here $a_k(N,n)$ is generated by

$$\left(T_{N-1}(x)\right)^n=\left(\sum_{k=0}^{N-1}\frac{x^k}{k!}\right)^n=\sum_{k=0}^{n(N-1)}a_k(N,n)x^k. \tag{2.7}$$

Proof. With $a_k(N,n)$ as defined in (2.7), we rewrite $f_n(N,s,a)$ as follows:

$$\begin{aligned} f_n(N,s,a) &= \frac{1}{\Gamma(s+N-1)}\int_0^\infty x^{s+N-2}T_{N-1}^n(x)e^{-(n+a)x}\,dx \\ &= \frac{1}{\Gamma(s+N-1)}\int_0^\infty x^{s+N-2}\left(\sum_{k=0}^{a(N-1)}a_k(N,n)x^k\right)e^{-(n+a)x}\,dx \\ &= \frac{1}{\Gamma(s+N-1)}\frac{1}{n^{s+N-1}}\int_0^\infty\left(\sum_{k=0}^{n(N-1)}a_k(N,n)\frac{x^{s+k+N-2}}{(n+a)^k}\right)e^{-x}\,dx \\ &= \frac{1}{\Gamma(s+N-1)}\frac{1}{n^{s+N-1}}\left(\sum_{k=0}^{n(N-1)}\frac{a_k(N,n)}{(n+a)^k}\int_0^\infty x^{s+k+N-2}e^{-x}\,dx\right) \end{aligned}$$

Then reversing the order of the inner summation with the integration, we obtain

$$\begin{aligned} f_n(N,s,a) &= \frac{1}{\Gamma(s+N-1)}\frac{1}{n^{s+N-1}}\left(\sum_{k=0}^{n(N-1)}\frac{a_k(N,n)}{(n+a)^k}\int_0^\infty x^{s+k+N-2}e^{-x}\,dx\right) \\ &= \frac{1}{n^{s+N-1}}\left(\sum_{k=0}^{n(N-1)}\frac{a_k(N,n)}{(n+a)^k}\frac{\Gamma(s+N+k-1)}{\Gamma(s+N-1)}\right) \\ &= \frac{1}{(n+a)^{s+N-1}}\left(\sum_{k=0}^{n(N-1)}\frac{a_k(N,n)}{(n+a)^k}(s+N-1)_k\right) \\ &= \frac{\mu_N(n,s,a)}{(n+a)^{s+N-1}}. \end{aligned}$$

Formula (2.5) now follows from (2.2). □

We note that the coefficients $\mu_N(n,s,a)$ have no closed form in general (at least the authors are not aware of one); however, for the particular case where $s=1$, the following formula holds.

Lemma 2.5.

$$\mu_N(n,1,a) = \sum_{k=0}^{n(N-1)}\frac{a_k(N,n)}{(n+a)^k}(N)_k = (n+a)^{N-1} \tag{2.8}$$

Proof. Since $f_n(N,s,a) = \mu_N(n,s,a)/(n+a)^{s+N-1}$ holds even at $s=1$, we have from (2.4) that

$$\mu_N(n,1,a) = (n+a)^N f_n(N,1,a) = (n+a)^{N-1}.$$

□

3. ANALYTIC CONTINUATION

In this section we follow Riemann by using contour integration to develop the analytic continuation of $\zeta_N(s,a)$. As in [2], we consider the contour integral

$$I_N(s,a) = \frac{1}{2\pi i}\int_\gamma \frac{(-w)^{s+N-1}e^{(1-a)w}}{e^w - T_{N-1}(w)}\frac{dw}{w}, \tag{3.1}$$

where the contour γ is taken to be along the real axis from ∞ to $\delta > 0$, then counterclockwise around the circle of radius δ, and lastly along the real axis from δ to ∞ (cf. Figure 1). Moreover, we let $-w$ have argument $-\pi$ backwards along ∞ to δ and argument π when going to ∞. Also, we choose the radius δ to be sufficiently small (depending on N) so that there are no roots of $e^w - T_{N-1}(w) = 0$ inside the circle of radius δ besides the trivial root $z_0 = 0$. This follows from the fact that $z_0 = 0$ is an isolated zero. It is then clear from this assumption that $I_N(s,a)$ must converge for all complex s and therefore defines an entire function.

Remark 3.1.

(a) To be precise the contour γ should be taken as a limit of contours γ_ϵ as $\epsilon \to 0$, where the portions running along the x-axis are positioned at heights $\pm\epsilon$. Moreover, the poles of the integrand in (3.1) cannot accumulate inside this strip due to the asymptotic exponential growth of the zeros of $e^w - T_{N-1}(w) = 0$ (see [2]).

(b) Since we are most interested in the properties of $I_N(s,a)$ in the limiting case when $\delta \to 0$, we will also write $I_N(s,a)$ to denote $\lim_{\delta\to 0} I_N(s,a)$.

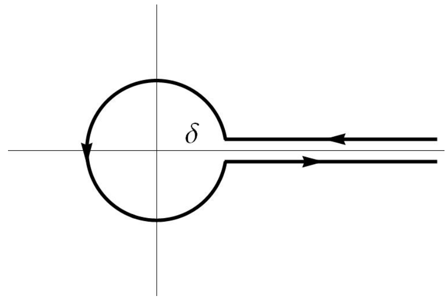

FIGURE 1. Contour γ.

We begin by evaluating $I_N(s,a)$ at integer values of s. To this end, we decompose it as follows:

$$\begin{aligned} I_N(s,a) &= \frac{1}{2\pi i}\int_\infty^\delta \frac{e^{(1-a)x}e^{(s+N-1)(\log x-\pi i)}}{e^x-T_{N-1}(x)}\frac{dx}{x} \\ &\quad + \frac{1}{2\pi i}\int_{|w|=\delta}\frac{(-w)^{s+N-1}e^{(1-a)w}}{e^w-T_{N-1}(w)}\frac{dw}{w} \\ &\quad + \frac{1}{2\pi i}\int_\delta^\infty \frac{e^{(1-a)x}e^{(s+N-1)(\log x+i\pi)}}{e^x-T_{N-1}(x)}\frac{dx}{x}. \end{aligned} \tag{3.2}$$

Now, for integer $s=n$, the two integrations along the real axis in (3.2) cancel and we are left with just the middle integral around the circle of radius δ:

$$I_N(n,a) = \frac{1}{2\pi i}\int_{|w|=\delta}\frac{(-w)^{n+N-1}e^{(1-a)w}}{e^w-T_{N-1}(w)}\frac{dw}{w}.$$

Since the expression $e^{(1-a)w}w^N\left(e^w-T_{N-1}(w)\right)^{-1}$ inside the integrand has a removable singularity at the origin, it follows by Cauchy's Theorem that for integers $n>1$,

$$I_N(n,a)=0.$$

For integers $n\le 1$, we consider the power series expansion

$$\frac{w^N e^{(1-a)w}/N!}{e^w-T_{N-1}(w)} = \sum_{m=0}^\infty \frac{B_{N,m}(1-a)}{m!}w^m. \tag{3.3}$$

It now follows from the Residue Theorem that

$$\begin{aligned} I_N(n,a) &= \frac{1}{2\pi i}\int_{|w|=\delta}\frac{(-w)^{n+N-1}e^{(1-a)w}}{e^w-T_{N-1}(w)}\frac{dw}{w} \\ &= (-1)^{n+N-1}\frac{N!}{2\pi i}\int_{|w|=\delta}\left(\sum_{m=0}^\infty \frac{B_{N,m}(1-a)}{m!}w^m\right)\frac{dw}{w^{2-n}} \\ &= \frac{(-1)^{n+N-1}N!B_{N,1-n}(1-a)}{(1-n)!}. \end{aligned} \tag{3.4}$$

We now express $\zeta_N(s,a)$ in terms of $I_N(s,a)$. For $\Re(s)=\sigma>1$, the middle integral in (3.2) goes to zero as $\delta\to 0$. It follows that

$$\begin{aligned} I_N(s,a) &= \left(\frac{e^{\pi i(s+N-1)}-e^{-\pi i(s+N-1)}}{2\pi i}\right)\int_0^\infty \left(e^x - T_{N-1}(x)\right)^{-1} e^{(1-a)x}x^{s+N-2}dx \\ &= \frac{\sin\left[\pi(s+N-1)\right]}{\pi}\Gamma(s+N-1)\zeta_N(s,a). \end{aligned}$$

Now, by using the functional equation for the gamma function:

$$\Gamma(1-(s+N-1))\Gamma(s+N-1)=\frac{\pi}{\sin[\pi(s+N-1)]}$$

we obtain

$$\zeta_N(s,a)=\Gamma(1-(s+N-1))I_N(s,a). \tag{3.5}$$

Remark 3.2. Equation (3.5) and the fact that $0<a<1$ imply that the zeros of $I_N(s,a)$ at positive integers $n>1$ are simple since we know by definition (1.1) that $\zeta_N(n,a)>1$ for $n>1$.

We close this section by proving

Theorem 3.1. *$\zeta_N(s,a)$ is analytic on the entire complex plane except for simple poles at $\{2-N,3-N,\cdots,1\}$ whose residues are*

$$Res\left(\zeta_N(s,a),s=n\right)=(2-n)\binom{N}{2-n}B_{N,1-n}(1-a) \qquad (2-N\le n\le 1). \tag{3.6}$$

Furthermore, for negative integers n less than $2-N$, we have

$$\zeta_N(n,a)=(-1)^{-n-N+1}\binom{1-n}{N}^{-1}B_{N,1-n}(1-a). \tag{3.7}$$

Proof. Since $\Gamma(1-(s+N-1))$ has only simple poles at $s=2-N,3-N,\cdots$, and $I_N(s,a)$ has simple zeros at $s=2,\ 3,\cdots$, it follows from (3.5) that $\zeta_N(s,a)$ is analytic on the whole plane except for simple poles at $s=n$, $2-N\le n\le 1$. Recalling the fact that the residue of $\Gamma(s)$ at negative integer n is $(-1)^n/|n|!$, it follows from (3.4) that

$$\begin{aligned} \mathrm{R}es\left(\zeta_N(s,a),s=n\right) &= \lim_{s\to n}(s-n)\zeta_N(s,a) \;=\; \lim_{s\to n}\left[(s-n)\Gamma(1-(s+N-1))I_N(s,a)\right] \\ &= -\frac{(-1)^{2-N-n}}{(2-N-n)!}I_N(n,a) \;=\; -\frac{(-1)^{2-N-n}}{(2-N-n)!}\frac{(-1)^{n+N-1}N!B_{N,1-n}(1-a)}{(1-n)!} \\ &= (2-n)\binom{N}{2-n}B_{N,1-n}(1-a), \end{aligned}$$

which proves (3.6). For $n<2-N$, (3.4), (3.5), and the fact that $\Gamma(1-(n+N-1))=(1-N-n)!$ imply

$$\begin{aligned} \zeta_N(n,a) &= \Gamma(1-(n+N-1)I_N(n,a)=\frac{(-1)^{n+N-1}N!(1-N-n)!B_{N,1-n}(1-a)}{(1-n)!} \\ &= (-1)^{-n-N+1}\binom{1-n}{N}^{-1}B_{N,1-n}(1-a), \end{aligned}$$

which is (3.7) . This completes the proof of the theorem. $\square$

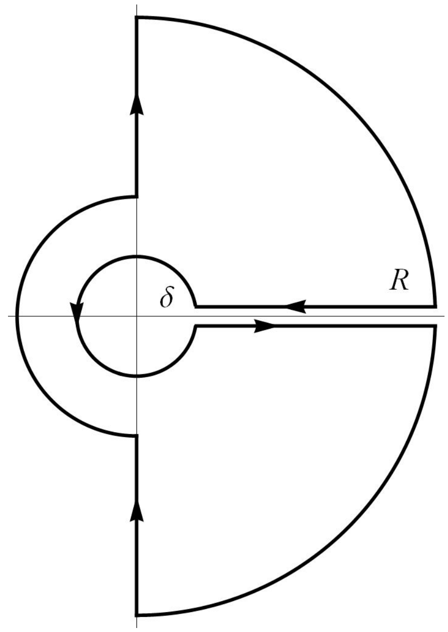

FIGURE 2. Contour γ_M.

4. MOMENTS OF HYPERGEOMETRIC HURWITZ ZETA FUNCTIONS

In the present section, we discuss a 'pre-functional equation' satisfied by $\zeta_N(s,a)$ and use it to derive a formula for its moments. Let γ_R be the contour shown in figure 2 where the outer circular region is part of a circle of radius $R = (2M+1)\pi$, (M is a positive integer so that the poles of the integrand are not on the contour), the inner circle has radius $\delta < 1$, the vertical line is $\Re(z) = -1$. The outer semi-circle is traversed clockwise, the imaginary axis is traversed from bottom to top, the inner circle counterclockwise and the radial segment along the positive real axis is traversed in both directions. Then define

$$I_{\gamma_R}(s,a) = \frac{1}{2\pi i}\int_{\gamma_R} \frac{(-z)^{s+N-1}e^{(1-a)z}}{e^z - T_{N-1}(z)}\frac{dz}{z}. \tag{4.1}$$

We claim that $I_{\gamma_R}(s,a)$ converges to $I_N(s,a)$ as $R\to\infty$ for $\Re(s) < 0$. To prove this, observe that the portion of $I_{\gamma_R}(s,a)$ around the outer circle and the imaginary axis tends to zero as $R\to\infty$ on the same domain. To prove this we first choose a constant $P > 0$ such that

$$A|z|^{N-1} \le |T_{N-1}(z)| \le B|z|^{N-1}, \quad \text{for all} \quad |z| > P.$$

But then on the outer circle defined by $|z| = |R(\cos\theta + i\sin\theta)| = (2M+1)\pi$, if we choose $R > P$ then

$$|e^z - T_{N-1}| \ge \eta\left(e^x - C|z|^{N-1}\right),$$

where $\eta = \pm 1$ and $C = A$ or B. Thus

$$\left|\frac{z^{N-1}e^{(1-a)z}}{e^z - T_{N-1}(z)}\right| \le \frac{e^{(1-a)x}|z|^{N-1}}{\eta\left(e^x - C|z|^{N-1}\right)} = \frac{R^{N-1}e^{-aR\cos\theta}}{\eta\left(1 - CR^{N-1}e^{-R\cos\theta}\right)}$$

converges to zero as $R\to\infty$, since $-\pi/2 < \theta < \pi/2$. On the imaginary axis $z = iy$, we have

$$\left|\frac{z^{N-1}e^{(1-a)z}}{e^z - T_{N-1}(z)}\right| \le \frac{|y|^{N-1}}{A|y|^{N-1} - 1}$$

which converges to $1/A$ as $|y| \to \infty$. Since $|(-z)^s/z| < |z|^{\Re(s)-1}$ and $\Re(s) < 0$, we conclude that the integrals on the outer circle and the vertical lines both converge to zero as $R \to \infty$. On the circle of radius 2δ the integrand is bounded and hence the integral vanishes as $\delta \to 0$.

$$I_N(s,a) = \lim_{R\to\infty} I_{\gamma_R}(s,a). \tag{4.2}$$

On the other hand, we have by residue theory

$$I_{\gamma_R}(s,a) = -\sum_{k=1}^{K}\left[\mathrm{Res}\left(\frac{(-z)^{s+N-2}e^{(1-a)z}}{e^z - T_{N-1}(z)}, z = z_k\right) + \mathrm{Res}\left(\frac{(-z)^{s+N-2}e^{(1-a)z}}{e^z - T_{N-1}(z)}, z = \bar{z}_k\right)\right]. \tag{4.3}$$

Here, $z_k = r_k e^{i\theta_k}$ and $\bar{z}_k = r_k e^{-i\theta_k}$ are the complex conjugate roots of $e^z - T_{N-1}(z) = 0$ and $K = K_M$ is the number of roots inside γ_R in the upper-half plane. Clearly z_k depends on N. We will make this assumption throughout and use the same notation z_k instead of the more cumbersome notation $z_k(N)$. Moreover, we arrange the roots in ascending order so that $|z_1| < |z_2| < |z_3| < \cdots$, since none of the roots can have the same length (see [2]). Now, to evaluate the residues, we call upon Cauchy's Integral Formula:

$$\mathrm{Res}\left(\frac{(-z)^{s+N-2}e^{(1-a)z}}{e^z - T_{N-1}(z)}, z = z_k\right) = (-z_k)^{s+N-2}e^{(1-a)z_k}\lim_{z\to z_k}\frac{z - z_k}{e^z - T_{N-1}(z)}.$$

Since

$$\lim_{z\to z_k}\frac{z - z_k}{e^z - T_{N-1}(z)} = \frac{1}{e^{z_k} - T_{N-2}(z_k)} = \frac{(N-1)!}{z_k^{N-1}},$$

it follows that

$$\mathrm{Res}\left(\frac{(-z)^{s+N-2}e^{(1-a)z}}{e^z - T_{N-1}(z)}, z = z_k\right) = (-1)^{N-1}(N-1)!(-z_k)^{s-1}e^{(1-a)z_k}.$$

Therefore,

$$I_{\gamma_R}(s,a) = (-1)^{N-1}(N-1)!\sum_{k=1}^{K}\left[(-z_k)^{s-1}e^{(1-a)z_k} + (-\bar{z}_k)^{s-1}e^{(1-a)\bar{z}_k}\right]. \tag{4.4}$$

Since $K \to \infty$ as $R \to \infty$, we have by (4.2) and (4.4),

$$\begin{aligned} I_N(s,a) &= \lim_{R\to\infty} I_{\gamma_R}(s,a) \\ &= 2(-1)^{N-1}(N-1)!\sum_{k=1}^{\infty}\left[(-z_k)^{s-1}e^{(1-a)z_k} + (-\bar{z}_k)^{s-1}e^{(1-a)\bar{z}_k}\right]. \end{aligned} \tag{4.5}$$

Combining (3.5) and (4.5) we have proved the following *pre-functional* equation for $\zeta_N(s,a)$.

Theorem 4.1. *For $\Re(s) < 0$,*

$$\zeta_N(s,a) = 2(-1)^{N-1}(N-1)!\Gamma(1-(s+N-1))\sum_{k=1}^{\infty}\left[(-z_k)^{s-1}e^{(1-a)z_k} + (-\bar{z}_k)^{s-1}e^{(1-a)\bar{z}_k}\right]. \tag{4.6}$$

As an application of our pre-functional equation, we consider moments of hypergeometric Hurwitz zeta functions. Our main result, Theorem 4.2 below, extends the formula for moments of the classical Hurwitz zeta functions given by Espinosa and Moll ([1], Theorem 3.7). The following lemma will be useful in our proof of the theorem.

Lemma 4.1. *If z_0 is a root of $e^z - T_{N-1}(z) = 0$, then*

$$\int_0^1 a^M e^{(1-a)z_0} da = \begin{cases} 0 & \text{if } M = N-1, \\ M! \displaystyle\sum_{n=M+1}^{N} \frac{z_0^{n-M-1}}{n!} & \text{if } M < N-1, \\ -M! \displaystyle\sum_{n=N}^{M} \frac{z_0^{n-M-1}}{n!} & \text{if } M \geq N. \end{cases} \tag{4.7}$$

Proof. Integration by parts yields

$$\int_0^1 a^M e^{(1-a)z_k} da = M! \frac{e^{z_k} - T_M(z_0)}{z_k^{M+1}} = \begin{cases} M! \dfrac{e^{z_0} - T_{N-1}(z_0)}{z_0^{M+1}} & \text{if } M = N-1, \\ M! \dfrac{e^{z_0} - T_{N-1}(z_0) + \sum_{n=M+1}^{N} \frac{z_0^n}{n!}}{z_0^{M+1}} & \text{if } M < N-1, \\ M! \dfrac{e^{z_0} - T_{N-1}(z_0) - \sum_{n=N}^{M} \frac{z_0^n}{n!}}{z_0^{M+1}} & \text{if } M \geq N. \end{cases} \tag{4.8}$$

The lemma now follows from the fact that $e_0^z - T_{N-1}(z_0) = 0$. □

We now state our main result.

Theorem 4.2. *Let $N > 0$ and $M > 0$. Then for $\Re(s) < -N$,*

$$\int_0^1 a^M \zeta_N(s,a) da = \begin{cases} 0 & \text{if } M = N-1, \\ \displaystyle\sum_{n=M+1}^{N} \frac{(-1)^{M-n-1} M! \Gamma(1-(s+N-1))}{n! \Gamma(1-(s+N+n-M-2))} \zeta_N(s+n-M-1) & \text{if } M < N-1, \\ \displaystyle\sum_{n=N}^{M} \frac{(-1)^{M-n} M! \Gamma(1-(s+N-1))}{n! \Gamma(1-(s+N+n-M-2))} \zeta_N(s+n-M-1) & \text{if } M \geq N. \end{cases} \tag{4.9}$$

Proof. Denote by $A = 2(-1)^{N-1}(N-1)!\Gamma(1-(s+N-1))$. Then

$$\begin{aligned} \int_0^1 a^M \zeta_N(s,a) da &= A \int_0^1 a^M \sum_{k=1}^{\infty} \left[(-z_k)^{s-1} e^{(1-a)z_k} + (-\bar{z}_k)^{s-1} e^{(1-a)\bar{z}_k} \right] da && (4.10) \\ &= A \sum_{k=1}^{\infty} \left[(-z_k)^{s-1} \int_0^1 a^M e^{(1-a)z_k} da + (-\bar{z}_k)^{s-1} \int_0^1 a^M e^{(1-a)\bar{z}_k} da \right] && (4.11) \end{aligned}$$

To justify the interchange of the sum and the integral above, we note that $z_k = x_k + iy_k$ with $x_k \geq 0$ and $y_k \geq 0$. Then

$$\begin{aligned} \left| (-z_k)^{s-1} e^{(1-a)z_k} \right| &= |z_k|^{\sigma-1} \left| e^{z_k} \right| e^{-ax_k} \\ &\leq |z_k|^{\sigma-1} \left| T_{N-1}(z_k) \right| \\ &\leq \beta \left| z_k \right|^{\sigma-1} \left| z_k \right|^{N-1} \end{aligned}$$

where we have used the fact that z_k is a root of $e^z - T_{N-1}(z) = 0$ and $|T_{N-1}(z_k)| \leq \beta |z_k|^{N-1}$ for some positive constant β.

Now, recall from the work in [2] that $|z_k| \geq \alpha k$ for some positive constant α. Consequently,

$$\left| (-z_k)^{s-1} e^{(1-a)z_k} \right| < \alpha^{\sigma-1} \beta k^{\sigma+N-2} := M_k$$

for all z_k and $\sigma + N - 2 < 0$. Since

$$\sum_{k=1}^{\infty} M_k = \alpha^{\sigma-1}\beta\zeta(2-N-\sigma) < \infty,$$

we see that this infinite series is uniformly convergent and hence the interchange mentioned previously is admissible.

Next, we apply Lemma (4.1) to the roots z_k to obtain the formula

$$\int_0^1 a^M e^{(1-a)z_k} da = \begin{cases} 0 & \text{if } M = N-1, \\ M! \displaystyle\sum_{n=M+1}^{N} \frac{z_k^{n-M-1}}{n!} & \text{if } M < N-1, \\ -M! \displaystyle\sum_{n=N}^{M} \frac{z_k^{n-M-1}}{n!} & \text{if } M \geq N. \end{cases} \tag{4.12}$$

A similar integral formula holds for the roots $\bar{z}_k$. Then substituting these formulas into (4.11) yields the lemma. For example, if $M \geq N$, then

$$\begin{aligned} \int_0^1 a^M \zeta_N(s,a) da &= A \sum_{k=1}^{\infty} \left\{ \sum_{n=N}^{M} (-1)^{M-n} \frac{M!}{n!} \left[(-z_k)^{s+n-M-2} + (-\bar{z}_k)^{s+n-M-2} \right] \right\} \\ &= \sum_{n=N}^{M} \frac{(-1)^{M-n} M! \Gamma(1-(s+N-1))}{n! \Gamma(1-(s+N+n-M-2))} \left\{ \sum_{k=1}^{\infty} \left[(-z_k)^{s+n-M-2} + (-\bar{z}_k)^{s+n-M-2} \right] \right\} \\ &= \sum_{n=N}^{M} \frac{(-1)^{M-n} M! \Gamma(1-(s+N-1))}{n! \Gamma(1-(s+N+n-M-2))} \zeta_N(s+n-M-1) \end{aligned} \tag{4.13}$$

The argument for the cases $M = N-1$ and $M < N-1$ are completely analogous. □

Thus, we have shown in this paper that many of the properties satisfied by the classical Hurwitz zeta function extends to hypergeometric Hurwitz zeta functions. However, the interesting and more challenging problem of finding a functional equation for the latter remains open, as is the case for hypergeometric zeta functions.

References

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