POLAR ISOTROPY ACTIONS ON COMPACT WEAKLY SYMMETRIC SPACES

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Abstract. Let $M = G/H$ be a compact weakly symmetric space with G a connected simple Lie group and H a closed connected subgroup of G. Consider the isotropy action of H on the tangent space of M at the origin. We classify those actions that are polar in the sense of Conlon [C], Dadok [D] and Palais and Terng [PT]. Our proof use M. Kramer's classification of spherical pairs and Dadok's result on reducible polar actions to perform a case-by-case study of the corresponding isotropy action of H.

1. Introduction

Let M be a complete Riemannian manifold and H a compact Lie group acting on M by isometries. It is an interesting problem to classify those actions that are hyperpolar since L. Conlon has shown in [C] that such actions are variationally complete . Following Palais and Terng [PT], we shall say that the action of H on M is *polar* if there exists a submanifold A of M (called a *section*) that meets every H-orbit and meets them orthogonally. Moreover, if A is flat with respect to the Riemannian metric on M, then the action is called *hyperpolar*. If M is symmetric and H is a symmetric subgroup of the isometry group of M, then the action of H on M is hyperpolar (cf. [HPTT]). Heintze, Palais, Terng, and Thorberg in [HPTT2] have shown that if M is a homogeneous space with H a group acting on it with a fixed point and such that the action is hyperpolar, then M must be locally symmetric. If $M = \mathbb{R}^n$ and the action is linear, then all polar actions have been classified by Dadok [D] in the sense that the H-orbit structure is equivalent to that of an s-representation, i.e. the isotropy action of a symmetric space. In this case, observe that the notions of polar and hyperpolar agree since sections are automatically flat being subspaces of $\mathbb{R}^n$.

If $M = G/H$ is a homogeneous Riemannian manifold, then Conlon has proven in [C] that a polar action of H on M with section A induces a polar action of H on $T_o(M)$ with section $\Omega = T_o(A)$ via the isotropy representation (the converse is false however; a counterexample is given in Lemma 3.8). This theorem allows us to partially classify hyperpolar actions on M by classifying polar actions on $T_o(M)$. Known results in this matter can be found in [C], [D], [PT], [HPTT], [PTh], and [Ko].

This paper investigates polar isotropy actions on compact weakly symmetric spaces, a natural class of manifolds to consider since it includes all known examples of polar isotropy

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actions, including symmetric spaces, isotropic spaces, and exceptional manifolds related to the Cayley plane. Recall that a manifold $M = G/H$ is *weakly symmetric* if each tangent vector can be reversed by the isotropy action of H modulo a fixed isometry μ. Our recent work on weakly symmetric spaces in [N1] and [N2] has revealed that for certain nonsymmetric and nonisotropic manifolds, weak symmetry essentially follows from the fact that they are polar; a tangent vector is reversed by mapping it to a section where μ is involutive. This suggests viewing polar isotropy actions as special cases of weak symmetry even though the former depends on the Riemannian metric and the latter does not. Our main result is

Theorem 1.1. *Let $M = G/H$ be a compact weakly symmetric space with G a connected compact simple Lie group and H a closed connected subgroup of G. Moreover, provide M with the naturally reductive Riemannian metric induced by the Killing-Cartan form defined on the Lie algebra of G. Then the isotropy action of H on $T_o(M)$ is polar if and only if M is one of the following spaces:*

(i) M is a symmetric space.
(ii) M is an S^1-bundle on an hermitian symmetric space of nontube type.
(iii) M is an S^2-bundle on $\mathbb{H}P^n$.
(iv) M is one of the following isotropic spaces: $SO(8)/Spin(7)$, $SO(7)/G_2$, or $G_2/SU(3)$.
(iv) M is either $SO(9)/Spin(7)$ or $SO(8)/G_2$.

Our proof makes use of Kramer's classification of spherical pairs in [Kr] and their equivalence with weakly symmetric spaces (cf. [N2]). As a result, this allows us to perform a case-by-case analysis of the orbit structure of each isotropy action. Our work is summarized in Table 3.

2. Preliminaries

Let H be a compact Lie group acting isometrically on a complete Riemannian manifold M with metric g.

Definition 2.1. The action of H on M is said to be *polar* if there exists a closed smooth submanifold A of M which meets every H-orbit of M orthogonally. In that case, we shall call A a *section* of M. If A is flat under g, then the action is called *hyperpolar*.

Consider the situation where M is homogeneous, i.e. $M = G/H$. Assume that the natural action of H on M is polar with section A. By homogeneity we may assume that A passes through the origin $o = eH$.

Lemma 2.2. *([C], Theorem 3.7) If the natural action of H on $M = G/H$ is polar with section A, then the isotropy representation of H on $T_o(M)$ is polar with section $\Omega = T_o(A)$.*

The following lemma will be useful in our arguments.

Lemma 2.3. *([D], Theorem 4) Let $\pi : H \to O(V)$ be a polar representation of a connected compact Lie group G. Assume that $V = V_1 \oplus V_2$ is a H-stable decomposition. Then:*

(i) $\pi_i : H \to O(V_i)$, $i = 1, 2$, *are polar representations. Every section* Ω *of* V *is of the form* $\Omega = \Omega_1 \oplus \Omega_2$ *with* Ω_i *being a section of* V_i.

(ii) Fix a section $\Omega = \Omega_1 \oplus \Omega_2$. *Let* $\mathfrak{h}_1 = Z(\Omega_2)$ *be the centralizer of* Ω_2 *in* $\mathfrak{h}$ *and* $\mathfrak{h}_2 = Z(\Omega_1)$ *be the centralizer of* Ω_1 *in* $\mathfrak{h}$. *If* H_i *is a connected Lie group having* $\mathfrak{h}_i$ *as its Lie algebra, then the action* $\rho : H_1 \times H_2 \to SO(V_1 \oplus V_2)$ *defined by*

$$\rho(h_1, h_2)(v_1, v_2) = (\pi(h_1)v_1, \pi(h_2)v_2)$$

is a polar representation and the orbits of ρ *coincide with the orbits of* π.

Definition 2.4. We say that $M = G/H$ is *weakly symmetric* if there exists an isometry μ (not necessarily in G) satisfying $\mu G \mu^{-1} = G$, $\mu(eH) = eH$, $\mu^2 \in H$, and given any tangent vector $v \in T_o(M)$, there exists an element $h \in H$ such that $d(h \circ \mu)_o(v) = -v$.

Consider the natural projection map $\pi : G \to G/H$. We identify $T_o(M)$, the tangent space of $M = G/H$ at $o = eH$, with $\mathfrak{q}$ via π. It follows that the isotropy action of H on $T_x(M)$ is precisely the adjoint action of H on $\mathfrak{q}$. Moreover, the H-orbit of $v \in \mathfrak{q}$ has tangent space $\mathfrak{k} \cdot v = [\mathfrak{k}, v]$ with $[\cdot, \cdot]$ being the Lie bracket on $\mathfrak{g}$.

3. Proof of Theorem

Assume now that G is a connected compact simple Lie group and H a closed connected subgroup of G so that $M = G/H$ is a compact homogeneous manifold. Under these assumptions, the author has proven in [N2] that the classification of compact weakly symmetric spaces and that of compact spherical pairs coincide. This allows us then to prove Theorem 1.1 by dividing M. Kramer's classification of all such spherical pairs in [Kr] into six families (categorized below) and determining which ones are polar.

I. Symmetric spaces, including $SO(8)/(SU(2) \cdot Sp(2))$.
II. S^1-bundles over hermitian symmetric spaces of nontube type:
$SU(m+n)/U(SU(m) \times SU(n))$, $SO(2n)/SU(n)$ and E_6/D_5.
III. S^2-bundles over $\mathbb{H}P^n$:
$Sp(n+1)/(Sp(n)U(1))$.
IV. Isotropic spaces:
$G_2/SU(3)$, $SO(7)/G_2$ and $SO(8)/Spin(7)$.
V. Spaces of Cayley-type:
$SO(8)/G_2$, $SO(9)/Spin(7)$ and $SO(10)/(SO(2) \times Spin(7))$.
VI. Spaces of orthogonal structures:
$SO(2n+1)/U(n)$, $SU(2n+1)/Sp(n)$ and $SU(2n+1)/(Sp(n) \cdot U(1))$.

The proof below shows that almost all of these families are polar, the exceptions being the spaces of orthogonal structures and $SO(10)/(SO(2) \times Spin(7))$.

Proof of Theorem 1.1. Our proof consists of case-by-case arguments as follows.

3.1. **I. Symmetric spaces.** If $M = G/K$ is a symmetric space, then it is well known isotropy representation $\pi_s : K \to O(T_x(M))$, called an *s-representation*, is polar with sections being maximally abelian subspaces $\Omega = \mathfrak{a} \subset \mathfrak{p}$. Here, $\mathfrak{p}$ is such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g} = Lie(G)$. (cf. [C], [D]).

3.2. **II. S^1-bundles on hermitian symmetric spaces.** Let G be a connected semisimple matrix Lie group with finite center and K a maximal compact subgroup of G such that $D = G/K$ is a hermitian symmetric space. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to a Cartan involution σ, where $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebras of G and K, respectively. Then $\mathfrak{k} = \mathfrak{h} + \mathfrak{z}_{\mathfrak{k}}$, where $\mathfrak{h}$ is the semisimple part of $\mathfrak{k}$ and $\mathfrak{z}_{\mathfrak{k}}$ the one-dimensional center of $\mathfrak{k}$. Let H be the subgroup of K with Lie algebra $\mathfrak{h}$ and Z_K the center of K. Then H is connected and $K = HZ_K^0$, where Z_K^0 is the connected component of Z_K.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. The following lemma will prove useful.

Lemma 3.3. *(*[N2]*, Lemma 3.7) If G/K is not of tube type, then $Ad(H)(\mathfrak{a}) = \mathfrak{p}$.*

Consider now the weakly symmetric space $M = G/H$ with tangent space $\mathfrak{h} \oplus \mathfrak{z}_{\mathfrak{k}}$. It has isotropy representation $\pi = \pi_s \oplus \mathrm{Id}$ where π_s is the s-representation of G/K on $\mathfrak{p}$ and Id is the one-dimensional trivial representation on $\mathfrak{z}_{\mathfrak{k}}$. It follows from Lemma 3.3 that every H-orbit of π intersects $\Omega = \mathfrak{a} \oplus \mathfrak{z}_{\mathfrak{k}}$. Moreover, $\langle \mathfrak{h} \cdot \Omega, \Omega \rangle = 0$, since $\langle [\mathfrak{k}, \mathfrak{a}], \mathfrak{a} \rangle = 0$ and $[\mathfrak{h}, \mathfrak{z}_{\mathfrak{k}}] = 0$. Hence, the intersection is orthogonal and so π is polar with section Ω.

3.4. **III. S^2-bundles on quaternionic projective space.** Let $M = Sp(n)/(Sp(n-1)U(1))$. Then $T_x(M) = \mathbb{H}^n \oplus \mathbb{R}^2$. As for the isotropy action action of $Sp(n-1)U(1)$ on $T_x(M)$, it is known that $Sp(n)$ acts on $\mathbb{H}^n$ as ν_n and trivially on $\mathbb{R}^2$. Moreover, $U(1)$ acts on $\mathbb{R}^2$ by rotations (its action on $\mathbb{H}^n$ is nontrivial but we shall not care to know what it is). It easily follows that π is polar with section $\Omega = \mathbb{R}_1 \oplus \mathbb{R}_2$ where $\mathbb{R}_1$ and $\mathbb{R}_2$ are one-dimensional lines in $\mathbb{H}^n$ and $\mathbb{R}^2$, respectively. This is because the actions of $Sp(n)$ and $U(1)$ are transitive on spheres in $\mathbb{H}^n$ and $\mathbb{R}^2$, respectively.

3.5. **IV. Isotropic spaces.** It is well known that for such spaces the isotropy representation π is transitive on the unit tangent sphere. Obviously, π is polar with one-dimensional sections.

3.6. **V. Spaces of Cayley-type.** (i) $Spin(9)/Spin(7) = S^{15}$: Here, $T_x(M) = \mathbb{R}^7 \oplus \mathbb{R}^8$ and $\pi = \rho_7 \oplus \Delta_7$. It is known that $Spin(7)$ acts transitively on $S^6 \times S^7 \subset \mathbb{R}^7 \oplus \mathbb{R}^8$ with isotropy subgroup $SU(3)$ (cf. [Z]). Hence, π is polar with two-dimensional sections $\Omega = \mathbb{R} \oplus \mathbb{R}$.

(ii) $Spin(8)/G_2 = S^7 \times S^7$ (not polar): Here, $T_x(M) = \mathbb{R}^7 \oplus \mathbb{R}^7$ and $\pi = \sigma_7 \oplus \sigma_7$. Assume on the contrary that π is polar. Since σ_7 is transitive on spheres in $\mathbb{R}^7$, Lemma 2.3 forces the section to be two-dimensional and to decompose as $\Omega = mathbbR \oplus \mathbb{R}$. The normalizers H_1 and H_2 are both easily seen to be $SU(3)$. However, the orbits of $H_1 \times H_2 = SU(3) \times SU(3)$ acting on $\mathbb{R}^7 \oplus \mathbb{R}^7$ are not the same as those of $H = G_2$ acting on $\mathbb{R}^7 \oplus \mathbb{R}^7$. This is because the orbits of H contain $S^5 \times S^6$ and those of $H_1 \times H_2$ do not; the former follows from G_2 being transitive on two-frames in $V_2(\mathbb{R}^7$ and the latter from the fact that $SU(3)$ is *not* transitive on spheres in $\mathbb{R}^7$. As this contradicts Lemma 2.3, we conclude that π is *not* polar.

(iii) $SO(10)/(SO(2) \times Spin_{\pm}(7))$ (not polar): Here, $T_x(M) = \mathbb{R}^7 \oplus (\mathbb{R}^8 \otimes \mathbb{R}^2)$ and $\pi = \rho_7 \oplus \Delta_7 \otimes \rho_2$. Let $v = w + (y, z) \in \mathbb{R}^7 \oplus \mathbb{R}^8 \otimes \mathbb{R}^2$ where $y, z \in \mathbb{R}^8$. To show that π is not polar, let us assume the contrary. Then π being polar implies that its restrictions, namely namely ρ_7 acting on $\mathbb{R}^7$ and $\Delta_7 \otimes \rho_2$ acting on $\mathbb{R}^8 \otimes \mathbb{R}^2$ are polar by Lemma 2.3. Now, it is clear that ρ_7 is transitive on spheres and hence has one-dimensional sections Ω_1. On the other hand, $\Delta_7 \otimes \rho_2$ is forced to have at least a two-dimensional section, Ω_2, because its action is not transitive on spheres. If we assume $\Omega_2 = (\mathbb{R}e_1, \mathbb{R}e_2)$ with $\{e_1, e_2, ..., e_8\}$ representing the standard unit vectors in $\mathbb{R}^8$. Now, let $H_1 = N(\Omega_2)$ and $H_2 = N(\Omega_1)$ be the normalizers of Ω_2 and Ω_1 in H, respectively. Then the action of $H_1 \times H_2$ on $\mathbb{R}^7 \oplus (\mathbb{R}^8 \otimes \mathbb{R}^2)$ is polar with sections $\Omega_1 \oplus \Omega_2$ and the orbits of $H_1 \times H_2$ are the same as those of π. However, $H_1 = SU(3)$ and $H_2 = Spin(6) \times SO(2)$ with the action of $Spin(6) = SU(4)$ being that of μ_4 on $\mathbb{R}^8 = \mathbb{C}^4$. From here it is easy to check that the action of H_2 on $\mathbb{R}^8 \otimes \mathbb{R}^2$ is not polar since not every H_2-orbit intersects $\Omega_2 = (\mathbb{R}e_1, \mathbb{R}e_2)$. This contradicts our conclusion that the action of $H_1 \times H_2$ is polar by part (ii) of Lemma 2.3. Hence, π cannot be polar.

3.7. **VI. Spaces of orthogonal structures.** (i) $G/H = SO(2n+1)/U(n)$: Let M_{n+1} be the symmetric space $SO(2n+2)/U(n+1)$ and denote by π^s_{n+1} the s-representation of $U(n+1)$ on $T_x(M_{n+1})$ with section $\mathfrak{a}_{n+1}$. However, M_{n+1} also has the homogeneous presentation $M_{n+1} = SO(2n+1)/U(n)$ with isotropy action of $U(n)$ given by $\pi = \pi^s_{n+1}|_{U(n)}$. Moreover, its tangent space decomposes as $T_x(M) = T_{x'}(M_n) \oplus \mathbb{C}^n$ $(x' \in M_n)$ under $\pi = \pi^s_n \oplus \mu_n$.

We now prove that π is not polar by again employing Lemma 2.3. Assume the contrary that π *is* polar. Since π^s_n is an s-representation and μ_n is transitive on spheres, Lemma 2.3 forces π to have section of the form $\mathfrak{a}_n \oplus \mathbb{R}$, where $\mathfrak{a}_n \subset T_{x'}(M_n)$ is a section for π^s_n and $\mathbb{R} \subset \mathbb{C}^n$ is a one-dimensional section for μ_n. Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be the centralizers of $\mathbb{R}$ and $\mathfrak{a}_n$ in $\mathfrak{h} = Lie(H)$, respectively. Denote by H_1 and H_2 to be the connected Lie groups with Lie algebras $\mathfrak{h}_1$ and $\mathfrak{h}_2$, respectively. Then $H_1 = U(n-1)$ and H_2 is a proper subgroup of $U(n)$ whose identity component is either $SU(2)^{\frac{1}{2}n}$, i.e. $\frac{1}{2}n$-copies of $SU(2)$, if n is even or $SU(2)^{\frac{1}{2}(n-1)} \times U(1)$ if n is odd (cf. [Kn], p. 530). But it is clear in either case that the action $H_2 \to SO(\mathbb{C}^n)$ is not transitive on spheres and therefore cannot have one-dimensional sections. This contradicts the conclusion that $\rho : H_1 \times H_2 \to SO(\mathfrak{a}_n \oplus \mathbb{C}^n)$ is polar by part (ii) of Lemma 2.3. Hence, π cannnot be polar.

(ii) $G/H = SU(2n+1)/Sp(n)$: Let M_{n+1} be the symmetric space $SU(2n+2)/Sp(n+1)$ and denote by π^{n+1}_s the isotropy representation of $Sp(n+1)$ on $T_x(M_{n+1})$ with section $\mathfrak{a}_{n+1}$. However, M_{n+1} also has the homogeneous presentation $M_{n+1} = SU(2n+1)/Sp(n)$ where the isotropy action of $Sp(n)$ here is $\pi = \pi^{n+1}_s|_{Sp(n)}$. Moreover, its tangent space decomposes as $T_x(M_{n+1}) = T'_x(M_n) \oplus \mathbb{H}^n \oplus \mathbb{R}$ $(x' \in M_n)$ under $\pi = \pi^n_s \oplus \nu_n \oplus \mathrm{Id}$ (ν_n is the standard representation of $Sp(n)$ on $\mathbb{H}^n$).

To see that π is not polar, we run through the same argument as that used for $SO(2n+1)/U(n)$. Assume the contrary that π *is* polar. If we ignore the trivial summand of π, then π must have sections of the form $\mathfrak{a}_n \oplus \mathbb{R}$, where $\mathfrak{a}_n \subset T_{x'}(M_n)$ is a section for π^s_n and $\mathbb{R} \subset \mathbb{H}^n$ is a section for ν_n. Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be the centralizers of $\mathbb{R}$ and $\mathfrak{a}_n$ in $\mathfrak{h} = Lie(H)$,

Table 3 - Polar isotropy actions on compact weakly symmetric spaces

Weakly symmetric space $M = G/H$	Isotropy action $\pi : H \to SO(T_x(M))$	Section $\Omega \subset T_x(M)$
I. Symmetric spaces	s-representation	$\mathfrak{a} \subset \mathfrak{p}$
II. S^1-bundles over hermitian symmetric spaces	$s \oplus \mathrm{Id}$	$\mathfrak{a} \oplus \mathbb{R} \subset \mathfrak{p} \oplus \mathbb{R}$
III. S^2-bundle over $\mathbb{H}P^n$	$(\nu_n \otimes (\cdot)) \oplus \mu_1$	$\mathbb{R} \oplus \mathbb{R} \subset \mathbb{H}^n \oplus \mathbb{R}^2$
IV. Isotropic spaces		
$SO(8)/Spin(7)$	ρ_7	$\mathbb{R} \subset \mathbb{R}^7$
$SO(7)/G_2$	σ_7	$\mathbb{R} \subset \mathbb{R}^7$
$G_2/SU(3)$	μ_3	$\mathbb{R} \subset \mathbb{R}^6$
V. Spaces of Cayley-type		
$SO(9)/Spin(7)$	$\rho_7 \oplus \Delta_7$	$\mathbb{R} \oplus \mathbb{R} \subset \mathbb{R}^7 \oplus \mathbb{R}^8$
$SO(8)/G_2$	$\sigma_7 \oplus \sigma_7$	not polar
$SO(10)/SO(2) \times Spin(7)$	$\rho_7 \oplus (\Delta_7 \otimes \rho_2)$	not polar
VI. Spaces of orthogonal structures		
$SO(2n+1)/U(n)$	$\wedge^2 \mu_n \oplus \mu_n$	not polar
$SU(2n+1)/Sp(n)$	$\wedge^2 \nu_n \oplus \nu_n$	not polar
$SU(2n+1)/(Sp(n)U(1))$	$(\wedge^2 \nu_n - \mathrm{Id}) \oplus \nu_n$	not polar

respectively. Denote by H_1 and H_2 to be the connected Lie groups with Lie algebras $\mathfrak{h}_1$ and $\mathfrak{h}_2$, respectively. Then $H_1 = Sp(n-1)$ and $H_2 = SU(2)^n$. But it is clear that the action $H_2 \to SO(\mathbb{H}^n)$ is not transitive on spheres and therefore cannot have one-dimensional sections. This contradicts part (ii) of Lemma 2.3. Hence, π is not polar.

(iii) $G/H = SU(2n+1)/Sp(n)U(1)$: As a corollary to the $SU(2n+1)/Sp(n)$ case, it follows that the isotropy representation for $SU(2n+1)/Sp(n)U(1)$ cannot be polar either since its isotropy representation is the same as that for $SU(2n+1)/Sp(n)$ minus the trivial summand.

This concludes the proof of Theorem 1.1. We refer the reader to Table 3 for a summary of our results. $\square$

As an application, we prove that there are actions which are polar at the tangent space level but not hyperpolar at the manifold level.

Lemma 3.8. *If $M = G/K_s$ is an S^1-bundle on an hermitian symmetric space of nontube type, then the action of K_s on $T_o(M)$ is polar but the action of K_s on M is not hyperpolar.*

Proof. Our classification in Theorem 1.1 proves that the action of K_s on $T_o(M)$ is polar. To prove that the action of K_s on M is not hyperpolar, assume on the contrary that A is a flat section for this action. Then $\Omega = T_o(A) \cong \mathfrak{a} \oplus \mathfrak{z}_{\mathfrak{k}}$ must be an abelian subspace of $T_o(M) \cong \mathfrak{q}$ since its sectional curvature is determined by the Lie bracket defined on $\mathfrak{q}$. However, it can be easily verified that Ω is not abelian, namely $[\mathfrak{z}_{\mathfrak{k}}, \mathfrak{a}] \neq 0$. Hence, the action of K_s on M cannot be hyperpolar. □

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