TAUTOCHRONES ON RIEMANNIAN MANIFOLDS

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ABSTRACT. The method of fractional calculus is used to study tautochrones on Riemannian manifolds. It is mathematically proven that the spring-like physical explanation for tautochrones is the correct one. Furthermore, various examples of tautochrones are provided, including one in Euclidean space that demonstrates that the tautochrone problem does not always possess a unique solution.

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I. Introduction

Let M be a C^∞-manifold endowed with a Riemannian metric ds^2. Consider a conservative force field $\mathbf{F} = -\nabla V$ on M defined by a scalar potential V. Fix any two points P and Q on M. Then a C^1-path γ connecting them is called a *tautochrone* if it has the property that any massive object moving along γ, while under the influence of F, will reach the terminal point Q (or P depending on which point is "higher") in the same amount of time regardless of the object's starting point on γ. The tautochrone problem is to determine γ. When M is the Euclidean plane and F the force of gravity, the solution is classical and due to Abel (1823), who found γ to be a cycloid (cf. [8]). Furthermore, the physical explanation behind this solution was found to be in the following *spring-like* property of the cycloid: the tangential force, i.e. that along the cycloid and experienced by the object, is directly proportional to the length of the cycloid between the object and Q (cf. [7]). Locally, this property is analagous to that described by Hooke's law for a mass-spring system.

It is clear that any path with this spring-like property must be a tautochrone; however, the same cannot be said for the converse, namely that every tautochrone must be spring-like. The purpose of this paper is to therefore establish that the converse does indeed hold.

Abel's solution to the classical tautochrone problem was obtained via fractional calculus. This method was later generalized by Kamath [4], who solved the relativistic tautochrone problem. Recently, Flores-Osler [1] similarly extended Abel's method, but independently of Kamath, to obtain tautochrones under arbitrary potentials on the Euclidean plane. Their idea is to specify the terminal point Q and a path for γ in which its arclength s can be found by formula. It suffices then to find a potential V satisfying $V = s^2$, a sufficient condition for γ to be a tautochrone. In this paper, we demonstrate that their method is valid for obtaining tautochrones on Riemannian manifolds as well. We also prove that the same physical explanation continues to hold for these tautochrones as it does in the classical case, namely that

$$F_\gamma = \frac{\pi^2 m}{4T^2} s,$$

where F_γ is the tangential force, m is the mass of the object, and T the time of descent. Moreover, equating this result with Newton's law of motion, we discover that s satisfies the differential equation

$$\frac{d^2 s}{dt^2} = -\frac{\pi^2}{4T^2} s, \qquad (0 \le t \le T),$$

and therefore, its solution is sinusoidal as a function of time.

Lastly, we provide examples of tautochrones on some classical Riemannian manifolds (cf. Table 3.1), including an example in $\mathbb{R}^3$ that demonstrates the tautochrone problem does not always possess a unique solution with respect to a fixed potential and fixed time of descent.

II. Fractional Calculus

Let $\gamma(\alpha)$ be a tautochrone between P and Q parametrized by the variable α so that $\gamma(0) = Q$ and $\gamma(L) = P$. If R denotes an arbitrary starting point between P and Q for an object moving along γ and heading towards Q, then let β be such that $\gamma(\beta) = R$. Assume that the object obeys the law of conservation of energy as it moves over time. Following Flores-Osler [1] and writing $V(\alpha) := V(\gamma(\alpha))$ to ease our notation, it follows that

$$\frac{1}{2}mv^2 = V(\beta) - V(\alpha), \ 0 \leq \alpha \leq \beta.$$

Here, $v = -\frac{ds}{dt}$ is the velocity of the object along γ with respect to time t and $s(\alpha)$ is the arclength of γ from 0 to α with respect to the Riemannian metric ds^2. In other words,

$$\frac{-1}{\sqrt{V(\beta) - V(\alpha)}}\frac{ds}{dt} = \sqrt{\frac{2}{m}}.$$

Integrating with respect to t from R to Q and dividing the result by $\Gamma(1/2) = \sqrt{\pi}$ yields

$$\frac{1}{\Gamma(1/2)}\int_0^T \frac{-\frac{ds}{dt}}{\sqrt{V(\beta) - V(\alpha)}}dt = \sqrt{\frac{2}{\pi m}}\, T, \tag{1}$$

where T here is the travel time required for the object to reach Q regardless of its starting point R. Assuming V is such that the improper integral in (1) is convergent, we then rewrite the left-hand side as follows using fractional calculus:

$$\begin{aligned}
\frac{1}{\Gamma(1/2)}\int_0^T \frac{\frac{ds}{dt}}{\sqrt{V(\beta)-V(\alpha)}}dt &= \frac{1}{\Gamma(1/2)}\int_0^s \frac{\frac{ds}{dV}\frac{dV}{d\alpha}}{\sqrt{V(\beta)-V(\alpha)}}d\alpha \\
&= D^{-1/2}_{V(\beta)}[\tfrac{ds}{dV}] \\
&= D^{-1/2}_{V(\beta)}D^{1}_{V(\beta)}[s] \\
&= D^{1/2}_{V(\beta)}[s].
\end{aligned}$$

Equating the right-hand side immediately above with that in (1), we then apply $D^{-1/2}_{V(\beta)}$ to the resulting equation:

$$s(\beta) = \sqrt{\frac{2}{\pi m}}\, T D^{-1/2}_{V(\beta)}[1] = \sqrt{\frac{2}{m}}\frac{2T}{\pi}\sqrt{V(\beta)}$$

or

$$V(\beta) = \frac{\pi^2 m}{8T^2}s^2(\beta). \tag{2}$$

As noticed by Flores-Osler [1], it is easier to obtain tautochrones by specifying an arbitrary path γ. It suffices then to compute the arclength function $s(\beta)$ and substitute it into (2) in order to realize the necessary potential for having γ as a solution. In fact, we shall do precisely this in section III to find various examples of tautochrones.

Next, we differentiate (2) with respect to s to yield the tangential force experienced by the object:

$$F_\gamma := ds(\mathbf{F},\mathbf{u}(s)) = -\nabla_{\mathbf{u}(s)}V = \frac{dV}{ds} = \frac{\pi^2 m}{4T^2}s, \tag{3}$$

where $\mathbf{u}(s) = -\frac{d\gamma}{ds}$ is the unit tangent vector of γ parametrized by s. This proves that F_γ is proportional to s and provides the correct physical explanation for the spring-like behavior of tautochrones.

Note that if Newton's law of motion holds on M, then $F_\gamma = -m\frac{d^2s}{dt^2}$. Therefore, equating this with (3) leads to the second-order differential equation

$$\frac{d^2s}{dt^2} = -\frac{\pi^2}{4T^2}s. \tag{4}$$

However, (4) cannot be solved uniquely for s in terms of t until T is found and therein lies the difficulty. But (4) is still useful for describing the general motion of the object, and coupled with the boundary condition $s = 0$ at $t = T$, it is easily seen to be sinusoidal. More precisely,

$$s(t) = K\cos\frac{\pi t}{2T}, \qquad (0 \le t \le T)$$

where the amplitude K represents the length of γ from R to Q; the initial condition $\frac{ds}{dt} = 0$ at $t = 0$ cannot help us find K here.

On the other hand, differentiating (2) with respect to β gives

$$\frac{ds}{d\beta} = \frac{\sqrt{2}T}{\pi}\frac{V'(\beta)}{\sqrt{V(\beta)}}.$$

Now, express any point $x \in M$ in local coordinates as $x = (x_1, x_2, ..., x_n)$ and write the Riemannian metric ds^2 in component form as

$$ds^2 = \sum_{i,j}^{n} g_{ij}(x)dx_i dx_j.$$

We then equate (4) with (5) to obtain the nonlinear ordinary differential equation

$$\sum_{i,j}^{n} g_{ij}(x)\frac{dx_i}{d\beta}\frac{dx_j}{d\beta} = \frac{2T^2}{\pi^2}\frac{[V'(\beta)]^2}{V(\beta)}.$$

As shown by Flores-Osler [1], when $M = \mathbb{R}^2$, V a one-dimensional potential, and T a fixed time of descent, this differential equation becomes separable and therefore the solution for γ can be expressed in integral form. It follows that if γ exists, then it must be unique. However, we shall provide a counterexample in $\mathbb{R}^3$ in the next section to demonstrate that the solution in general is *not* unique for Riemannian manifolds.

III. Examples

In this section, we produce in Table I examples of tautochrones on some classical Riemannian manifolds. We normalize the mass of our object m and time of descent T so that $\frac{\pi^2 m}{8T^2} = 1$. Furthermore, $\gamma(0)$ always specifies the terminal point Q. Notice that the tautochrones in examples 1a, 2 and 3 are geodesics and the potential corresponding to each is easy to compute because arclength for minimal(i.e. distance-minimizing) geodesics is precisely determined by the distance function on the manifold (cf. (ii) below). We offer two interesting comments about the examples in Table I:

(i) Example 1c illustrates that in general the solution to the tautochrone problem with V and T fixed is not unique, namely there exists two distinct tautochrones γ_1 and γ_2 connecting $P = \gamma_i(\pi)$ and $Q = \gamma_i(0)$ (both are helices but spiral in opposite directions).

(ii) All examples can be generalized to arbitrary dimensions. For instance, by following example 1b, we can consider the spring or harmonic oscillator potential on $\mathbb{R}^n$ as a scalar product:

$$V(x) = x \cdot x, \quad x \in \mathbb{R}^n.$$

Then all tautochrones with the origin as the terminal point must be straight lines. Similarly, in generalizing $\mathbb{CP}^1$ to $\mathbb{CP}^n$, we use the distance function that gives rise to the Fubini-Study metric on $\mathbb{CP}^n$. If $[x]$ and $[y]$ are two elements of $\mathbb{CP}^n$ representing projective lines passing through the points x and y in $\mathbb{C}^{n+1}$, respectively, then the distance between them, $d([x],[y])$, satisfies

$$\cos\left(d([x],[y])\right) = \frac{\langle x, y\rangle\langle y, x\rangle}{\langle x, x\rangle\langle y, y\rangle}, \tag{5}$$

where $\langle\cdot,\cdot\rangle$ is the standard positive-definite Hermitian inner product on $\mathbb{C}^{n+1}$. To obtain geodesic tautochrones, it suffices then to define $\gamma(\alpha)$ as a minimal geodesic connecting $Q = [x]$ and $P = [y]$ and use the distance formula defined by (5) to compute its arclength as $s(\alpha) = d(Q, \gamma(\alpha))$ in order to determine the correct potential.

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