Weakly Symmetric Spaces and Bounded Symmetric Domains

Hiếu Đức Nguyễn

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Xin được thân tặng cho Ba Mẹ

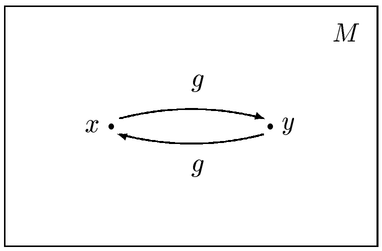
Contents

Chapter 1

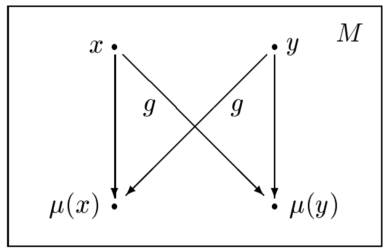
Bounded Symmetric Domains

1.1 Introduction

The classic notion of a symmetric space M dates back to Riemann and Cartan: given any two points x and y of M, there exists an isometry g which maps each point to the other. This deep and yet simple idea helped spurred vigorous research in the early 1900's which resulted in a classification of the irreducible symmetric spaces (see [9]). Then in 1956, Atle Selberg introduced in his seminal paper [23] the notion of a weakly symmetric space which naturally generalizes that of a symmetric space. He called a space M weakly symmetric if it has a fixed involutive isometry μ such that given any two points x and y on M, there exists an isometry g which maps each point to the image of the other point under μ. If μ can be chosen to be the identity, then M is in fact symmetric. The diagram below gives a picture of this generalization:



Symmetry



Weak Symmetry

A symmetric space possesses the crucial property that its space of invariant linear operators is commutative. As a result, harmonic analysis becomes interesting and fruitful on such spaces. Fortunately, Selberg [23] has proven that this property is retained for weakly symmetric spaces. It is mainly because of this fact that makes Selberg's generalization worth investigating. However, its significance depends on whether or not there are examples of weakly symmetric spaces that are not symmetric. Unfortunately, such examples are few in number and a classification seems far at hand. In fact, the only method known to us for obtaining concrete examples was described by Selberg in [15], where he informally constructed circle bundles on the Siegel half-space. Furthermore, he found applications for these circle bundles in his investigation of automorphic forms.

It is our thesis that new examples of weakly symmetric spaces can be obtained by generalizing Selberg's circle bundle construction to bounded symmetric domains. By using the symmetry and fine structure of bounded symmetric domains, it is proven that the corresponding circle bundles are indeed weakly symmetric. Applications of these weakly symmetric spaces to harmonic analysis, representation theory, and automorphic forms are presented.

The contents of our thesis are as follows: Chapter I lays out the structure theory for simple Lie groups of hermitian type and irreducible bounded symmetric domains. Let G be such a Lie group and $D \cong G/K$ the corresponding bounded symmetric domain. We shall assume that D is of classical type. The factor of automorphy κ and a kernel function K are introduced. As they play an important role in our thesis, the rest of the chapter is devoted to finding explicit expressions for κ and K. Proofs are given for the complex quadric since we were unable to find references for it. Other classical results about the Bergman kernel and the Bergman on D are provided as background for our weakly symmetric spaces.

Chapter II is the beginning of new material and new results. First, we describe the construction of a circle bundle $\mathbb{D}^1 \to D$. Let $\mathbb{G}^1$ and $\mathbb{D}^1$ be circle extensions of G and D, respectively. Then a twisted action of $\mathbb{G}^1$ on $\mathbb{D}^1$ is defined using the factor of automorphy κ. It is shown that this action is well-defined and transitive due to a cocycle condition satisfied by κ. This makes $\mathbb{D}^1$ a homogeneous riemannian manifold and the $\mathbb{G}^1$-invariant metric Ω is computed for $\mathbb{D}^1$ of classical type.

Let σ denote complex conjugation on D and extend it to $\mathbb{D}^1$. Chapter III then presents the main result of our thesis:

Main Theorem. $(\mathbb{D}^1, \mathbb{G}^1, \sigma)$ is a weakly symmetric space.

Furthermore, in case $G = SU(m,n)$ with $m \neq n$, it happens that $\mathbb{G}^1$ can be replaced with G as the transitive group of isometries of $\mathbb{D}^1$. We also prove that the universal cover of a weakly symmetric space is also weakly symmetric. It follows that $(\mathbb{D}, \mathbb{G}, \sigma)$ is also weakly symmetric, where $\mathbb{D}$ and $\mathbb{G}$ denote the universal covers of $\mathbb{D}^1$ and $\mathbb{G}^1$, respectively.

We go on in this chapter to describe the Cayley transformation of $\mathbb{D}^1$. This is an extension of the Cayley transform c on D which maps it to its unbounded realization H. As a result, $\mathbb{D}^1$ should have a similar unbounded realization $\mathcal{H}_\nu$. It is shown that $\mathcal{H}_\nu$ can be constructed independently, so that via the Cayley transform, is seen to be precisely the unbounded version of $\mathbb{D}^1$. We obtain as new results expressions for the riemannian metric on $\mathcal{H}_\nu$ in the two cases where H is either a tube domain with $G = SU(n,n)$ or complex hyperbolic space with $G = SU(n,1)$. Lastly, it is verified that when G is the symplectic group, $\mathcal{H}_\nu$ is exactly the circle bundle constructed by Selberg.

Chapter 4 then discusses some applications of our circle bundles. We hope to convince the reader of their usefulness even though the results that we obtain are not new. The first application deals with $\mathbb{D}^1$ as a commutative space, or equivalently, viewing $(\mathbb{G}^1, \mathbb{K}^1)$ as a Gelfand pair. This means that $\mathbf{D}(\mathbb{G}^1/\mathbb{K}^1)$, the space of $\mathbb{G}^1$-invariant linear operators on $\mathbb{D}^1$, is commutative since $\mathbb{D}^1$ is weakly symmetric. However, this result was already proven by Flensted-Jensen in [2]. There, he actually defined the same extensions $\mathbb{G}^1$ and $\mathbb{K}^1$ as we did and performed harmonic analysis on the function algebra $C^\infty(\mathbb{G}^1/\mathbb{K}^1)$. A direct proof is given of the commutativity of that $C^\infty(\mathbb{G}^1//\mathbb{K}^1)$, the bi-$\mathbb{K}_\nu$-invariant functions on $\mathbb{G}^1$. This implies that $(\mathbb{G}^1, \mathbb{K}^1)$ is a Gelfand pair and the commutativity of $\mathbf{E}(\mathbb{G}^1/\mathbb{K}^1)$ now follows from certain equivalent definitions for a Gelfand pair. We show that his method can be used to prove that the function algebra $C_c^\infty(\mathbb{G}//\mathbb{K}_\nu)$ is also commutative, hence $(\mathbb{G}, \mathbb{K}_\nu)$ is also a Gelfand pair.

We then make use of Flensted-Jensen's results about spherical functions on $\mathbb{G}^1$ (with respect to $\mathbb{K}^1$) to study the relationship between spherical kernels and spherical functions on $\mathbb{D}^1$. Namely, we derive the important property that joint eigenfunctions of invariant different operators $\mathbf{D}(\mathbb{G}^1/\mathbb{K}^1)$ are joint eigenfunctions of the invariant integral operators $\mathbf{E}(\mathbb{G}^1/\mathbb{K}^1)$.

The last application shortly describes using $\mathbb{D}^1$ to study representations of G and automorhic forms on D via the Selberg trace formula. Let $\mathcal{E}$ be a joint eigenspace of $\mathbf{D}(\mathbb{G}^1/\mathbb{K}^1)$. Now, every $F \in \mathcal{E}$ can be written as $F(z,t) = |K(z,z)|^{\frac{k}{2}} f(z) e^{ikt}$, where $K(z,z)$ is a fractional power of the Bergman kernel. Consider the operator $T : F \mapsto f$. Then the natural action of $\mathbb{G}^1$ on F by left translation means that G acts on f under a transformation via the factor of automorphy. The operator T becomes an intertwining operator between the eigenspace representation of $\mathbb{G}^1$ and the corresponding discrete series representation of G.

Let Γ be a discrete subgroup of G such that $\Gamma\backslash D$ is compact. Fix a one-dimensional representation τ_ν of K and a one-dimensional representation χ of Γ. Let $\mathcal{A}$ be the space consisting of square-integrable functions on D that transform automorphically under Γ as follows:

$$f(mz) = \chi(m)\tau^k(\kappa(m,z))f(z), \quad m \in \Gamma,\ z \in D,\ k \in \mathbb{Z}. \tag{1.1}$$

The important connection is from observing that if $F \in \mathcal{E}$ is invariant under Γ, then $T(F) = f$ is in $\mathcal{A}$. We construct an appropriate $\mathbb{G}^1$-invariant integral operator I_p on $\mathbb{D}^1$ and go on to prove that its kernel p is spherical in the sense of Selberg [5]. A dimension formula for $\mathcal{A}$ can be obtained by applying the Selberg trace method to I_p (as in [23]).

1.2 Preliminaries

The following treatment is taken from R. Herb and J.A. Wolf [12]. Let G be a connected, simply-connected real simple Lie group of Hermitian type. Denote by Z_G the center of G. Fix a Cartan involution θ of G. The fixed point set $K = G^\theta$ contains the center Z_G of G and K/Z_G is a maximal compact subgroup of G/Z_G. The space $D = G/K$ is an irreducible bounded symmetric domain.

If $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebras of G and K, respectively, then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition under θ. Now, $\mathfrak{k} = \mathfrak{k}_s \oplus \mathfrak{z}_{\mathfrak{k}}$ where $\mathfrak{k}_s = [\mathfrak{k}, \mathfrak{k}]$ is the semisimple part of $\mathfrak{k}$, and $\mathfrak{z}_{\mathfrak{k}}$ is its one-dimensional center. Let K_s denote the connected closed subgroup of K with Lie algebra $\mathfrak{k}_s$ and Z_K denote the center of K. Then K_s is compact, simply-connected, and normal in K. The vector subgroup $Z_K \cong E \times Z_K^0$, where E is a finite abelian subgroup and $Z_K^0 \cong \mathbb{R}$ the identity component of Z_K, and $K = K_s \cdot Z_K^0$. Also, $Z = Z_G \cap Z_K^0$ is an infinite cyclic group.

Let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of $\mathfrak{g}$. Extend θ to $\mathfrak{g}_{\mathbb{C}}$ so that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_{\mathbb{C}}$. If $\mathfrak{h}$ is a Cartan of subalgebra of $\mathfrak{k}$, then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and its complexification $\mathfrak{h}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Let Φ be root system for $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Then $\mathfrak{g}_{\mathbb{C}}$ has the root decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} + \sum_{\alpha\in\Phi} \mathfrak{g}_\alpha$. Decompose $\Phi = \Phi_c \cup \Phi_n$ into subsets of compact and noncompact roots. Choose a root ordering for Φ so that Φ^+ and Φ^- are the set of positive and negative roots, respectively. This allows us to write $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ + \mathfrak{p}_-$, where $\mathfrak{p}_+$ (resp. $\mathfrak{p}_-$) is the holomorphic (resp. antiholomorphic) tangent space. Furthermore, let Z_0 be the element in the center of $\mathfrak{k}$ which defines this complex structure. Set $\Phi_c^{\pm} = \Phi^{\pm} \cap \Phi_c$ and $\Phi_n^{\pm} = \Phi^{\pm} \cap \Phi_n$.

Let $G_{\mathbb{C}}$ be the connected simply connected Lie group for $\mathfrak{g}_{\mathbb{C}}$. Denote by $G_{\mathbb{R}}$, $K_{\mathbb{R}}$, $K_{\mathbb{C}}$, P_+ and P_- to be analytic subgroups of $G_{\mathbb{C}}$ corresponding to $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{k}_{\mathbb{C}}$, $\mathfrak{p}_+$ and $\mathfrak{p}_-$. Let $q : G \to G_{\mathbb{R}}$ be the projection map. Then $K = q^{-1}(K_{\mathbb{R}})$. If $x \in P_+K_{\mathbb{R}}P_-$, then we write its $P_+K_{\mathbb{R}}P_-$ decomposition as $x = p_+(x) \cdot \kappa_o(x) \cdot p_-(x)$. This allows us to define the map $\zeta : P_+K_{\mathbb{C}}P_- \to \mathfrak{p}_+$ by requiring that $p_{\pm}(x) = \exp \zeta(x)$, where $\exp : \mathfrak{p}_+ \to P_+$ is the exponential map. Since $G_{\mathbb{R}} \subset P_+K_{\mathbb{C}}P_- \subset G_{\mathbb{C}}$, we write $g \in \exp z \cdot K_{\mathbb{C}}P_-$ so that the restriction map $\zeta : G_{\mathbb{R}} \to \mathfrak{p}_+$ given by $g \mapsto z$ gives the Harish-Chandra embedding of $G_{\mathbb{R}}/K_{\mathbb{R}}$ onto a bounded domain D in the complex vector space $\mathfrak{p}_+$.

We now lift this picture up to G. The embedding $G_{\mathbb{R}} \subset P_+K_{\mathbb{C}}P_-$ lifts to a corresponding decomposition map $G \to P_+\widetilde{K}_{\mathbb{C}}P_-$ by lifting $p_{\pm}$ and κ_o to G via the universal covering $q : G \to G_{\mathbb{R}}$. This gives $G/K \cong G_{\mathbb{R}}/K_{\mathbb{R}}$. Now, let $q_K : \widetilde{K}_{\mathbb{C}} \to K_{\mathbb{C}}$ be the universal covering group. Then $\widetilde{K}_{\mathbb{C}}$ can be thought of as the complexification of K and $q_K|_K = q|_K$. As a result, $\kappa_o : G \to K_{\mathbb{C}}$ lifts to $\tilde{\kappa}_o : G \to \widetilde{K}_{\mathbb{C}}$ such that $\tilde{\kappa}_o|_K : K \hookrightarrow \widetilde{K}_{\mathbb{C}}$. This gives the embedding $G \to P_+\widetilde{K}_{\mathbb{C}}P_-$ as defined in [12].

The picture can be made explicit when we choose $G_{\mathbb{R}}$ to be a matrix Lie group. Since our results pertain only to the case where $G_{\mathbb{R}}$ is of classical type, we shall always make this assumption when referring to $G_{\mathbb{R}}$. There are four such classical bounded symmetric domains and are numbered according to their type. Following ([9]), they can be described as follows:

T*ype*	$G_{\mathbb{R}}/K_{\mathbb{R}}$	$D \subset \mathfrak{p}_+$
I.	$\mathbf{SU}(m,n)/S(\mathbf{U}(m) \times \mathbf{U}(n))$	$\{Z \in \mathbf{M}_{mn}(\mathbb{C}) : I - Z^*Z \gg 0\}$
II.	$\mathbf{Sp}(n,\mathbb{C}) \cap \mathbf{U}(n,n)/\mathbf{U}(n)$	$\{Z \in \mathbf{M}_{nn}(\mathbb{C}) : Z^t = Z \text{ and } I - Z^*Z \gg 0\}$
III.	$\mathbf{SO}^*(2n,\mathbb{C})/\mathbf{U}(n)$	$\{Z \in \mathbf{M}_{nn}(\mathbb{C}) : Z^t = -Z \text{ and } I - Z^*Z \gg 0\}$
IV.	$\mathbf{SO}_o(n,2)/(\mathbf{SO}(n) \times \mathbf{SO}(2))$	$\left\{Z \in \mathbf{M}_{n1}(\mathbb{C}) : \begin{array}{c} 1 + \lvert {}^tZZ\rvert^2 - 2Z^*Z > 0 \\ \text{and } \ 1 - Z^*Z > 0 \end{array}\right\}$

1.3 The Factor of Automorphy

Recall that the action of G and $G_{\mathbb{R}}$ on D agree (via q). If $g \in G$, then we write its projection to $G_{\mathbb{R}}$ via q as $q(g) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. The function κ_o defined for G and $G_{\mathbb{R}}$ above is called the factor of automorphy and $\tilde{\kappa}_o$ the universal factor of automorphy. However, we will need to generalize κ_o slightly as it will be used later to help us define a certain group action. The following description is from Satake [21]. We extend κ_o to a map $\kappa : G_{\mathbb{R}} \times D \to K_{\mathbb{C}}$ satisfying the following relation [1]:

$$g \cdot \exp z \in \exp gz \cdot \kappa(g,z) \cdot P_- \tag{1.2}$$

Let $o \in D$ be the coset element $eK_{\mathbb{R}}$. Then κ has the following properties:

Lemma 1.3.1 *i)* $\kappa(g,o) = \kappa_o(g)$, *for all* $g \in G_{\mathbb{R}}$.
ii) $\kappa(k,z) = k$, *for all* $k \in K_{\mathbb{C}}$ *and* $z \in D$.
iii) $\kappa(g_1 g_2, z) = \kappa(g_1, g_2 z)\kappa(g_2, z)$, *for all* $g_1, g_2 \in G_{\mathbb{R}}$ *and* $z \in D$.

Our immediate goal will be to compute the factor of automorphy for those bounded symmetric domains of classical type.

Proposition 1.3.2 *The factor of automorphy has the expression*
Types I-III: $\kappa(g,Z) = \begin{pmatrix} A-(gZ)C & 0 \\ 0 & D \end{pmatrix}$, $gZ = (AZ+B)(CZ+D)^{-1}$
Type IV: $\kappa(g,Z) = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$, *with* U *and* V *given as follows:*
Express $gZ = \dfrac{1}{(1,i)(CZ_1+DZ_2)}(AZ_1+BZ_2)$ *with* $Z_1 = 2iZ$ *and* $Z_2 = \begin{pmatrix} 1+\ {}^tZZ \\ i-i{}^tZZ \end{pmatrix}$.
Then

$$U = A - B{}^tZ'_+ + (gZ)[CZ'_+ + D(I+\tfrac{1}{2}Z''_+]{}^tW,$$
$$V = (I+\tfrac{1}{2}(gZ)''_+)\{{}^t(gZ)'_+[AZ'_+ + B(I+\tfrac{1}{2}Z''_+)] + [CZ'_+ + D(I+\tfrac{1}{2}Z''_+)]\}(I-\tfrac{1}{2}W''_-),$$

where $W = -\dfrac{1}{2iv}{}^t(C - D{}^tZ'_+)\begin{pmatrix} i \\ 1 \end{pmatrix}$, *as given below in (1.4), (1.5).*

The proof really boils down to finding the $P_+K_{\mathbb{C}}P_-$-decomposition of $G_{\mathbb{R}}$. Fortunately, the decomposition for Types I-III is rather easy and the answer is given below without proof (see [12], p. 5). On the other hand, the Type IV case requires quite a bit more wrangling. Since the author has failed to find a reference for this decomposition (and probably for good reason because the calculations get very messy), complete details will be given.

Lemma 1.3.3 *Let* $G_{\mathbb{R}}$ *be of Types I-III. Then the* $P_+K_{\mathbb{C}}P_-$*-decomposition of* g *is*

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & Z \\ 0 & I \end{pmatrix}\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}\begin{pmatrix} I & 0 \\ W & 0 \end{pmatrix},$$

where $Z = BD^{-1}$, $U = A - BD^{-1}C$, $V = B$, *and* $W = D^{-1}C$.

We begin with some preliminaries. Let $g \in \mathbf{SO}_o(n,2)$. Then according to [28],

$$\mathfrak{p}_+ = \left\{ \widetilde{Z} = \begin{pmatrix} 0 & Z'_+ \\ -{}^tZ'_+ & 0 \end{pmatrix} : Z'_+ = (iZ, Z) \text{ where } Z \text{ is } n \times 1 \right\},$$

[1]The universal factor of automorphy can be extended similarly to $\tilde{\kappa} : G \times D \to \widetilde{K}_{\mathbb{C}}$ (due to Tirao [25], p. 64).

and the embedding of $Z \in D \subset \mathfrak{p}_+$ is given naturally as $Z \mapsto \widetilde{Z}$. Similarly,

$$\mathfrak{p}_- = \widetilde{W} = \left\{ \begin{pmatrix} 0 & W'_- \\ -{}^tW'_- & 0 \end{pmatrix} : W'_- = (iW, -W) \text{ where } W \text{ is } n \times 1 \right\}.$$

To obtain P_+ and P_- now requires exponentiating $\mathfrak{p}_+$ and $\mathfrak{p}_-$, respectively. Use the fact that

$$\widetilde{Z}\widetilde{Z} = \begin{pmatrix} 0 & 0 \\ 0 & Z''_+ \end{pmatrix} \text{ where } Z''_+ = \begin{pmatrix} {}^tZZ & -i{}^tZZ \\ -i{}^tZZ & -{}^tZZ \end{pmatrix},$$

and $(\widetilde{Z})^k = 0$ for $k \geq 3$ to compute $\exp \widetilde{Z} = \sum_{k=0}^{\infty} \frac{1}{k!}(\widetilde{Z})^k$. This gives

$$P_+ = \left\{ \exp \widetilde{Z} = \begin{pmatrix} I & Z'_+ \\ -{}^tZ'_+ & I + \frac{1}{2}Z''_+ \end{pmatrix} : \widetilde{Z} \in \mathfrak{p}_+ \right\}$$

A similar calculation shows that

$$P_- = \left\{ \exp \widetilde{W} = \begin{pmatrix} I & W'_- \\ -{}^tW'_- & I + \frac{1}{2}W''_- \end{pmatrix} : \widetilde{W} \in \mathfrak{p}_- \right\},$$

where W''_- now takes the form $W''_- = \begin{pmatrix} {}^tWW & i{}^tWW \\ i{}^tWW & -{}^tWW \end{pmatrix}$. Furthermore, $K_{\mathbb{C}}$ has the form

$$K_{\mathbb{C}} = \left\{ \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} : U \in \mathbf{SO}(n), V \in \mathbf{SO}(2) \right\}$$

Lemma 1.3.4 *Let $G_{\mathbb{R}} = SO_o(n, 2)$. Then the $P_+K_{\mathbb{C}}P_-$-decomposition of $g \in G_{\mathbb{R}}$ is*

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & Z'_+ \\ -{}^tZ'_+ & I + \frac{1}{2}Z''_+ \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} I & W'_- \\ -{}^tW'_- & I + \frac{1}{2}W''_- \end{pmatrix},$$

where

$$Z'_+ = (iZ, Z), \quad Z = \frac{1}{2id}B \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{with} \quad d = \frac{1}{2i} \begin{pmatrix} i & 1 \end{pmatrix} D \begin{pmatrix} 1 \\ i \end{pmatrix},$$
$$W'_- = (iW, -W), \quad W = -\frac{1}{2id}{}^tC \begin{pmatrix} i \\ 1 \end{pmatrix},$$
$$\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} = \begin{pmatrix} A + Z'_+D^tW'_- & 0 \\ 0 & (I + \frac{1}{2}Z''_+)({}^tZ'_+B + D)(I - \frac{1}{2}W''_-) \end{pmatrix}.$$

Proof. Multiplying the three matrices on the right hand side above together and comparing entries leads to the set of equations

$$\begin{aligned} U - Z'_+V^tW'_- &= A, & UW'_- + Z'_+V(I + \tfrac{1}{2}W''_-) &= B, \\ -{}^tZ'_+U - (I + \tfrac{1}{2}Z''_+)V^tW'_- &= C, & -{}^tZUW'_- + (I + \tfrac{1}{2}Z''_+)V(I + \tfrac{1}{2}W''_+) &= D. \end{aligned} \tag{1.3}$$

The two equations in the second column give $V = (I - \frac{1}{2}Z''_+)^{-1}({}^tZ'_+B + D)(I + \frac{1}{2}W''_-)^{-1}$. It can be checked that $\det(I - \frac{1}{2}Z''_+) = \det(I - \frac{1}{2}W''_-) = 1$ so that the inverses above makes sense. In fact, we easily see that

$$(I - \tfrac{1}{2}Z''_+)^{-1} = (I + \tfrac{1}{2}Z''_+), \quad (I + \tfrac{1}{2}W''_-)^{-1} = I - \tfrac{1}{2}W''_-,$$

and so

$$V = (I + \tfrac{1}{2}Z''_+)({}^tZ'_+B + D)(I - \tfrac{1}{2}W''_-).$$

Now use this expression to obtain $U = A + Z'_+D{}^tW'_-$, which follows from recognizing that $Z'_+{}^tZ'_+ = W'_-{}^tW'_- = 0$.

It remains to compute what Z'_+ is in terms of g. This requires noticing that the map

$$\Theta : V \mapsto v = \frac{1}{2i}\begin{pmatrix} i & 1 \end{pmatrix} V \begin{pmatrix} 1 \\ i \end{pmatrix}$$

identifies $\mathbf{SO}(2)$ with $\mathbf{GL}(1) = \mathbb{C}^*$, and so v is never 0. This is because $\mathbf{SO}(2)$ consists of the matrices

$$\mathbf{SO}(2) = \left\{ V = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{C} \right\},$$

and it is easy to check that $\Theta(V) = e^\theta$. We then use our expression for V above to obtain

$$\Theta(V) = v = \frac{1}{2i}\begin{pmatrix} i & 1 \end{pmatrix} V \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2i}\begin{pmatrix} i & 1 \end{pmatrix} D \begin{pmatrix} 1 \\ i \end{pmatrix} = d,$$

where we have used the relations $\begin{pmatrix} i & 1 \end{pmatrix} Z''_+ = 0$, $W''_- \begin{pmatrix} 1 \\ i \end{pmatrix} = 0$. Now, the equation in (1.3) involving B gives

$$d \cdot Z = \frac{1}{2i} Z \begin{pmatrix} i & 1 \end{pmatrix} V \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2i} Z'_+ V \begin{pmatrix} 1 & i \end{pmatrix} = \frac{1}{2i} B \begin{pmatrix} 1 & i \end{pmatrix},$$

from which we obtain the desired expression for Z after dividing by d. A similar argument can be used to find

$$W = -\frac{1}{2id}\, {}^tC \begin{pmatrix} i \\ 1 \end{pmatrix}$$

and will be left for the reader. This completes the lemma. □

Proof. (of proposition) By definition, $g \exp \widetilde{Z} = \exp(\widetilde{gZ})\kappa(g, Z) \exp \widetilde{W}$. For Types I-III, this means finding the $K_{\mathbb{C}}$-component of

$$g \exp Z = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & Z \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & AZ + B \\ C & CZ + D \end{pmatrix}.$$

Now use our $P_+K_{\mathbb{C}}P_-$ decomposition to get the desired expression.

For Type IV, we now have

$$g \exp \widetilde{Z} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & Z'_+ \\ -{}^tZ'_+ & I + \frac{1}{2}Z''_+ \end{pmatrix} = \begin{pmatrix} A - B{}^tZ'_+ & AZ'_+ + B(I + \frac{1}{2}Z''_+) \\ C - D{}^tZ'_+ & CZ'_+ + D(I + \frac{1}{2}Z''_+) \end{pmatrix}.$$

We use our formula for the $P_+K_{\mathbb{C}}P_-$ decomposition of $G_{\mathbb{R}} = \mathbf{SO}(n,2)$ to get the $K_{\mathbb{C}}$-component of $g \exp \widetilde{Z}$, which is precisely the factor of automorphy

$$\kappa(g, Z) = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}.$$

First, by rewriting $CZ'_+ + D(I + \frac{1}{2}Z''_+) = \frac{1}{2}\begin{pmatrix} CZ_1 + CZ_2 \end{pmatrix}\begin{pmatrix} 1 & -i \end{pmatrix} + \frac{1}{2}D\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$, we get

$$v = \Theta(V) = \frac{1}{2i}\begin{pmatrix} i & 1 \end{pmatrix}[CZ'_+ + D(I + \frac{1}{2}Z''_+)]\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2i}\begin{pmatrix} i & 1 \end{pmatrix}(CZ_1 + DZ_2).$$

Then a short calculation shows that we get the correct answer for g acting on Z:

$$gZ = \frac{1}{2iv}[AZ'_+ + B(I + \frac{1}{2}Z''_+]\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\begin{pmatrix} i & 1 \end{pmatrix}(CZ_1 + DZ_2)}(AZ_1 + BZ_2).$$

By denoting $W = -\frac{1}{2iv}{}^t(C - D{}^tZ'_+)\begin{pmatrix} i \\ 1 \end{pmatrix}$, we can finally write the expression for U and V as

$$V = (I + \tfrac{1}{2}(gZ)''_+)\{{}^t(gZ)'[AZ'_+ + B(I + \tfrac{1}{2}Z''_+)] + [CZ'_+ + D(I + \tfrac{1}{2}Z''_+)]\}(I - \tfrac{1}{2}W''_-), \tag{1.4}$$

and the relation $(gZ)''_+ {}^t(gZ)' = 0$ is used to simplify the answer for U:

$$U = A - B{}^tZ'_+ + (gZ)[CZ'_+ + D(I + \tfrac{1}{2}Z''_+)]^tW. \tag{1.5}$$

This completes the proof. □

1.4 The Kernel Function

Our next goal will be to define a certain kernel function K for the classical domains. Now, we can view $K_{\mathbb{C}} = K_{\mathbb{C}}^+ \times K_{\mathbb{C}}^-$ so that

$$K_{\mathbb{C}} = \left\{ \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} : U \in K_{\mathbb{C}}^+, V \in K_{\mathbb{C}}^- \text{ and } \det(U)\det(V) = 1 \right\}.$$

If $k \in K_{\mathbb{C}}$, we label $k^+ = U$ and $k^- = V$.

Then, following Satake [21], we define the matrix kernel function $\mathcal{K} : D \times D \to K_{\mathbb{C}}$:

$$\mathcal{K}(z, w) = \kappa_o((\exp \overline{z})^{-1}(\exp w))^{-1},$$

where $\overline{z}$ represents conjugation with respect to the complex structure on $\mathfrak{p}$, so that elements of $\mathfrak{p}_+$ are sent to $\mathfrak{p}_-$ (and vice versa). For the classical domains, this means that if the element $z \in D \subset \mathfrak{p}_+$ is viewed as a matrix, then the complex structure is complex conjugate transpose of matrices:

$$z \leftrightarrow \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix}, \quad \overline{z} \leftrightarrow \begin{pmatrix} 0 & 0 \\ Z^* & 0 \end{pmatrix}.$$

Then the kernel function $K : D \times D \to \mathbb{C}^*$ is defined as:

$$K(z,w) = \begin{cases} \det(\mathcal{K}(z,w)^-), & \text{Types I} - \text{III} \\ \Theta(\mathcal{K}(z,w)^-), & \text{Type IV} \end{cases}$$

Proposition 1.4.1 *The kernel function K for the classical domains is given as*

Type I: $K(Z,W) = \det(I - Z^*W)^{-1}$.
Type II: $K(Z,W) = \det(I - \overline{Z}W)^{-1}$.
Type III: $K(Z,W) = \det(I + \overline{Z}W)^{-1}$.
Type IV: $K(Z,W) = (1 + {}^t\overline{Z}\overline{Z}\,{}^tWW - 2Z^*W)^{-1}$.

Proof. For Types I, II, III, just multiply the following matrices

$$(\exp Z^*)^{-1} \exp W = \begin{pmatrix} I & 0 \\ -Z^* & 0 \end{pmatrix} \begin{pmatrix} I & W \\ 0 & I \end{pmatrix}) = \begin{pmatrix} I & W \\ -Z^* & I - Z^*W \end{pmatrix}$$

to easily see that

$$\mathcal{K}(Z,W) = \begin{pmatrix} (I + W(I - Z^*W)^{-1}Z^*)^{-1} & 0 \\ 0 & (I - Z^*W)^{-1} \end{pmatrix},$$

and so $K(Z,W) = \det(I - Z^*W)^{-1}$. Now use the fact that ${}^tZ = Z$ for Type II and ${}^tZ = -Z$ for Type III to get the right expressions for K in these cases.

For Type IV, it can be checked that

$$\begin{aligned}(\exp(\widetilde{Z}')^*)^{-1} \exp \widetilde{W} &= \begin{pmatrix} I & -\overline{Z}'_- \\ {}^t\overline{Z}'_- & I + \frac{1}{2}\overline{Z}''_- \end{pmatrix} \begin{pmatrix} I & W'_+ \\ -{}^tW'_+ & I + \frac{1}{2}W''_+ \end{pmatrix} \\ &= \begin{pmatrix} I + \overline{Z}'_-\,{}^tW'_+ & W'_+ - \overline{Z}'_-(I + \frac{1}{2}W''_+) \\ {}^t\overline{Z}'_- - (I + \frac{1}{2}\overline{Z}''_-){}^tW'_+ & {}^t\overline{Z}'W'_+ + (I + \frac{1}{2}\overline{Z}'')(I + \frac{1}{2}W''_+) \end{pmatrix}.\end{aligned}$$

Now, recall that if

$$\kappa_o \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix},$$

then $\Theta(V) = \frac{1}{2i} \begin{pmatrix} i & 1 \end{pmatrix} D \begin{pmatrix} 1 \\ i \end{pmatrix}$. Therefore, by using the identities

$$\overline{Z}''_- \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} i & 1 \end{pmatrix} W''_+ = 0, \quad \begin{pmatrix} i & 1 \end{pmatrix} \overline{Z}''_- W''_+ \begin{pmatrix} 1 \\ i \end{pmatrix} = 8i\,{}^t\overline{Z}\overline{Z}\,{}^tWW$$

we arrive at

$$\begin{aligned}\Theta(V) &= \tfrac{1}{2i}\begin{pmatrix} i & 1\end{pmatrix}[{}^t\overline{Z}'_- W'_+ + (I+\tfrac12\overline{Z}''_-)(I+\tfrac12 W''_+)]\begin{pmatrix}1\\ i\end{pmatrix}\\ &= 1+\overline{{}^tZZ}\,{}^tWW - 2Z^*W.\end{aligned}$$

Now use the definition of $K(Z,W)=\Theta(V^{-1})=\Theta(V)^{-1}$ to obtained the desired expression for K. $\square$

We will also need to define a determinant factor of automorphy:

$$j(g,z)=\begin{cases}\det(\kappa(g,z)^-)^{-1}, & \text{Types I} - \text{III}\\ \Theta(\kappa(g,z)^-)^{-1}, & \text{Type IV}\end{cases}$$

Lemma 1.4.2 *The kernel function K enjoys the following properties:*
i) $K(z,w)=\overline{K(w,z)}$ *and* $K(z,z)>0$.
ii) $K(gz,gw)=\overline{j(g,z)}^{-1}K(z,w)j(g,w)^{-1}$

Proof. Property i) is obvious. Property ii) for Types I-III follows from the relations

$$(gZ)^*I_{m,n}(gW)=Z^*I_{m,n}W \iff (AZ+B)^*(AW+B)-(CZ+D)^*(CW+D)=Z^*W$$

and

$$I-(gZ)^*(gW)=(CZ+D)^{*-1}(I-Z^*W)(CW+D)^{-1},\;\; j(g,Z)=(CZ+D)^{-1}.$$

The identity for Type IV requires a much more complicated expression. To simplify the notation, recall the definition of Z_1 and Z_2 earlier and denote

$$\hat{Z}=\begin{pmatrix}Z_1\\ Z_2\end{pmatrix},\;\; U=AZ_1+BZ_2,\;\; V=\begin{pmatrix}CZ_1+DZ_2\end{pmatrix}.$$

Then we define $g\hat{Z}$ as normal matrix multiplication to obtain

$$g\hat{Z}=\begin{pmatrix}U\\ V\end{pmatrix},\;\; gZ=Uv^{-1},\;\; v=\begin{pmatrix}i & 1\end{pmatrix}V.$$

In what follows, we use z and w subscripts to distinguish the terms defined above for two different elements Z and W. First, some preliminary identities will be needed:

$$\begin{aligned}&\hat{Z}^*g^*I_{n,2}\,g\hat{W}=\hat{Z}^*\hat{W} &&\iff U_z^*U_w-V_z^*V_w=Z_1^*W_1-Z_2^*W_2\\ &{}^t\hat{Z}\,{}^tgg\hat{Z}={}^t\hat{Z}\hat{Z}=0, &&\iff {}^tUU+{}^tVV=0.\end{aligned}$$

Also, the following three equalities can be easily verified:

$$Z_1W_1+Z_2W_2=4(1+\overline{{}^tZZ}\,{}^tWW-2Z^*W),$$
$$(v_z^*v_w)^2-2v_z^*V_z^*V_wv_w=v_z^*V_z^*\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}V_wv_w-2v_z^*V_z^*V_wv_w=-v_z^*V_z^*\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}V_wv_w,$$
$$\overline{{}^tV_zV_z}\,{}^tV_wV_w=\overline{{}^tV_z}\begin{pmatrix}-i\\ 1\end{pmatrix}V_z^*\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}V_w\begin{pmatrix}i & 1\end{pmatrix}V_w=v_z^*V_z^*\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}V_wv_w.$$

This gives $(v_z^* v_w)^2 - 2v_z^* V_z^* V_w v_w + \overline{{}^tV_z V_z}\,{}^tV_w V_w = 0$. Using $j(g,Z) = (v_z/2i)^{-1}$ and $j(g,W) = (q_w/2i)^{-1}$, we get

$$\begin{aligned}
&1 + \overline{{}^t(gZ)(gZ)}\,{}^t(gW)(gW) - 2(gZ)^*(gW) = \\
&= \frac{1}{v_z^* v_w}[v_z^* v_w + \frac{1}{v_z^* v_w}(\overline{{}^tU_z U_z}\,{}^tU_w U_w) - 2U_z^* U_w] \\
&= \frac{1}{v_z^* v_w}[v_z^* v_w + \frac{1}{v_z^* v_w}(\overline{{}^tV_z V_z}\,{}^tV_w V_w) - 2(Z_1^* W_1 - Z_2^* W_2 + V_z^* V_w)] \\
&= \frac{1}{(v_z^* v_w)^2}((v_z^* v_w)^2 - 2v_z^* V_z^* V_w v_w + \overline{{}^tV_z V_z}\,{}^tV_w V_w) + \frac{4}{v_z^* v_w}(1 + \overline{{}^tZZ}\,{}^tWW - 2Z^* W) \\
&= 0 + j(g,Z)^*(1 + \overline{{}^tZZ}\,{}^tWW - 2Z^*W)j(g,W).
\end{aligned}$$

The transformation property of $K(gZ, gW)$ for Type IV now follows. □

Let $K_{\mathbb{C}}$ act on $\mathfrak{p}_+$ by the adjoint action and consider the linear map $J(g,z) = \mathrm{Ad}_{\mathfrak{p}_+}\kappa(g,z)$. It is well-known that $d(gz) = J(g,z)dz$ (see [21], Ch. II, §5), Then the jacobian of $d(gz)$ is $\det(J(g,z)) = j(g,z)^p$, where the exponent p depends on the bounded symmetric domain (see the lemma below).

Let $B(Z,W)$ be the Bergman kernel on D. Then B can be written as a constant factor (namely the volume of D) of our kernel function K raised to an appropriate exponent. Here are explicit formulas for B in the classical cases:

Lemma 1.4.3 *(Hua [11]) The Bergman kernel can be expressed as*

$$B(Z,W) = \mathrm{vol}(D)^{-1} K(Z,W)^p$$

where the values for $\mathrm{vol}(D)$ *and* p *are given by the table below:*

Type	$\mathrm{vol}(D)$	p
I	$\frac{1!2!\ldots(m-1)!1!2!\ldots(n-1)!}{1!2!\ldots(m+n-1)!}\pi^{m+n}$	$m+n$
II	$\frac{2!4!\ldots(2n-2)!}{n!(n+1)!\ldots(2n-1)!}\pi^{\frac{n(n+1)}{2}}$	$n+1$
II	$\frac{2!4!\ldots(2n-4)!}{(n-1)!n!\ldots(2n-3)!}\pi^{\frac{n(n-1)}{2}}$	$n-1$
IV	$\frac{1}{2^{n-1}n!}\pi^n$	n

The Bergman kernel gives rise to the hermitian (or Bergman) metric Ψ on D defined by the fundamental 2-form $i\partial\overline{\partial}\log B(Z,Z)$ (see [21], II. §6 and [26], §1):

Lemma 1.4.4 *The Bergman metric* Ψ *on* D *is given by:*

Type I: $\Psi = ip\,\mathrm{Tr}\{(I - Z^*Z)^{-1}dZ^*(I - ZZ^*)^{-1}dZ\}$.
Type II: $\Psi = ip\,\mathrm{Tr}\{(I - \overline{Z}Z)^{-1}d\overline{Z}(I - Z\overline{Z})^{-1}dZ\}$.
Type III: $\Psi = ip\,\mathrm{Tr}\{(I + \overline{Z}Z)^{-1}d\overline{Z}(I + Z\overline{Z})^{-1}dZ\}$.
Type IV: $\Psi = ip\,(1 + \overline{{}^tZZ}\,tZZ - 2Z^*Z)^{-2}(2dZ^*dZ - d(\overline{{}^tZZ})d({}^tZZ))$.

Lastly, the riemannian metric ω on D is obtained by taking the imaginary part of Ψ.

1.5 One-Dimensional K-Types

Lastly, we need to review how the factor of automorphy can be composed with representations of K to obtain "generalized" factors of automorphy. These take the form $\tilde{\tau}(\tilde{\kappa}(g,z))$, where $\tilde{\tau}$ is a representation of K extended to its complexification $\widetilde{K}_{\mathbb{C}}$. For our needs however, we will restrict $\tilde{\tau}$ to be one-dimensional, hence a character of K. Since $K = K_s Z_K^0$ and K_s is semisimple, these characters are in fact prescribed by characters of $Z_K^0 = \{\exp lZ_0 : l \in \mathbb{R}\}$. Any such character of Z_K^0 can be parametrized as $\tilde{\tau}_\nu$, where $\nu \in \mathbb{R}$ and

$$\frac{d}{dl}\tilde{\tau}_\nu(\exp lZ_0) = il\nu.$$

We then extend $\tilde{\tau}_\nu$ to $K = K_s Z_K^0$ by making it trivial on K_s. Complexifying $\tilde{\tau}_\nu$ then gives us a complex character of $\widetilde{K}_{\mathbb{C}}$. It is clear that $\tilde{\tau}_\nu^{-1} = \tilde{\tau}_{-\nu}$.

Suppose a character $\tilde{\tau}_\nu$ of K was fixed. Then recall the covering $q : G \to G_{\mathbb{R}}$. The discrete subgroup $Q = q^{-1}(e) \cap Z_K^0$ lies inside $Z = Z_G \cap Z_K^0$. The identity component of the center of $K_{\mathbb{R}}$, $Z_{K_{\mathbb{R}}}^0$, is then a circle isomorphic to Z_K^0/Q. For $\tilde{\tau}_\nu$ to push down to a well-defined character τ_ν of $K_{\mathbb{R}}$ means that τ_ν should be trivial on Q. This can be accomplished by properly parametrizing Q inside Z_K^0. Furthermore, we shall set $\tau_\nu(k)$ to agree with the determinant factor of automorphy $j(k,o)^\nu$ on $K_{\mathbb{R}}$. The last step is to extend τ_ν to $K_{\mathbb{R}}$ and complexify to a character of $K_{\mathbb{C}}$. The generalized factors of automorphy are then given as

Type I-III: $\tau_\nu(\kappa(g,Z)) = j(g,Z)^\nu = \det(CZ+D)^{-\nu}$.
Type IV: $\tau_\nu(\kappa(g,Z)) = j(g,Z)^\nu = [\frac{1}{2i}\begin{pmatrix} i & 1\end{pmatrix}\begin{pmatrix} CZ_1 + DZ_2\end{pmatrix}]^{-\nu}$.

It will then be necessary to define an argument function $\arg\tilde{\tau}_\nu(k)$ on K as follows. Since τ maps into $\mathbb{C}^*$ and K is simply-connected, we can properly define

$$\arg\tilde{\tau}_\nu(k) = \frac{1}{2i}(\log\tilde{\tau}_\nu(k) - \log\overline{\tilde{\tau}_\nu(k)})$$

by fixing a branch point of our logarithm so that $\arg\tilde{\tau}_\nu(\exp lZ_0) = l\nu$ on Z_K^0. Since $\widetilde{K}_{\mathbb{C}}$ is also simply-connected, this argument function makes sense when $\tilde{\tau}_\nu$ is extended to $\widetilde{K}_{\mathbb{C}}$.

Then having fixed our branch point means we can push the argument function down to $\tau_\nu(k)$ for $k \in K_{\mathbb{R}}$ via the covering $q : K \to K_{\mathbb{R}}$. This gives the standard argument function for complex numbers.

Chapter 2

Circle Bundles

2.1 A Circle Bundle Construction

The construction of our circle bundles will proceed in three steps. The first is to define line extensions $\mathbb{G}$ and $\mathbb{D}$ of G and D, respectively. This will give an action Φ_ν of $\mathbb{G}$ on $\mathbb{D}$ determined from a one-dimensional K-type $\tilde{\tau}_\nu$. From there, we push Φ_ν down to an action Φ_ν^1 of $\mathbb{G}^1$ acting on our circle bundle $\mathbb{D}^1$, where $\mathbb{G}^1$ is a quotient of $\mathbb{G}$. This is the second step. Similar, pushing Φ_ν^1 down to an action of $\mathbb{G}^1_{\mathbb{R}}$ (a circle extension of $G_{\mathbb{R}}$) on $\mathbb{D}^1$ gives us the desired setting in completing our third step.

Let $\mathbb{D} = D \times \mathbb{R}$ be the product manifold of D and the real line. If $z \in D$ and $t \in \mathbb{R}$, we shall write (z,t) or z_t to denote the corresponding element of $\mathbb{D}$. Let $\mathbb{G} = G \times \mathbb{R}$ be the direct product of G and the real line $\mathbb{R}$ as Lie groups. We write out its elements as $g_s = (g,s)$, $g \in G$ and $s \in \mathbb{R}$ and define the multiplication as $(g_1,s_1)(g_2,s_2) = (g_1g_2, s_2+s_1)$. Fix a nontrivial one-dimensional representation $\tilde{\tau} = \tilde{\tau}_\nu$ of K (ν a nonzero number) and extend $\tilde{\tau}$ to $\widetilde{K}_{\mathbb{C}}$.

Lemma 2.1.1 *i) The action $\Phi_\nu : \mathbb{G} \times \mathbb{D} \to \mathbb{D}$ defined by*

$$g_s z_t = (gz, \arg \tilde{\tau}^{-1}(\tilde{\kappa}(g,z)) + t + s) \tag{2.1}$$

is transitive with isotropy subgroup $\mathbb{K}_\nu = \{(k, \arg\tilde{\tau}(k)) : k \in K\}$. Furthermore, Φ_ν restricted to G (viewed as a subgroup of $\mathbb{G}$) is also transitive with isotropy subgroup K_s, and $\mathbb{D} \cong \mathbb{G}/\mathbb{K}_\nu \cong G/K_s$ as homogeneous spaces under this action.

Proof. i) Let $g_s = (g,s) \in \mathbb{G}$ and $z_t = (z,t) \in \mathbb{D}$. Now, define an action $\Phi_\nu : \mathbb{G} \times \mathbb{D} \to \mathbb{D}$ as follows:

$$g_s z_t = (gz, \arg \tilde{\tau}^{-1}(\tilde{\kappa}(g,z)) + t + s) \tag{2.2}$$

Observe that Φ_ν is indeed an action:

$$\begin{aligned}
g_s \tilde{g}_{\tilde{s}} z_t &= (g\tilde{g}z, \arg\tilde{\tau}^{-1}(\tilde{\kappa}(g\tilde{g},z)) + t + s + \tilde{s}) \\
&= (g\tilde{g}z, \arg\tilde{\tau}^{-1}(\tilde{\kappa}(g,\tilde{g}z)\cdot\tilde{\kappa}(\tilde{g},z)) + t + s + \tilde{s}) \\
&= (g\tilde{g}z, \arg\tilde{\tau}^{-1}(\tilde{\kappa}(g,\tilde{g}z)) + (\arg\tilde{\tau}^{-1}(\tilde{\kappa}(\tilde{g},z)) + t + \tilde{s}) + s) \\
&= g_s(\tilde{g}z, \arg\tilde{\tau}^{-1}(\tilde{\kappa}(\tilde{g},z)) + t + \tilde{s}) \\
&= g_s(\tilde{g}_{\tilde{s}} z)
\end{aligned}$$

and so Φ_ν is well-defined.

Next, we show that Φ_ν is transitive by showing that Φ_ν restricted to G (considered as a subgroup of $\mathbb{G}$) is transitive on $\mathbb{D}$. Also, since G is already transitive on D, it suffices to prove that G is transitive on each slice $z \times \mathbb{R} \subset \mathbb{D}$. Recall the covering $q : G \to G_{\mathbb{R}}$ and pick $g \in G$ so that $q(g) = \exp z \in P_+$. If $k \in K$, then $\tilde{\kappa}(gkg^{-1}, z) = k$ since $q(gkg^{-1})\exp z = q(g)q(k) = (\exp z)q(k)$. So $gkg^{-1}(z,0) = (z, \arg\tilde{\tau}(k))$ and the orbit $gKg^{-1}(z,0)$ is precisely $z \times \mathbb{R}$. This proves transitivity.

Let us now find the isotropy subgroup $\mathbb{K}_\nu$ of Φ_ν at the origin $(o,0) \in \mathbb{D}$, where o is the origin in D. For an element g_s to stabilize $(o,0)$ means

$$g_s(o,0) = (go, \arg\tilde{\tau}^{-1}(\tilde{\kappa}(g,o)) + s) = (o,0).$$

Since K is the isotropy subgroup of o and $\tilde{\kappa}(g,o) = g$ whenever g is in K, this implies $g \in K$ and $s = \arg\tilde{\tau}(g)$. Clearly then $\mathbb{K}_\nu = \{k_\nu = (k, \arg\tilde{\tau}(k)) : k \in K\}$. As for the isotropy subgroup of Φ_ν restricted to G, we instead have $g \in K$ and $\arg\tilde{\tau}(g,o) = 0$. If we decompose $g = kz$, with $k \in K_s$ and $z \in Z_K^0$, then this forces $z = e$. This means $g \in K_s$ and shows that the isotropy subgroup is precisely K_s. Hence, $\mathbb{D} \cong \mathbb{G}/\mathbb{K}_\nu \cong G/K_s$. □

The second step is to define $\mathbb{D}^1 = D \times S^1_\nu$ be the product manifold of D and the unit circle S^1_ν. Here, the subscript ν on S^1_ν indicates that S^1_ν has circumference $2\pi\nu$ and parametrized by the interval $[0, 2\pi\nu)$. Therefore, an element (z,t) of $\mathbb{D}^1$ means $z \in D$ and $z \in [0, 2\pi\nu)$. Now let $G_\mathbb{R}$ be a real form of $G_\mathbb{C}$ and $q : G \to G_\mathbb{R}$ the projection. Then $K = q^{-1}(K_\mathbb{R})$, where $K_\mathbb{R}$ is the maximal compact subgroup of $G_\mathbb{R}$. Again, define $\widetilde{Q} = q^{-1}(e)$ and $Q = \widetilde{Q} \cap Z_K^0$. Now define the discrete subgroup $\mathbb{Q}_\nu = \{(q, \arg\tilde{\tau}_\nu(q)) : q \in Q\}$. Then $\mathbb{Q}_\nu$ is a normal subgroup of $\mathbb{G}$ and so the quotient $\mathbb{G}^1 = \mathbb{G}/\mathbb{Q}_\nu$ becomes a reductive Lie group with finite center and $\mathbb{K}^1 = \mathbb{K}_\nu/\mathbb{Q}_\nu$ is a compact subgroup of $\mathbb{G}^1$. Let $q^1 : \mathbb{G} \to \mathbb{G}^1$ denote the covering map. We shall freely write $g_s = (g,s)$ to denote elements of $\mathbb{G}$ or $\mathbb{G}^1$. If it is needed to distinguish g_s as a coset element in $\mathbb{G}^1$, we shall then write $[g_s] = g_s\mathbb{Q}_\nu$.

Let Q have a suitable parametrization in Z_K^0 so that $\tilde{\tau}_\nu$ pushes down to a character τ_ν of $K_\mathbb{R}$, i.e. τ_ν should be trivial on Q.

Lemma 2.1.2 *The action Φ_ν projects to a transitive action $\Phi^1_\nu : \mathbb{G}^1 \times \mathbb{D}^1 \to \mathbb{D}^1$ given by*

$$[g_s]z_t = (gz, \arg\tau_\nu^{-1}(\kappa(g,z)) + t + s) \tag{2.3}$$

is transitive with isotropy subgroup $\mathbb{K}^1 = K_sQ^1_\nu/\mathbb{Q}_\nu$, and so $\mathbb{D}^1 \cong \mathbb{G}^1/\mathbb{K}^1$.

Proof. To show that Φ^1_ν is well-defined, it suffices to prove that Φ_ν is trivial on $\mathbb{Q}_\nu$ so that it factors to an action of $\mathbb{G}^1$ on $\mathbb{D}^1$. Let $k_\nu = (k, \arg\tilde{\tau}(k)) \in \mathbb{Q}_\nu$. Note then that $kz = z$ and $\tilde{\kappa}(k,z) = k$ for all $z \in D$. Therefore

$$\begin{aligned}
k_\nu z_t &= (kz, \arg\tilde{\tau}^{-1}(\tilde{\kappa}(k,z)) + t + \arg\tilde{\tau}(k)) \\
&= (z, -\arg\tilde{\tau}(k) + t + \arg\tilde{\tau}(k))) \\
&= z_t
\end{aligned}$$

and hence the action of Φ^1_ν is well-defined on $\mathbb{D}^1$.

Transitivity of Φ^1_ν follows from Φ_ν being transitive. As for the isotropy subgroup of Φ^1_ν, we have

$$[g_s](o,0) = (go, \arg\tau^{-1}(\kappa(g,o)) + s) = (o,0).$$

Again, it must be that $g \in K$, but now $s = \arg\tau(\kappa(g,o))$ modulo $2\pi\nu$. Define

$$Q^1_\nu = \{q \in Z_K^0 : \arg\tau(q) = 2\pi l\nu \text{ for some } l \in \mathbb{Z}\}.$$

Then g must have the form kq, with K_s and $q \in Q^1_\nu$ since $\arg\tau(q) = 2\pi l\nu = 0 \pmod{2\pi\nu}$. So the isotropy subgroup must be $\mathbb{K}^1 = K_sQ^1_\nu/\mathbb{Q}_\nu$ and hence $\mathbb{D}^1 \cong \mathbb{G}^1/\mathbb{K}^1$. Note that $\mathbb{K}^1$ is a compact subgroup of $\mathbb{G}^1$. □

Our last step is to now push Φ^1_ν down to $\mathbb{G}^1_\mathbb{R} = G_\mathbb{R} \times S^1_\nu$. This requires the projection map $q^1_\mathbb{R} : \mathbb{G} \to \mathbb{G}^1_\mathbb{R}$, where $q^1_\mathbb{R} = q \times p_\nu$. Here, $q : G \to G_\mathbb{R}$ is the usual covering map and $p_\nu : \mathbb{R} \to S^1_\nu$ is defined as $t \mapsto t\nu$.

Lemma 2.1.3 *The action $\Phi^\mathbb{R}_\nu : \mathbb{G}^1_\mathbb{R} \times \mathbb{D}^1 \to \mathbb{D}^1$ given by*

$$g_s z_t = (gz, \arg \tau_\nu^{-1}(\kappa(g,z)) + t + s)$$

is transitive with isotropy subgroup $\mathbb{K}^1_\mathbb{R} = \{(k, \arg \tau(k)) : k \in K_\mathbb{R}\}$. Furthermore, $\Phi^\mathbb{R}_\nu$ restricted to $G_\mathbb{R}$ is also transitive with isotropy subgroup $[K_\mathbb{R}, K_\mathbb{R}]$ and so $\mathbb{D}^1 \cong \mathbb{G}^1_\mathbb{R}/\mathbb{K}^1_\mathbb{R} \cong G_\mathbb{R}/[K_\mathbb{R}, K_\mathbb{R}]$.

Proof. The proof of these statements follows exactly the same argument as for the action Φ_ν We have $(g,s)(o,0) = (o,0)$. We leave it for the reader to check the details. □

We mention again that the actions Φ_ν, Φ^1_ν, and $\Phi^\mathbb{R}_\nu$ defined above depend on $\tilde{\tau}_\nu$. Because of this dependence, it is appropriate to attach the subscript ν to $\mathbb{D}_\nu$ and $\mathbb{D}^1_\nu$ whenever there is a need to identify them as homogeneous spaces of the form $\mathbb{D}_\nu \cong \mathbb{G}/\mathbb{K}_\nu$ and $\mathbb{D}^1_\nu \cong \mathbb{G}^1/\mathbb{K}^1 = \mathbb{G}^1_\mathbb{R}/\mathbb{K}^1_\mathbb{R}$ under the action of Φ_ν and Φ^1_ν (or $\Phi^\mathbb{R}_\nu$), respectively. Furthermore, the reason for consider the two different actions Φ^1_ν and $\Phi^1_\mathbb{R}$ on $\mathbb{D}^1$ is as follows. First, view $\mathbb{D}^1$ with the homogeneous structure $\mathbb{G}^1/\mathbb{K}^1$. Then the extensions $\mathbb{G}^1$ and $\mathbb{K}^1$ are precisely the same extensions defined by Flensted-Jensen [2] for doing harmonic analysis of functions on $\mathbb{G}^1$ that are bi-$\mathbb{K}^1$-invariant. However, no reference is made of the manifold $\mathbb{D}^1$. This makes our approach different from his since our objective is to show weak symmetry of $\mathbb{D}^1$.

It is clear the $\mathbb{D}$ is the universal cover of $\mathbb{D}^1$. Furthermore, it is easy to show that $\mathbb{D}$ (resp. $\mathbb{D}^1$) can be viewed as a line (resp. circle) bundle over D under their natural projection maps. Define $\pi : \mathbb{D} \to D$ given by $\pi(z_t) = z$ and a right action on $\mathbb{D}$ given by

$$\begin{array}{rcl} \rho : \mathbb{D} \times \mathbb{R} & \to & \mathbb{D} \\ z_t \cdot \theta & \mapsto & (z, t + \theta). \end{array}$$

A similar projection map π^1 and right action ρ^1 can be defined for $\mathbb{D}^1$.

Lemma 2.1.4 $(\mathbb{D}, \pi)$ *(resp. $(\mathbb{D}^1, \pi^1)$) is a principal bundle over D with structure group $\mathbb{R}$ (resp. S^1) and principal action ρ (resp. ρ^1).*

Proof. It is clear that ρ is an action and that $\pi(z_t \cdot \theta) = \pi(z_t)$. Also, ρ commutes with Φ,

$$\begin{array}{rcl} g_s(z_t \cdot \theta) & = & g_s(z, t+\theta) \\ & = & (gz, t + \theta + s + \arg \tilde{\tau}^{-1}(\tilde{\kappa}(g,z))) \\ & = & (gz, t + s + \alpha + \arg \tilde{\tau}^{-1}(\tilde{\kappa}(g,z))) \cdot \theta \\ & = & (g_s z_t) \cdot \theta \end{array},$$

hence $\pi\rho = \rho\pi$. We note that the fiber at each $z \in D$ is $\pi^{-1}(z) \cong \mathbb{R}$. A similar argument holds for the case of $\mathbb{D}^1$. □

Now, $\mathbb{G}$, $\mathbb{G}^1$, and $\mathbb{G}^1_{\mathbb{R}}$ have the same Lie algebra $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}^1$, where $\mathfrak{g}^1 = \mathfrak{g} + \mathbb{R}$, $\mathfrak{p}^1 = \mathfrak{p} + \mathbb{R}$, and $\mathfrak{k}^1 = \mathfrak{k}_s + \{\tau(X) : X \in \mathbb{R}\}$ (the Lie algebra of $\mathbb{K}_\nu$, $\mathbb{K}^1$, and $\mathbb{K}^1_{\mathbb{R}}$). We also have the relations

$$[\mathfrak{k}^1, \mathfrak{p}^1] \subset \mathfrak{p}^1 \text{ and } [\mathfrak{p}^1, \mathfrak{p}^1] \subset \mathfrak{k}^1. \tag{2.4}$$

Lemma 2.1.5 *$\mathbb{D}$ is Riemannian manifold with the $\mathbb{G}$-invariant metric Ω_ν. The corresponding metric on $\mathbb{D}^1$ invariant under $\mathbb{G}^1_{\mathbb{R}}$ is the one induced from Ω_ν.*

Proof. $\mathbb{G}/\mathbb{K}_\nu$ has $\mathfrak{p}$ as its tangent space. The relations (2.4) implies that $Ad(\mathbb{K}_\nu)\mathfrak{p} \subset \mathfrak{p}$ and so $\mathbb{D}$ is a reductive homogeneous space. Any such space has a $\mathbb{G}$-invariant Riemannian metric Ω_ν (see [18], Prop. 6.58). This makes $\mathbb{G}$ a group of isometries for the Riemannian manifold $\mathbb{D}$ under Ω_ν. Since $\mathbb{G}^1_{\mathbb{R}}$ and $\mathbb{G}^1$ are quotients of $\mathbb{G}$ and $\mathbb{D}$ is the universal cover of $\mathbb{D}^1$, the Riemannian metric on $\mathbb{D}^1$ must be the one induced from Ω_ν. $\square$

Later on, we shall give an explicit expression for Ω_ν when G is of classical type. Now, define $\mathbb{Z}_\nu = \{k_\nu = (k, \arg\tilde{\tau}(k)) : k \in Z^0_K\}$ to be a subgroup of $\mathbb{G}$.

Lemma 2.1.6 *The injection $j : G \to \mathbb{G}$ defined by $j(g) = [(g, 0)]$ gives diffeomorphisms of G with $\mathbb{G}/\mathbb{Z}_\nu$ and G/K_s with $\mathbb{G}/\mathbb{K}_\nu$. Furthermore, we have G/Q diffeomorphic to $\mathbb{G}^1/q(\mathbb{Z}_\nu)$ and G/K_sQ with $\mathbb{G}^1/q(\mathbb{K}^1)$.*

Proof. Let $\pi : \mathbb{G} \to \mathbb{G}/\mathbb{Z}_\nu$ be a projection map. Then it is clear that $\pi \circ j$ is one-to-one and onto with inverse given by $(\pi \circ j)^{-1}(g_s\mathbb{Z}_\nu) = g\exp(t\nu\eta)$, where $\tilde{\tau} = \tilde{\tau}_\nu$ is the fixed character of K which parametrizes Q. This gives the desired diffeomorphism between G and $\mathbb{G}/\mathbb{Z}_\nu$. $\square$

Remarks. Note that if we view $\mathbb{D} \cong G/K_s$, then the riemannian metric Ω_ν does **not** correspond to the standard product metric induced on G/K_s via the diffeomorphism $G/K_s \cong G/K \times \mathbb{R}$. This occurs only when $\nu = 0$. In that case, S^1_ν reduces to a point and $\mathbb{D}^1 = D$ becomes a symmetric space. We shall avoid this trivial situation by always assuming that $\nu \neq 0$.

2.2 The Riemannian Metric

Let G be defined as before and $G_{\mathbb{R}}$ be a real form of $G_{\mathbb{C}}$. Fix a one-dimensional K-type τ_ν and define the line bundle $\mathbb{D}$ and the circle bundle $\mathbb{D}^1_\nu$ as before. Here are the promised expressions for the $\mathbb{G}^1_{\mathbb{R}}$-invariant metric on $\mathbb{D}^1_\nu$ for the classical domains. If z is a complex number, let $\operatorname{Re} z$ and $\operatorname{Im} z$ denote its real and imaginary parts, respectively. Also, we write $\operatorname{Tr}\{A\}$ to denote the trace of the matrix A.

Proposition 2.2.1 *Let ω be the riemannian metric on D invariant under $G_{\mathbb{R}}$. Then the riemannian metric Ω_ν on $\mathbb{D}^1_\nu$ invariant under $\mathbb{G}^1_{\mathbb{R}}$ is*

$$\Omega_\nu = \omega + (\frac{dt}{\nu} - \delta)^2, \tag{2.5}$$

where $\delta = \delta(Z)$ at $(Z, t) \in \mathbb{D}^1_\nu$ is expressed as

Type I: $\delta = \operatorname{Im}\operatorname{Tr}\{Z^* dZ(1 - Z^*Z)^{-1}\}$.
Type II: $\delta = \operatorname{Im}\operatorname{Tr}\{\overline{Z} dZ(1 - \overline{Z}Z)^{-1}\}$.
Type III: $\delta = \operatorname{Im}\operatorname{Tr}\{-\overline{Z} dZ(1 + \overline{Z}Z)^{-1}\}$.
Type IV: $\delta = \operatorname{Im}\left\{\dfrac{2Z^*dZ - 2q(Z)^*dq(Z)}{1 + \overline{{}^tZZ}\,{}^tZZ - 2Z^*Z}\right\}, \quad q(Z) = \dfrac{Z_2}{\begin{pmatrix} i & 1 \end{pmatrix} Z_2}$.

The heart of the proof will rely on the following lemma.

Lemma 2.2.2 *The following identity holds for the classical domains:*

$$d \arg j(g, Z) = \delta(Z) - \delta(gZ) \tag{2.6}$$

Proof. For Type I, we have $j(g, Z) = \det(CZ + D)^{-1}$ and so

$$\begin{aligned} d \arg j(g, Z) &= (1/2i)(d \log \det(CZ + D)^* - d \log \det(CZ + D)) \\ &= (1/2i)\operatorname{Tr}\{d(CZ + D)^*(CZ + D)^{*-1} - d(CZ + D)(CZ + D)^{-1}\}. \end{aligned}$$

By setting $U = (AZ + B)$ and $V = (CZ + D)$ to simplify our notation, it suffices to find an expression for $d(CZ + D)(CZ + D)^{-1} = dVV^{-1}$. First, notice that $gZ = UV^{-1}$. Then consider the following equivalent relations:

$$\begin{pmatrix} Z^* & I \end{pmatrix} g^* I_{m,n} g d \begin{pmatrix} Z \\ I \end{pmatrix} = \begin{pmatrix} Z^* & I \end{pmatrix} d \begin{pmatrix} Z \\ I \end{pmatrix} \iff U^*dU - V^*dV = Z^*dZ.$$

Now,

$$\begin{aligned} dVV^{-1} &= V^{*-1}(V^*dV)V^{-1} \\ &= V^{*-1}(U^*dU - Z^*dZ)V^{-1} \\ &= V^{*-1}U^*(dU)V^{-1}) - V^{*-1}Z^*dZV^{-1} \\ &= V^{*-1}U^*(UV^{-1}dVV^{-1} + d(UV^{-1})) - V^{*-1}Z^*dZV^{-1} \\ &= (gZ)^*(gZ)dVV^{-1} + (gZ)^*d(gZ) - V^{*-1}Z^*dZV^{-1}, \end{aligned}$$

or equivalently

$$(I - (gZ)^*(gZ))dVV^{-1} = (gZ)^*d(gZ) - V^{*-1}Z^*dZV^{-1},$$

and so

$$dVV^{-1} = (I - (gZ)^*(gZ))^{-1}[(gZ)^*d(gZ) - V^{*-1}Z^*dZV^{-1}].$$

We multiply through and take the trace, simplifying the second term on the right hand side to

$$\begin{aligned} \operatorname{Tr}\{(I - (gZ)^*(gZ))^{-1}V^{*-1}Z^*dZV^{-1}\} &= \operatorname{Tr}\{V^{-1}(I - (gZ)^*(gZ))^{-1}V^{*-1}Z^*dZ\} \\ &= \operatorname{Tr}\{(I - Z^*Z)^{-1}Z^*dZ\}, \end{aligned}$$

where the identity $(I - (gZ)^*(gZ))^{-1} = V^*(I - Z^*Z)^{-1}V$ has been used. We then get

$$\operatorname{Tr}\{dVV^{-1}\} = \operatorname{Tr}\{(I - (gZ)^*(gZ))^{-1}(gZ)^*d(gZ)\} - \operatorname{Tr}\{(I - Z^*Z)^{-1}Z^*dZ\}.$$

We now use this trace equation to compute

$$\begin{aligned} d\arg j(g,Z) &= d\arg\det(CZ+D)^{-1} \\ &= (1/2i)\mathrm{Tr}\{(dVV^{-1})^* - dVV^{-1}\} \\ &= -\mathrm{Im}\,\mathrm{Tr}\{(I-(gZ)^*(gZ))^{-1}(gZ)^*d(gZ)\} + \mathrm{Im}\,\mathrm{Tr}\{(I-Z^*Z)^{-1}Z^*dZ\} \\ &= \delta(Z) - \delta(gZ), \end{aligned}$$

as desired.

For Types II and III, we observe in these two cases that the argument parallels that used for Type I. This follows from recognizing that the expression for the determinant factor of automorphy $j(g,Z)$ doesn't change. Also, D lies inside $\{Z \in \mathbf{M}_{nn}(\mathbb{C}) : I - Z^*Z >> 0\}$ and $G_{\mathbb{R}}$ is a subgroup of $\mathbf{SU}(n,n)$, so $d\arg j(g,Z)$ produces the same identity formula as in Type I case. Now use the identities ${}^tZ = Z$ and ${}^tZ = -Z$ for Types II and III, respectively, to simplify the formula to the desired expressions as stated in the lemma.

Again, the Type IV argument is complicated and must be derived separately. First, we recall some of the notation used earlier in showing the transformation property of the kernel function K for the Type IV case:

$$U = AZ_1 + BZ_2, \quad V = CZ_1 + DZ_2, \quad v = \begin{pmatrix} i & 1 \end{pmatrix} V.$$

We shall often write v^* to mean $\bar{v}$ even though v is not a matrix. Secondly, define $q(Z) = Z_2 z_2^{-1}$, where $z_2 = \begin{pmatrix} i & 1 \end{pmatrix} Z_2$. Then $q(gZ) = Vv^{-1}$ and $2q(Z)^*dq(Z) = \overline{{}^tZZ}d({}^tZZ)$. Now, $j(g,Z) = (v/2i)^{-1}$ and so

$$\begin{aligned} d\arg j(g,Z) &= \tfrac{1}{2i}(\log\det(v/2i)^* - \log\det(v/2i)) \\ &= \tfrac{1}{2i}\mathrm{Tr}\{(dvv^{-1})^* - dvv^{-1}\}. \end{aligned}$$

Therefore, to find an expression for dvv^{-1}, note that $gZ = Uv^{-1}$ and check the following equalities:

$$\begin{aligned} \hat{Z}^*g^*I_{n,2}\,d(g\hat{Z}) = {}^t\hat{Z}I_{n,2}\hat{Z} &\iff U^*dU - V^*dV = Z_1^*dZ_1 - Z_2^*dZ_2, \\ {}^t\hat{Z}\,{}^tgId(g\hat{Z}) = {}^t\hat{Z}d\hat{Z} = 0 &\iff {}^tUdU + {}^tVdV = 0. \end{aligned}$$

Then compute

$$\begin{aligned} dvv^{-1} &= v^{*-1}v^*dvv^{-1} \\ &= v^{*-1}V^*\begin{pmatrix} i & 1 \end{pmatrix}^*\begin{pmatrix} i & 1 \end{pmatrix}dVv^{-1} \\ &= v^{*-1}V^*dVv^{-1} + v^{*-1}V^*\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}dVv^{-1}. \end{aligned}$$

Add and subtract the term $2q(gZ)^*dq(gZ)$ to the right hand side above, but replace the addition by the equivalent expansion below instead:

$$\begin{aligned} 2q(gZ)^*d(gZ) &= 2v^{*-1}V^*d(Vv^{-1}) \\ &= 2v^{*-1}V^*(dVv^{-1} - Vdvv^{-2}) \\ &= (2v^{*-1}V^*dVv^{-1} - dvv^{-1}) - (2v^{*-1}V^*Vv^{-1} - 1)dvv^{-1} \\ &= v^{*-1}V^*\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}dVv^{-1} - \overline{{}^t(gZ)(gZ)}\,{}^t(gZ)(gZ)dvv^{-1} \\ &= v^{*-1}V^*dVv^{-1} + v^{*-1}V^*\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}dVv^{-1} - \overline{{}^t(gZ)(gZ)}\,{}^t(gZ)(gZ)dvv^{-1}, \end{aligned}$$

where the identity $v^*v^*vv^{-1} - 2v^*V^*Vv^{-1} + \overline{{}^tVV}\,{}^tVV = 0$ (proved earlier in our discussion of the kernel function) is used below to justify the previous substitution:

$$\begin{aligned}\overline{{}^t(gZ)(gZ)}\,{}^t(gZ)(gZ) &= v^{*-2}\overline{{}^tUU}\,{}^tUUv^{-2}\\ &= v^{*-2}(\overline{{}^tUU}\,{}^tUU)v^{-2}\\ &= 2v^{*-1}V^*Vv^{-1} - 1.\end{aligned}$$

If our directions above are correctly put into practice, then our expression for dvv^{-1} simplifies (namely from cancellation of the terms involving the 2×2 matrices):

$$dvv^{-1} = 2v^{*-1}V^*dVv^{-1} - \overline{{}^t(gZ)(gZ)}\,{}^t(gZ)(gZ)dvv^{-1} - 2q(gZ)^*dq(gZ).$$

Now, expanding the first term on the right hand side above as

$$\begin{aligned}2v^{*-1}V^*dVv^{-1} &= 2v^{*-1}(U^*dU - Z_1^*dZ_1 + Z_2^*dZ_2)v^{-1}\\ &= 2(gZ)^*dUv^{-1} - 2v^{*-1}(Z_1^*dZ_1 - Z_2^*dZ_2)v^{-1}\\ &= 2(gZ)^*(d(Uv^{-1}) + 2Uv^{-1}dvv^{-1}) - 2v^{*-1}(4Z^*dZ - 2\overline{{}^tZZ}d({}^tZZ))z_2^{-1}\\ &= 2(gZ)^*d(gZ) + 2(gZ)^*(gZ)dvv^{-1}\\ &\quad - j(g,Z)^*(2Z^*dZ - 2q(Z)^*dq(Z))j(g,Z),\end{aligned}$$

and bring terms involving dvv^{-1} on the right to the left hand side to get:

$$\begin{aligned}&(1 + \overline{{}^t(gZ)(gZ)}\,{}^t(gZ)(gZ) - 2(gZ)^*(gZ))dvv^{-1} =\\ &\quad 2(gZ)^*d(gZ) - 2q(gZ)^*dq(gZ) - j(g,Z)^*(2Z^*dZ - 2q(Z)^*dq(Z))j(g,Z).\end{aligned}$$

A division and application of the transformation property for the $K(gZ, gZ)$ term then brings our calculation for dvv^{-1} to an end:

$$dvv^{-1} = \frac{2(gZ)^*d(gZ) - 2q(gZ)^*d(gZ)}{1 + \overline{{}^t(gZ)(gZ)}\,{}^t(gZ)(gZ) - 2(gZ)^*(gZ)} - \frac{2Z^*dZ - 2q(Z)^*dq(Z)}{1 + \overline{{}^tZZ}\,{}^tZZ - 2Z^*Z}.$$

As a result,

$$\begin{aligned}d\arg j(g,Z) &= \tfrac{1}{2i}\mathrm{Tr}\{dv^*v^{*-1} - dvv^{-1}\}\\ &= -\delta(gZ) + \delta(Z).\end{aligned}$$

This completes the proof of the lemma. □

Remark. The reader may have expected the metric for Type IV to contain the term $\overline{{}^t(gZ)(gZ)}d({}^t(gZ)(gZ))$ instead of $q(gZ)dq(gZ)$. Oddly, it happens that

$$\overline{{}^t(gZ)(gZ)}d({}^t(gZ)(gZ)) = 4q(gZ)dq(gZ), \quad \text{but} \quad \overline{{}^tZZ}d({}^tZZ) = 2q(Z)dq(Z).$$

It is this difference by a factor of 2 between the identities above that makes the $\mathbb{G}^1_{\mathbb{R}}$-invariance of Ω_ν unclear when the expression $\overline{{}^t(gZ)(gZ)}d({}^t(gZ)(gZ))$ is used.

Proof. (of the proposition) We first reduce the proof to showing that only the second term in Ω_ν is invariant under $\mathbb{G}^1_{\mathbb{R}} = G_{\mathbb{R}} \times S^1_\nu$. This follows from the fact that since ω is the riemannian metric for D, it is $G_{\mathbb{R}}$-invariant, hence $G_{\mathbb{R}} \times S^1_\nu$-invariant, because the action of the circle S^1_ν on D is trivial.

We have $\mathbb{G}^1_{\mathbb{R}}$ acting on $\mathbb{D}^1$ as follows: $(g,s)(Z,t) = (gZ, \check{t})$, where $\check{t} = t + s + \arg \tau_\nu^{-1}(\kappa(g,Z))$. Then $d\check{t} = dt + d \arg \tau_\nu^{-1}(\kappa(g,Z))$ and by the previous lemma,

$$d \arg \tau_\nu^{-1}(\kappa(g,Z)) = -\nu d \arg j(g,Z) = \delta(gZ) - \delta(Z).$$

Hence,

$$\frac{d\check{t}}{\nu} - \delta(gZ) = \frac{dt}{\nu} - \delta(Z),$$

and shows the $\mathbb{G}^1_{\mathbb{R}}$-invariance of Ω_ν. □

We claim that the spaces $\mathbb{D}^1$ are all isometric to each other for any two different values of ν (excluding the case where $\nu = 0$).

Corollary 2.2.3 *If $\nu_1, \nu_2 \neq 0$ are integral, then the map $T_{\nu_1,\nu_2} : \mathbb{D}^1_{\nu_1} \to \mathbb{D}^1_{\nu_2}$ given by*

$$T_{\nu_1,\nu_2}(z, t_1) = (z, \frac{\nu_2}{\nu_1} t_1) \tag{2.7}$$

is an isometry.

Proof. We will write $T = T_{\nu_1,\nu_2}$ when there is no ambiguity. Clearly, T is a diffeomorphism. Then observe that $T^*(dz) = dz$ and $T^*(dt_2) = (\nu_2/\nu_1)dt_1$. Now check the equality $T^*\Omega_{\nu_2} = \Omega_{\nu_1}$ by using our expression for Ω_{ν_1} and Ω_{ν_2}. Hence, T is an isometry. □

Chapter 3

Weakly Symmetric Spaces

3.1 Weakly Symmetric Spaces

In this section we intend to characterize $\mathbb{D}^1_\nu$ as a weakly symmetric space in the sense of Selberg. Since all our circle bundles are isometric to each other, it suffices to consider just one of them by choosing $\nu = 1$. From now on, we shall assume this to be the case (unless otherwise mentioned) and urge the reader to bear this in mind. We also mention that the results in this section are new and extend those of Selberg in the symplectic case.

Definition. Let M be a riemannian manifold. Then (M, G, μ) is called a weakly symmetric space if
i) G is a transitive group of isometries on M.
ii) μ is an involutive isometry on M (not necessarily in G) with $\mu G \mu^{-1} \subset G$.
iii) Given any $x, y \in M$, there exists an element $g \in G$ such that $gx = \mu y$ and $gy = \mu x$.

Let σ be ordinary complex (not matrix) conjugation on D and extend it to $\mathbb{D}^1$ (also called σ) by $\sigma(Z, t) = (\sigma(Z), -t) = (\overline{Z}, -t)$. We are then ready to present the main result of our thesis.

Main Theorem 3.1.1 *I.* $(\mathbb{D}^1, \mathbb{G}^1_\mathbb{R}, \sigma)$ *is a weakly symmetric space.*
II. Let $G_\mathbb{R} = SU(m, n)$*. Then* $(\mathbb{D}^1, G_\mathbb{R}, \sigma)$ *is a weakly symmetric space if only if* $m \neq n$.

The proof will require the following two lemmas:

Lemma 3.1.2 *Let* $Z \in D$*. Then* Z *and* $\overline{Z}$ *are in the same* K*-orbit, i.e. there is an element* $k \in K$ *such that* $kZ = \overline{Z}$*. If* $G_\mathbb{R} = SU(m, n)$ *with* $m \neq n$ *and* $t \in [0, 2\pi)$*, then* k *can chosen so that* $\arg \tau(\kappa(k, o)) = t$.

Proof. We first prove the lemma for $G_\mathbb{R} = SU(1, 1)$ and then make use of the Polydisk theorem [28] to handle the general case. Decompose $z = re^{i\theta}$. Then define

$$k = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \in K_\mathbb{R}$$

so that $kz = re^{-i\theta} = \overline{z}$. For the general case, consideration the decomposition $D = K_\mathbb{R} \cdot G_\mathbb{R}[\Psi](x_0)$, where $G_\mathbb{R}[\Psi](x_0)$ is the polydisk defined as the product of $|\Psi|$ open unit discs constructed from each strongly orthogonal root in the maximal set Ψ. Write

$$Z = k_0 \cdot \prod_{\gamma \in \Psi} g_\gamma(x_0) = k_0 \cdot \Pi z_\gamma.$$

Then conjugation of Z means $\overline{Z} = \overline{k} \cdot \Pi \overline{z}_\gamma$. As in the $SU(1, 1)$ case, we choose $k_\gamma \in K_\mathbb{R}[\gamma]$ such that $k_\gamma z_\gamma = \overline{z}_\gamma$. Now define $k = \overline{k_0} \cdot \Pi k_\gamma \cdot k_0^{-1}$ to get $kZ = \overline{Z}$. This completes the proof of the first part.

As for the second part, we essentially give an alternative proof of the first part but reveals the difference between $G_\mathbb{R}$ of tube type and not of tube type.

This also avoids (masking would be more appropriate) the machinery of the polydisk theorem. Consider $G_{\mathbb{R}} = SU(m,n)$. We then have the decomposition $\mathfrak{p}_+ = \mathrm{Ad}(K_{\mathbb{R}})(\mathfrak{a}_+)$, where $\mathfrak{a}_+ \subset \mathfrak{p}_+$ is the subalgebra corresponding to the maximal abelian subalgebra $\mathfrak{a}$ in $\mathfrak{p}$ generated by the strong orthogonal roots. Then in view of the Harish-Chandra embedding $D \subset \mathfrak{p}_+$, we can assume $Z = \mathrm{Ad}(k_0)(X)$ for some $k_0 \in K$ and $X \in \mathfrak{a}_+$ in the following diagonal form:

$$X = \left(\begin{array}{cccc|c} r_1 e^{i\theta_1} & & & & \\ & r_2 e^{i\theta_2} & & & \\ & & \ddots & & 0 \\ & & & r_m e^{i\theta_m} & \end{array}\right),$$

where 0 is the $m \times (n-m)$ zero matrix. Let $t_0 = \arg \tau(\overline{k}_0 k_0^{-1})$. We now look for

$$k_1 = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}; \quad A = \begin{pmatrix} e^{i\alpha_1} & & & \\ & e^{i\alpha_2} & & \\ & & \ddots & \\ & & & e^{i\alpha_m} \end{pmatrix}, \quad B = \begin{pmatrix} e^{i\beta_1} & & & \\ & e^{i\beta_2} & & \\ & & \ddots & \\ & & & e^{i\beta_n} \end{pmatrix},$$

so that $\mathrm{Ad}(k_1)X = AXB^{-1} = \overline{X}$. This translates to the set of equations

$$\{\alpha_i + \theta_i - \beta_i = -\theta_i\}, \quad i = 1, 2, ..., m; \quad \sum_{i=1}^{m} \alpha_i + \sum_{j=1}^{n} \beta_j = 0.$$

Notice that if $m = n$, then a unique solution is specified, namely

$$\alpha_i = -\theta_i, \quad \beta_i = \theta_i, \quad i = 1, 2, ..., m; \quad \beta_j = 0, \quad i = m+1, ..., n,$$

which gives us the desired A and B. However, since we are assuming G is not of tube type, suppose $m < n$. Let $t \in [0, 2\pi)$ and denote

$$\delta = \frac{(t_0 - t) - \sum_{i=1}^{m} \theta_i}{m}, \quad \phi = \frac{2((t_0 - t) - \sum_{i=1}^{m} \theta_i)}{n - m}.$$

Then by choosing

$$\alpha_i = -(\theta_i + \delta), \quad \beta_i = \theta_i - \delta, \quad i = 1, 2, .., m; \quad \beta_j = \phi, \quad j = m+1, ..., n,$$

one can check that

$$\arg \tau(k_1) = \arg \det(B)^{-1} = t - t_0.$$

Finally, set $k = \overline{k_0} k_1 k_0^{-1}$ to get

$$kZ = \mathrm{Ad}(\overline{k_0} k_1 k_0^{-1})Z = \mathrm{Ad}(\overline{k_0} k_1)X = \mathrm{Ad}(\overline{k_0})\overline{X} = \overline{\mathrm{Ad}(k_0)X} = \overline{Z}.$$

It remains to check that

$$\begin{aligned}\arg\tau(\kappa(k,o)) &= \arg\tau(\overline{k}_0 k_1 k_0^{-1}) \\ &= \arg\tau(\overline{k}_0 k_0^{-1}) + \arg\tau(k_1) \\ &= t_0 + (t - t_0) \\ &= t,\end{aligned}$$

which completes the proof. □

Lemma 3.1.3 *Let $Z_1, Z_2 \in D$. Then there exists an element $g \in G_{\mathbb{R}}$ such that $gZ_1 = \overline{Z}_2$ and $gZ_2 = \overline{Z}_1$. Furthermore,* $\arg\tau(\kappa(g, Z_1)) = \arg\tau(\kappa(g, Z_2)) \pmod{2\pi}$.

Proof. To accomplish this, we first pick $h_1 \in G_{\mathbb{R}}$ such that $h_1 Z_1 = o$ and $h_1 Z_2 = Z$. Now choose $h_2 \in G_{\mathbb{R}}$ such that $h_2 o = \overline{Z}$ and $h_2 \overline{Z} = o$. By the lemma above, there exists $k \in K$ such that $ko = o$ and $kZ = \overline{Z}$. Then D a symmetric space means we can pick $h_2 \in G_{\mathbb{R}}$ which exchanges o and $\overline{Z}$, i.e. $h_2 o = \overline{Z}$ and $h_2 \overline{Z} = 0$. In summary, consider the chain of maps in the following diagram:

$$\begin{array}{ccccccccccccc} Z_1 & \to & o & \to & o & \to & \overline{Z} & \to & Z & \to & Z_2 & \to & \overline{Z}_2 \\ & h_1 & & k & & h_2 & & \sigma & & h_1^{-1} & & \sigma & \\ Z_2 & \to & Z & \to & \overline{Z} & \to & o & \to & o & \to & Z_1 & \to & \overline{Z}_1 \end{array}$$

Now, it can be easily checked that $\sigma G_{\mathbb{R}} \sigma = \overline{G}_{\mathbb{R}} = G_{\mathbb{R}}$, so that $\sigma h_1^{-1} \sigma = \overline{h}_1^{-1}$. Just set $g = \overline{h}_1^{-1} h_2 k h_1$ to get $gZ_1 = \overline{Z}_2$ and $gZ_2 = \overline{Z}_1$.

As for the proof of the second statement, consider the kernel function $K(Z, W)$ defined in Chapter I. Recall that $K(Z, W)$ satisfies $K(W, Z) = \overline{K(Z, W)}$ and has the following transformation property:

$$K(gZ, gW) = \overline{j(g, Z)}^{-1} K(Z, W) j(g, W)^{-1}.$$

To apply this to our situation, use the fact that $gZ_1 = \overline{Z_2}$ and $gZ_2 = \overline{Z_1}$ to get

$$K(gZ_1, gZ_2) = K(\overline{Z_2}, \overline{Z_1}) = \overline{K(Z_2, Z_1)} = K(Z_1, Z_2),$$

or

$$K(Z_1, Z_2) = K(gZ_1, gZ_2) = \overline{j(g, Z_1)}^{-1} K(Z_1, Z_2) j(g, Z_2)^{-1},$$

from which the fact $\overline{j(g, Z_1)} j(g, Z_2) = 1$ is clear. We conclude from $\tau(\kappa(g, Z)) = j(g, Z)$ that

$$\arg\tau(\kappa(g, Z_1)) = \arg\tau(\kappa(g, Z_2)) \pmod{2\pi}.$$

This finishes the proof of the lemma. □

Proof. (of main theorem) I. By construction, $\mathbb{G}^1_{\mathbb{R}}$ is transitive on $\mathbb{D}^1$, so property i) in the definition of a weakly symmetric space is satisfied. For ii), it is clear that σ is an involution and that it normalizes $\mathbb{G}^1_{\mathbb{R}}$ since

$$\begin{aligned}\sigma(g, s)\sigma^{-1}(Z, t) &= \sigma(g, s)(\overline{Z}, -t) \\ &= \sigma(g\overline{Z}, \arg\tau^{-1}(\kappa(g, \overline{Z})) - t + s) \\ &= (\overline{g}Z, \arg(\tau^{-1}(\kappa(\overline{g}, Z)) + t - s),\end{aligned}$$

and so $\sigma(g,s)\sigma^{-1} = (\overline{g}, -s) \in \mathbb{G}^1_{\mathbb{R}}$. To show that Ω is preserved under σ, observe that its first term is invariant under conjugation. As for its second term,

$$dt - \operatorname{Im}\operatorname{Tr}\{(I - Z^*Z)^{-1} dZ^* Z\}$$

is sent to its negative under conjugation, so under the square, is also invariant under σ. This proves ii).

To prove iii), let $(Z_1, t_1), (Z_2, t_2) \in \mathbb{D}^1$. Then it suffices to find an element $(g,s) \in \mathbb{G}^1_{\mathbb{R}}$ such that

$$(g,s)(Z_1,t_1) = \sigma(Z_2,t_2), \quad (g,s)(Z_2,t_2) = \sigma(Z_1,t_1).$$

This translates into the following equalities:

$$\begin{array}{c} gZ_1 = \overline{Z_2} \\ gZ_2 = \overline{Z_1} \end{array} \; ; \; \begin{array}{c} \arg^{-1}\tau(\kappa(g,Z_1)) + t_1 + s = -t_2 \\ \arg^{-1}\tau(\kappa(g,Z_2)) + t_2 + s = -t_1 \end{array} \; .$$

The first pair of equations is satisfied by choosing g from the previous lemma. For the second pair of equations, just set $s = -(t_1 + t_2 + \arg\tau^{-1}(\kappa(g, Z_1)))$ and again use the lemma to check that the equations hold. Hence, $(\mathbb{D}^1, \mathbb{G}^1_{\mathbb{R}}, \sigma)$ is weakly symmetric.

II. If we view $G_{\mathbb{R}}$ as a subgroup of $\mathbb{G}^1_{\mathbb{R}}$, then it suffices to show that (g,s) can be chosen with $s = 0$ and $\arg\tau^{-1}(\kappa(g, Z_i)) = -(t_1 + t_2)$. Denote

$$t_0 = \arg\tau(\kappa(\overline{h_1}^{-1}h_2, o) + \arg\tau(\kappa(h_1, Z_1)).$$

From the previous lemma, we can choose k obtained in part I so that $\arg\tau(\kappa(k,o)) = t_1 + t_2 - t_0$. Then

$$\begin{aligned} \kappa(g, Z_1) &= \kappa(\overline{h_1}^{-1}h_2 k h_1, Z_1) \\ &= \kappa(\overline{h_1}^{-1}h_2, o)\kappa(kh_1, Z_1) \\ &= \kappa(\overline{h_1}^{-1}h_2, o)\kappa(k, o)\kappa(h_1, Z_1). \end{aligned}$$

As a result, we can now choose $s = 0$ since

$$\arg\tau(\kappa(g, Z_1)) = t_0 + (t_1 + t_2 - t_0) = t_1 + t_2.$$

This completes the proof of our main theorem. □

Corollary 3.1.4 $(\mathbb{D}, \mathbb{G}, \sigma)$ *is a weakly symmetric space.*

This will follow from the proposition below which states that the universal cover of a weakly symmetric space is weakly symmetric. Let (M, G, μ) be a weakly symmetric space. Denote by $\widetilde{M}$ and $\widetilde{G}$ the universal covers of M and G, respectively, and $\pi : \widetilde{M} \to M$, $p : \widetilde{G} \to G$ their respective covering map. Also, let $\tilde{\mu}$ be an extension of μ to $\widetilde{M}$. If $x, y \in M$, then $d(x,y)$ shall mean the distance from x to y with respect to the riemannian metric on M invariant under G and μ.

Proposition 3.1.5 *If (M, G, μ) is weakly symmetric, then $(\widetilde{M}, \widetilde{G}, \tilde{\mu})$ is weakly symmetric.*

Proof. Write $M = G/K$. Let $H = p^{-1}(K)^0$ be the identity component of $p^{-1}(K)$. Then $\widetilde{M} = \widetilde{G}/H$. Suppose $\tilde{x}, \tilde{y} \in \widetilde{M}$ and $x = \pi(\tilde{x}), y = \pi(\tilde{y})$ their projection to M. Since M is weakly symmetric, we can choose $g \in G$ such $gx = \mu y$, $gy = \mu x$. Then since $\pi^{-1}(gx) = \pi^{-1}(\mu y) = \{\tilde{\mu}(w) : w \in \pi^{-1}(y)\}$ and $\tilde{y} \in \pi^{-1}(y)$, g lifts to a unique $\tilde{g}$ with $\tilde{g}(\tilde{x}) = \tilde{\mu}(\tilde{y})$. If $d(\cdot, \cdot)$ denotes the distance between two points on M (or $\widetilde{M}$), then we have

$$d(\tilde{\mu}(\tilde{y}), \tilde{g}(\tilde{y})) = d(\tilde{g}(\tilde{x}), \tilde{g}(\tilde{y})) = d(\tilde{\mu}(\tilde{y}), \tilde{\mu}(\tilde{x})),$$

which implies that $\widetilde{g}(\tilde{y}) = \widetilde{\mu}(\tilde{x})$, as desired. This completes the proof. $\square$

3.2 The Unit Disc

Let $G_{\mathbb{R}} = SU(1,1)$. The characterization that $\mathbb{D}^1$ $(\nu = 1)$ is weakly symmetric can be made more explicit. Let (z,t) and (w,r) be two distinct points in $\mathbb{D}^1$. Then without loss of generality, we can pick (w,r) to be the origin $(o,0)$. The strategy is to pick $h \in G_{\mathbb{R}}$ to switch o and z and compose with an element $k \in K_{\mathbb{R}}$ mapping z to $\bar{z}$ to obtain the desired g. The answer is

$$k = \begin{pmatrix} \sqrt{\bar{z}/z} & 0 \\ 0 & \sqrt{z/\bar{z}} \end{pmatrix}, \quad h = \frac{1}{d}\begin{pmatrix} i & -iz \\ i\bar{z} & -i \end{pmatrix}, \quad d = \sqrt{1 - \bar{z}z}.$$

This leads to the element (g,s) as being

$$g = \frac{1}{d}\begin{pmatrix} i\sqrt{\bar{z}/z} & -i\sqrt{\bar{z}z} \\ i\sqrt{\bar{z}z} & -i\sqrt{z/\bar{z}} \end{pmatrix}, \quad s = -t + \arg(i/z).$$

Now check that $\arg\tau(g,z) = \arg\tau(g,o) = -\arg(i/z)$ to obtain $(g,s)(o,0) = (\overline{z}, -t)$ and $(g,s)(z,t) = \mu(o,0)$.

We next compute the Laplacian Δ on $\mathbb{D}^1$. Write $z \in D$ as $z = x + iy$. By definition, any riemannian manifold (M, Ω) has a Laplacian Δ defined by

$$\Delta f = \text{div grad } f = \frac{1}{\|\Omega\|}\sum_l \frac{\partial}{\partial x_l}\left(\sum_k \Omega^{kl}\|\Omega\|\frac{\partial f}{\partial x_k}\right), \quad \|\Omega\| = |\det(\Omega_{ij})|.$$

Lemma 3.2.1 *If $G = SU(1,1)$, then*

$$\Delta = (1-x^2-y^2)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + (1-x^2-y^2)^{\frac{1}{2}}(y-x)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\frac{\partial}{\partial t} + (1+x^2+y^2)\frac{\partial^2}{\partial t^2}.$$

Proof. The metric Ω of $\mathbb{D}^1$ can be expressed in matrix form as

$$(\Omega_{ij}) = \begin{pmatrix} 1 + y^2/d^2 & -xy/d & -y/2\nu d \\ -xy/d & 1 + x^2/d^2 & x/2\nu d \\ -y/2\nu d & x/2\nu d & 1/4\nu^2 \end{pmatrix}.$$

It can be checked that

$$\|\Omega\| = \frac{1}{d^4}, \quad (\Omega^{ij}) = \begin{pmatrix} d^2 & 0 & 2y\nu d \\ 0 & d^2 & -2x\nu d \\ 2y\nu d & -2x\nu d & (1+x^2+y^2)4\nu^2 \end{pmatrix}.$$

It remains to just write out the definition of the Laplacian to obtain the desired formula in the lemma. We leave this tedious calculation for the interested reader. □

Remark. The expression for δ takes the form:

$$\delta(z) = \mathrm{Im}\left\{\frac{\bar{z}dz}{1-\bar{z}z}\right\} = \frac{xdy - ydx}{1-(x^2+y^2)}.$$

Then differentiating δ gives

$$d(\delta) = \mathrm{Im}\left\{\frac{d\bar{z}\wedge dz}{(1-\bar{z}z)^{-2}}\right\} = 2\frac{dx\wedge dy}{(1-(x^2+y^2))^2}.$$

We notice that the numerator term $dx \wedge dy$ above is the symplectic form on $\mathbb{R}^2$.

3.3 The Unbounded Realization

In this section we describe the unbounded realization of $\mathbb{D}^1$ by extending the Cayley transform on D to $\mathbb{D}^1$. Our goal is to verify that for $G_{\mathbb{R}} = \mathbf{Sp}(n,\mathbb{C}) \cap \mathbf{SU}(n,n)$ and $\nu = 1$, the Cayley transform of $\mathbb{D}^1$ is indeed the circle bundle constructed by A. Selberg in [15].

Fix a one-dimensional $\tilde{K}_{\mathbb{R}}$-type τ_ν with ν integral. Let c be the Cayley transform, $c(D) = H$ the unbounded realization, and $\widetilde{G}_{\mathbb{R}} = cG_{\mathbb{R}}c^{-1}$ the corresponding group of isometries of H. It is then possible to independently construct circle bundles $\mathbb{H} \to H$ in the same fashion as $\mathbb{D}^1 \to D$. We do this by first defining the group extension

$$\widetilde{\mathbb{G}}^1_{\mathbb{R}} = \widetilde{G}_{\mathbb{R}} \times S^1_\nu = c(G_{\mathbb{R}} \times S^1_\nu)c^{-1} = cG_{\mathbb{R}}c^{-1} \times S^1_\nu,$$

giving $\widetilde{\mathbb{G}}^1_{\mathbb{R}}$ the direct product structure. Now set $\mathbb{H} = H \times S^1_\nu$. Then define the following group action $\widetilde{\mathbf{\Phi}} : \widetilde{\mathbb{G}}^1_{\mathbb{R}} \times \mathbb{H} \to \mathbb{H}$:

$$(\tilde{g}, s)(w, r) = (\tilde{g}w, \arg \tau^{-1}(\kappa(\tilde{g}, w)) + r + s).$$

The point is that by properly extending c to a transformation of $\mathbb{D}^1$, its image should correspond precisely to our unbounded domain $\mathbb{H} = H \times S^1_\nu$ described above. Therefore, we define an extended Cayley transform $\mathbf{c}$ by

$$\mathbf{c}(z,t) = (c,0)(z,t) = (cz, \arg \tau^{-1}(\kappa(c,z)) + t) = (w,r).$$

Observe that this extension is not trivial on the circle component of $\mathbb{D}^1$ but twists it via a factor of automorphy. Of course, it is easily checked that this action is compatible with $\mathbf{c}$ in the sense that the following diagram is commutative:

$$\begin{array}{ccccc} & \mathbb{D}^1 & \xrightarrow{\mathbf{c}} & \mathbb{H} & \\ (g,s) & \big\downarrow & & \big\downarrow & (\tilde{g},s) \\ & \mathbb{D}^1 & \xrightarrow{\mathbf{c}} & \mathbb{H} & \end{array}$$

If $\tilde{\omega}$ is the metric on H, then call $\widetilde{\Omega}$ the corresponding metric on $\mathbb{H}$ invariant under $\widetilde{G}_{\mathbb{R}}$. We shall later explicitly compute $\widetilde{\Omega}$ and prove that indeed

$$c^*(\widetilde{\Omega}) = \Omega.$$

Theorem 3.3.1 $(\mathbb{H}, \widetilde{\mathbb{G}}^1_{\mathbb{R}}, \tilde{\sigma})$ *is weakly symmetric, where* $\tilde{\sigma} = c\sigma c^{-1}$ *and* σ *is complex conjugation on* $\mathbb{D}^1$.

It is straightforward to check that $\mathbb{H}$ satisfies all the conditions of a weakly symmetric space by transforming this situation over to $\mathbb{D}^1$ and use the fact that $(\mathbb{D}^1, \mathbb{G}^1_{\mathbb{R}}, \sigma)$ is weakly symmetric. However, it would be more interesting for us to explicitly characterize the action of $\widetilde{\mathbb{G}}^1_{\mathbb{R}}$ on $\mathbb{H}$, namely determine $\kappa(\tilde{g}, W)$, and compute the riemannian metric. We shall discover below that for tube domains of Types I ($m = n$) and II, this is possible. Afterwards, we check our results by applying the extended Cayley transform $\mathbf{c}$.

Let $G_{\mathbb{R}} = SU(n,n)$. Then the Cayley transform c is given by

$$c = \begin{pmatrix} iI & iI \\ -I & I \end{pmatrix},$$

and the group $\widetilde{G}_{\mathbb{R}} = cG_{\mathbb{R}}c^{-1}$ consists of following matrices:

$$\widetilde{G}_{\mathbb{R}} = \left\{ \tilde{g} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},\ \ A^*C = CA^*,\ A^*D - C^*B = I,\ B^*D = D^*B \right\}.$$

Let $M_n(\mathbb{C})$ be the space of $n \times n$ complex matrices and V the subset of all hermitian matrices. Then $cZ = i(Z+I)(I-Z)^{-1} = W$ maps D to

$$H = \{W = X + iY \in M_n(\mathbb{C}) : X, Y \in V \text{ and } Y >> 0\},$$

and $Y >> 0$ means that it is positive definite. The action of $\widetilde{G}_{\mathbb{R}}$ on H is $\check{W} := gW = (AW+B)(CW+D)^{-1}$. It can be checked that $\tilde{\sigma}$ acts on H as $\tilde{\sigma}W = -\overline{W} = -X + iY$.

We will need to compute the factor of automorphy $\kappa(\tilde{g}, w)$. By definition,

$$\tilde{g}\exp(W) \in \exp(\tilde{g}W)\kappa(\tilde{g}, W)P_-.$$

The decomposition below now tells us what $\kappa(\tilde{g}, W)$ should be:

$$\tilde{g}\exp(W) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} I & W \\ 0 & I \end{pmatrix} \in \begin{pmatrix} I & \tilde{g}W \\ 0 & I \end{pmatrix}\begin{pmatrix} A - (\tilde{g}W)C & 0 \\ 0 & CW + D \end{pmatrix}\begin{pmatrix} I & 0 \\ U & 0 \end{pmatrix}.$$

If $\tau = \tau_\nu$ is a one-dimensional K-type,

$$\tau_\nu(\kappa(\tilde{g}, W)) = \det(CW + D)^{-\nu}.$$

The following lemma will be needed to compute the metric on $\mathbb{H}$.

Lemma 3.3.2 *If $W_2, W_1 \in H$, then*

$$\check{W}_2 - \check{W}_1 = (W_2C^* + D^*)^{-1}(W_2 - W_1)(CW_1 + D)^{-1} \tag{3.1}$$

Proof. We first observe that $(\tilde{g}W)^* = \tilde{g}W^*$. Then rewriting it as $(\tilde{g}W^*)^* = \tilde{g}W$ and using the properties of $\widetilde{G}_{\mathbb{R}}$, we have

$$\begin{aligned}
\check{W}_2 - \check{W}_1 &= (AW_2 + B)(CW_2 + D)^{-1} - (AW_1 + B)(CW_1 + D)^{-1} \\
&= [(CW_2^* + D)^{-1}]^*(AW_2^* + B)^* - (AW_1 + B)(CW_1 + D)^{-1} \\
&= (W_2C^* + D^*)^{-1}[(W_2A^* + B^*)(CW_1 + D) \\
&\qquad -(W_2C^* + D^*)(AW_1 + B)](CW_1 + D)^{-1} \\
&= (W_2C^* + D^*)^{-1}[W_2(A^*C - C^*A)W_1 + W_2(A^*D - C^*B) \\
&\qquad -(D^*A - B^*C)W_1 + (B^*D - D^*B)](CW_1 + D)^{-1} \\
&= (W_2C^* + D^*)^{-1}(W_2 - W_1)(CW_1 + D)^{-1}
\end{aligned}$$

and so the lemma follows. □

The lemma above allows us to compute the differential of $\check{W}$. Set $W_2 = W + \Delta W$, $W_1 = W$ and define $\Delta\check{W} = \check{W}_2 - \check{W}_1$. Then using the lemma and allowing $\Delta W \to 0$, we have

$$\begin{aligned}
d\check{W} &= \lim_{\Delta W \to 0}[(W + \Delta W)C^* + D^*]^{-1}(\Delta W)(CW + D)^{-1} \\
&= (WC^* + D^*)^{-1}dW(CW + D)^{-1}
\end{aligned} \tag{3.2}$$

Now write $\check{W} = \check{X} + i\check{Y}$. If we let $W_2 = W$ and $W_1 = W^* = X^* - iY^* = X - iY$, then again from the lemma above

$$\begin{aligned}
\check{X} &= (WC^* + D^*)^{-1}X(CW^* + D)^{-1} \\
\check{Y} &= (WC^* + D^*)^{-1}Y(CW^* + D)^{-1}
\end{aligned} \tag{3.3}$$

Observe that $(\check{W})^* = (\check{X})^* - i(\check{Y})^* = \check{X} - i\check{Y}$.

Recall that the action of $\widetilde{G}^1_{\mathbb{R}}$ on $\mathbb{H}$ can now be written as

$$(\tilde{g}, s)(W, r) = (\tilde{g}W, r + s + \arg\det(CW + D)^\nu) = (\check{W}, \check{r}).$$

Write $\mathrm{Tr}\{W\}$ for the trace of the matrix W.

Lemma 3.3.3

$$\frac{d\check{r}}{\nu} - \frac{1}{2}\mathrm{Tr}\{\check{Y}^{-1}d\check{X}\} = \frac{dr}{\nu} - \frac{1}{2}\mathrm{Tr}\{Y^{-1}dX\} \tag{3.4}$$

Proof. Making use of (3.2) and (3.3), we see that

$$\begin{aligned}
d\check{W}\cdot\check{Y}^{-1} &= (WC^* + D^*)^{-1}dW(CW + D)^{-1}(CW^* + D)Y^{-1}(WC^* + D^*) \\
d\check{W}^*\cdot\check{Y}^{-1} &= (W^*C^* + D^*)^{-1}dW^*(CW^* + D)^{-1}(CW^* + D)Y^{-1}(WC^* + D^*)^{-1})
\end{aligned}$$

Then observing that the trace of matrix is preserved under conjugation, we have

$$\begin{aligned}
\mathrm{Tr}\{d\check{X}\cdot\check{Y}^{-1}\} &= \tfrac{1}{2}[\mathrm{Tr}\{d\check{W}\cdot\check{Y}^{-1}\} + \mathrm{Tr}\{d\check{W}^*\cdot\check{Y}^{-1}\}] \\
&= \tfrac{1}{2}[\mathrm{Tr}\{dW(CW + D)^{-1}(CW^* + D)Y^{-1}\} \\
&\qquad +\mathrm{Tr}\{(W^*C^* + D^*)^{-1}dW^*\cdot Y^{-1}(WC^* + D^*)\}].
\end{aligned}$$

Now write

$$(CW^* + D) = (CW + D) + (CW^* - CW) = (CW + D) - 2iCY$$
$$(WC^* + D^*) = (W^*C^* + D^*) + (WC^* - W^*C^*) = (W^*C^* + D^*) + 2iYC^*$$

and use conjugation again to see that

$$\begin{aligned}\mathrm{Tr}\{d\check{X} \cdot \check{Y}^{-1}\} &= \tfrac{1}{2}[\mathrm{Tr}\{dW \cdot Y^{-1}\} - 2i\mathrm{Tr}\{dW(CW+D)^{-1}C\} \\ &\quad + \mathrm{Tr}\{(dW^* \cdot Y^{-1}\} + 2i\mathrm{Tr}\{(W^*C^* + D^*)^{-1}dW^* \cdot C^*\}] \\ &= \mathrm{Tr}\{dX \cdot Y^{-1}\} - i[\mathrm{Tr}\{CdW(CW+D)^{-1}\} \\ &\quad - \mathrm{Tr}\{(W^*C^* + D^*)^{-1}dW^* \cdot C^*\}].\end{aligned}$$

We next note that

$$\begin{aligned}d\arg\det(CW+D)^\nu &= \tfrac{1}{2i}d[\log\det(CW+D)^\nu - \log\det((CW+D)^*)^\nu] \\ &= \tfrac{\nu}{2i}[\mathrm{Tr}\{CdW(CW+D)^{-1}\} - \mathrm{Tr}\{(W^*C^* + D^*)^{-1}dW^*C^*\}]\end{aligned}$$

and so using the fact that $d\check{r} = dr + d\arg\det(CW+D)^\nu$, we can then conclude

$$\mathrm{Tr}\{d\check{X} \cdot \check{Y}^{-1}\} = \mathrm{Tr}\{dX \cdot Y^{-1}\} + \frac{2}{\nu}(d\check{r} - dr).$$

Our proposition is now clear. □

It is well-known that the metric $\tilde{\omega}$ on H is the following:

$$\tilde{\omega} = \mathrm{Tr}\{Y^{-1}dX\}^2 + \mathrm{Tr}\{Y^{-1}dY\}^2.$$

Theorem 3.3.4 *The Riemannian metric $\widetilde{\Omega}_\nu$ on $\mathbb{H}$ given below is $\widetilde{\mathbb{G}}^1_{\mathbb{R}}$-invariant:*

$$\widetilde{\Omega}_\nu = \tilde{\omega} + (\frac{dr}{\nu} - \frac{1}{2}\mathrm{Tr}\{Y^{-1}dX\})^2 \tag{3.5}$$

Proof. Again, we use the reduction that the first term in (3.5) is just the metric on H and since it depends only on the action of $\widetilde{G}$, it is clearly $\widetilde{\mathbb{G}}^1_{\mathbb{R}}$-invariant. Proposition 3.3.3 then tells us that the second term is also invariant under $\widetilde{\mathbb{G}}^1_{\mathbb{R}}$ and so the same can be said for Ω_ν. It is clear that Ω_ν is positive definite and hence is Riemannian. □

We now verify that the metric induced by $\mathbf{c}$ on $\mathcal{H}_\nu = \mathbf{c}(\mathcal{H}_\nu)$ defined earlier coincides with $\widetilde{\Omega}$ computed above for Types I ($m = n$) and II. Denote by $\tilde{\delta} = \frac{1}{2}\mathrm{Tr}\{Y^{-1}dX\}$.

Proposition 3.3.5

$$\mathbf{c}^*(\widetilde{\Omega}_\nu) = \Omega_\nu \tag{3.6}$$

Proof. Writing out $\Omega_\nu = \omega + (dt/\nu - \delta)^2$. Classically, we already have $c^*(\tilde{\omega}) = \omega$, so it suffices to show that the differential map $\mathbf{c}^*$ satisfies that $\mathbf{c}^*(dr/\nu - \tilde{\delta}) = dt/\nu - \delta$. The proof then requires first finding $Z^*dZ(I - Z^*Z)^{-1}$ in terms of dW and secondly dt in terms of dr. We have $Z = c^{-1}(W) = (W - iI)(W + iI)^{-1}$

and so $I-Z=2i(W+iI)^{-1}$. Differentiating the inverse Cayley transform gives $dZ=2i(W+iI)^{-1}dW(W+iI)^{-1}$. Furthermore, the following identity is true:

$$\begin{aligned} I-Z^*Z &= I-(W+iI)^{*-1}(W-iI)^*(W-iI)(W+iI)^{-1} \\ &= (W+iI)^{*-1}[(W+iI)^*(W+iI)-(W-iI)^*(W-iI)](W+iI)^{-1} \\ &= (W+iI)^{*-1}(4Y)(W+iI)^{-1}. \end{aligned}$$

Use these facts to compute

$$\begin{aligned} \mathrm{Tr}\{Z^*dZ(I-Z^*Z)^{-1}\} &= \mathrm{Tr}\{\tfrac{i}{2}(W+iI)^{*-1}(W-iI)^*(W+iI)^{-1}dWY^{-1}(W+iI)^*\} \\ &= \mathrm{Tr}\{\tfrac{i}{2}(W+iI)^{-1}dWY^{-1}(W-iI)^*\}. \end{aligned}$$

To find dt in terms of dr, we first need to determine $\tau(\kappa(c,Z))$. This we can get from

$$c\exp Z = \begin{pmatrix} iI & iI \\ -I & I \end{pmatrix}\begin{pmatrix} I & Z \\ 0 & I \end{pmatrix} = \begin{pmatrix} iI & i(I+Z) \\ -I & I-Z \end{pmatrix},$$

so that $\tau(\kappa(c,Z))=\det(I-Z)^{-\nu}$. Now differentiate to get

$$\begin{aligned} d\arg\tau^{-1}(\kappa(c,Z)) &= \tfrac{\nu}{2i}(d\log\det(I-Z)-d\log\det(I-Z)^*) \\ &= \tfrac{\nu}{2i}\mathrm{Tr}\{-(I-Z)^{-1}dZ+dZ^*(I-Z)^{*-1}\} \\ &= -\nu\mathrm{Im}\,\mathrm{Tr}\{I-Z)^{-1}dZ\} \\ &= -\nu\mathrm{Im}\,\mathrm{Tr}\{(W+iI)(W+iI)^{-1}dW(W+iI)^{-1}\} \\ &= -\nu\mathrm{Im}\,\mathrm{Tr}\{(W+iI)^{-1}dW\}. \end{aligned}$$

Next, denote by $A=\mathrm{Tr}\{Z^*dZ(I-Z^*Z)^{-1}\}-\mathrm{Tr}\{(W+iI)^{-1}dW\}$. From the previous calculations, A simplifies to

$$\begin{aligned} A &= \mathrm{Tr}\{\tfrac{i}{2}(W+iI)^{-1}dWY^{-1}[(W-iI)^*+2iY]\} \\ &= \mathrm{Tr}\{\tfrac{i}{2}(W+iI)^{-1}dWY^{-1}(W+iI)\} \\ &= \mathrm{Tr}\{\tfrac{i}{2}dWY^{-1}\}, \end{aligned}$$

from whence we get after writing $W^*=X-iY$ the identity $\mathrm{Im}\,A=\frac{1}{2}\mathrm{Tr}\{Y^{-1}dX\}$. Since $dr=d\arg\tau^{-1}(\kappa(g,Z))+dt$, we conclude that

$$\begin{aligned} dt/\nu-\mathrm{Im}\,\mathrm{Tr}\{Z^*dZ(I-Z^*Z)^{-1}\} &= [dr-d\arg\tau^{-1}(\kappa(g,Z))]/\nu-\mathrm{Im}\,\mathrm{Tr}\{(I-Z^*Z)^{-1}dZ^*Z\} \\ &= dr/\nu-\mathrm{Im}\,\mathrm{Tr}\{Z^*dZ(I-Z^*Z)^{-1}-(W+iI)^{-1}dW\} \\ &= dr/\nu-\mathrm{Im}\,A \\ &= dr/\nu-\tfrac{1}{2}\mathrm{Tr}\{Y^{-1}dX\}, \end{aligned}$$

as desired. □

In case $G_{\mathbb{R}}=Sp(n,\mathbb{C})\cap SU(n,n)$, we have $\widetilde{G}_{\mathbb{R}}=Sp(n,\mathbb{R})$ and

$$H=\{W=X+iY:\ W=W^t \text{ and } Y>0\}$$

is the Siegel generalized upper-half plane. Now view $\widetilde{G}_{\mathbb{R}}$ as a subgroup of $SU(n,n)$ and H as a subspace of the corresponding domain for $SU(n,n)$ to see that the metric for $\mathbb{H}$ in this case is exactly the same:

$$\widetilde{\Omega}_\nu=\tilde{\omega}+(\frac{dr}{\nu}-\frac{1}{2}\mathrm{Tr}\{Y^{-1}dX\})^2.$$

When $\nu=1$, $\widetilde{\Omega}_1$ given above is precisely the metric computed by Selberg for his circle bundle (see [15]) as promised.

3.4 Hyperbolic Space

We shall compute $\widetilde{\Omega}_\nu$ (and assume $\nu = 1$ for simplicity) when D is a complex hyperbolic space by using the Cayley transform $\mathbf{c}$. Let $Z = {}^t(z_1, ..., z_n)$ denote the elements of D and $W = {}^t(w_1, ..., w_n)$ denotes the elements of H. The Cayley transform $c(Z) = W$ (and its inverse) is then explicitly given by

$$w_j = \frac{2iz_j}{1-z_n},\ 1 \le j < n;\ w_n = i\frac{1+z_j}{1-z_n}; \qquad z_j = \frac{w_j}{w_n+i},\ 1 \le j < n;\ z_n = \frac{w_n - i}{w_n + i}.$$

Write $w_j = x_j + iy_j$, for $1 \le j \le n$. Then the unbounded realization is then described as $H = \{W \in \mathbb{C}^n : 4y_n - \sum_{j=1}^{n-1} |w_j|^2 > 0\}$.

Lemma 3.4.1 *Let $\mathcal{H}_\nu$ be complex hyperbolic space. The riemannian metric $\widetilde{\Omega}$ is given as*

$$\widetilde{\Omega} = \tilde{\omega} + \left(dr - \frac{\sum_{j=1}^{n-1}(x_j dy_j - y_j dx_j) + 2dx_n}{4y_n - \sum_{j=1}^{n-1} |w_j|^2} \right)^2 \tag{3.7}$$

Proof. Using the Cayley transformation equations above, we obtain

$$dz_j = \frac{(w_n+i)dw_j - w_j dw_n}{(w_n+i)^2}, \quad dz_n = \frac{2idw_n}{(w_n+i)^2}.$$

The metric Ω_ν (assume $\nu = 1$ for simplicity) is given as $\Omega = \omega + (dt - \delta)^2$, where

$$\delta = \mathrm{Im}\,\mathrm{Tr} \left\{ \frac{Z^* dZ}{1 - Z^* Z} \right\} = \mathrm{Im} \left\{ \frac{\sum_{j=1}^n \bar{z}_j dz_j}{1 - \sum_{j=1}^n \bar{z}_j z_j} \right\}.$$

Then a short computation gives

$$\frac{1}{1 - Z^* Z} = \frac{1}{1 - \sum_{j=1}^n \bar{z} z} = \frac{|w_n + i|^2}{4y_n - \sum_{j=1}^{n-1} |w_j|^2}.$$

Now check that

$$\frac{Z^* dZ}{1 - Z^* Z} = \frac{\sum_{j=1}^n \bar{z}_j dz_j}{1 - \sum_{j=1}^n \bar{z}_j z_j} = \frac{\sum_{j=1}^{n-1} \bar{w}_j dw_j}{4y_n - \sum_{j=1}^{n-1} |w_j|^2} + \frac{2i(\bar{w}_n + i)dw_n - (\sum_{j=1}^{n-1} |w_j|^2) dw_n}{(w_n+i)(4y_n - \sum_{j=1}^{n-1} |w_j|^2)}.$$

We'll also need to determine $\tau^{-1}(\kappa(c, Z))$ since $dr = d \arg \tau^{-1}(\kappa(c, Z)) + dt$. First, observe that

$$c{\cdot}\exp Z = \begin{pmatrix} 2i & & & & \\ & \ddots & & & \\ & & 2i & & \\ & & & i & i \\ & & & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & & & & z_1 \\ & \ddots & & & \vdots \\ & & 1 & & z_{n-1} \\ & & & 1 & z_n \\ & & & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2i & & & & 2iz_1 \\ & \ddots & & & \vdots \\ & & 2i & & 2iz_{n-1} \\ & & & i & i(1+z_n) \\ & & & -1 & 1 - z_n \end{pmatrix}.$$

This gives $\tau^{-1}(\kappa(c,Z)) = 1 - z_n = 2i/(w_n + i)$ and

$$d \arg \tau^{-1}(\kappa(c,Z)) = -\mathrm{Im}\left\{\frac{dz_n}{1-z_n}\right\} = -\mathrm{Im}\left\{\frac{dw_n}{w_n+i}\right\}.$$

Then combining

$$\begin{aligned}\delta + d\arg\tau^{-1}(\kappa(c,Z)) &= \mathrm{Im}\left\{\frac{\sum_{j=1}^{n-1}\bar{w}_j dw_j + 2i dw_n}{4y_n - \sum_{j=1}^{n-1}|w_j|^2}\right\} \\ &= \frac{\sum_{j=1}^{n-1}(x_j dy_j - y_j dx_j) + 2dx_n}{4y_n - \sum_{j=1}^{n-1}|w_j|^2}.\end{aligned}$$

We can now write the answer for $\widetilde{\Omega}$ as

$$\begin{aligned}\widetilde{\Omega} &= \tilde{\omega} + (dr - d\arg\tau^{-1}(\kappa(c,Z)) - \delta)^2 \\ &= \tilde{\omega} + \left(dr - \frac{\sum_{j=1}^{n-1}(x_j dy_j - y_j dx_j) + 2dx_n}{4y_n - \sum_{j=1}^{n-1}|w_j|^2}\right)^2,\end{aligned}$$

as desired. □

Chapter 4

Applications

4.1 Gelfand Pairs

The notion of weakly symmetric spaces is intimately related to that of Gelfand pairs. Therefore, we first provide a treatment of Gelfand pairs in connection with our circle bundles $\mathbb{D}^1$. Denote by $C^\infty(G)$ as the space of infinitely differentiable functions on G and define the following subspaces of $C^\infty(G)$:

$$\begin{aligned}
&C_c^\infty(G) = \{f \in C^\infty(G) : f \text{ has compact support}\}\\
&C^\infty(G//K) = \{f \in C^\infty(G) : f(hg) = f(gh) = f(g) \text{ for all } h \in K\}\\
&C^K(G) = \{f \in C^\infty(G) : f(hgh^{-1}) = f(g) \text{ for all } h \in K\}
\end{aligned}$$

Also, we define $C^\infty(G/K)$ to be the space of infinitely differentiable functions on G/K. If $f_1, f_2 \in C_c^\infty(G)$, then define their convolution to be

$$(f_1 * f_2)(x) = \int_G f_1(g) f_2(g^{-1}x)dg$$

The spaces $C_c^\infty(G)$ and $C_c^\infty(G/K)$ become algebras under this convolution product.

Definition. We say that (G, K) is a **Gelfand pair** if the algebra $C_c^\infty(G//K) = C_c^\infty(G) \cap C^\infty(G//K)$ is commutative under the convolution product. In that case, the homogeneous space G/K is said to be **commutative**.

There are many other characterizations of a Gelfand pair (G, K) in terms of its spaces of G-invariant differential operators and integral operators. Let $\mathbf{E}(G/K)$ be the space of G-invariant endomorphisms of $C_c^\infty(G/K)$. Let $D \in \mathbf{E}(G/K)$ be such that, if given a function $f \in C_c^\infty(G/K)$ and an open subset V of G/H we have $f|_V = 0$ implies $Df|_V = 0$, then D is said to satisfy the local property and called a differential operator of $C_c^\infty(G/K)$.

Let $h \in L^1(G//K)$ and define a function p_f on $G/K \times G/K$ as follows:

$$p_h(xK, yK) = h(y^{-1}x) \tag{4.1}$$

Observe that p_h is well-defined since h is bi-K-invariant. Now, let I_{p_h} be an integral operator on $C^\infty(G/K) \cap L^2(G/K)$ with kernel p_h, where we define

$$I_{p_h} f(xK) = \int_{G/K} p_h(xK, yK) f(yK) dyK \tag{4.2}$$

Then I_{p_h} is well-defined and p_h is G-invariant, since

$$p_h(gxK, gyK) = h((gy)^{-1}gx) = h(y^{-1}x) = p_h(xK, yK) \tag{4.3}$$

for all $g \in G$ and hence, $I_{p_h} \in \mathbf{E}(G/K)$. The invariant kernel p_h is also called a point-pair invariant (see Selberg [23]).

We define the following subspaces of $\mathbf{E}(G/K)$:

$$\begin{aligned}
&\mathbf{D}(G/K) = \{D \in \mathbf{E}(G/K) : D \text{ satisfies the local property}\}\\
&\mathbf{I}(G/K) = \{I_{p_h} \in \mathbf{E}(G/K) : h \in L^1(G//K)\}
\end{aligned}$$

Note that $\mathbf{D}(G/K)$ is just the space of G-invariant differential operators and $\mathbf{I}(G/K) \cong L^1(G//K)$. Here are the awaited equivalent conditions for (G, K) to be a Gelfand pair.

Theorem 4.1.1 *The following are equivalent if G is a reductive Lie group and K a compact subgroup:*
(i) (G, K) is a Gelfand pair.
(ii) The space of double cosets $K\backslash G/K$ is commutative.
(iii) $\mathbf{E}(G/K)$ is commutative.
(iv) $\mathbf{D}(G/K)$ is commutative.
(v) $\mathbf{I}(G/K)$ is commutative.

These equivalence conditions are well known results; we refer the interested reader to [27], Chapter 8 and [4], Proposition 1.7.1 for details. We mention that it is still unknown whether weak symmetry is equivalent to the Gelfand pair conditions.

We next investigate the commutativity of certain function spaces on $\mathbb{D}$ and $\mathbb{D}^1$. It is then profitable to view $\mathbb{D} = \mathbb{G}/\mathbb{K}_\nu$ and $\mathbb{D}^1 = \mathbb{G}^1/\mathbb{K}^1$ as homogeneous in what follows. Define the subspace

$$C^K(G//K_s) = C^K(G) \cap C^\infty(G//K_s).$$

Then there is the following inclusion of subspaces:

$$C^\infty(G//K) \subset C^K(G//K_s) \subset C^\infty(G//K_s).$$

Lemma 4.1.2 *(Flensted-Jensen) $C_c^K(G//K_s)$ is commutative under convolution.*

Proof. See [2], Theorem 3.1. □

Recall the injection $j : G \to \mathbb{G}$ discussed in Section 2.1.

Lemma 4.1.3 *(cf. [2], Prop. 4.1) $C_c^K(G//K_s) \cong C_c^\infty(\mathbb{G}//\mathbb{K}_\nu)$ as function algebras under convolution.*

Proof. We mention that Flensted-Jensen [2] has shown that $C_c^K(G//K_s) \cong C_c^\infty(\mathbb{G}^1//\mathbb{K}^1)$. We extend his argument to $\mathbb{D}$. Let $f \in C_c^\infty(\mathbb{G}//\mathbb{K}_\nu)$. Then $f \circ j$ is just the restriction of f to G. Since $j(K_s) \subset \mathbb{K}_\nu$, this means $f \circ j \in C_c^\infty(G//K_s)$. Note that to show $f \circ j \in C_c^K(G//K_s)$ it suffices to show that it is invariant under Z_K^0. Let $k \in Z_K^0$. Then the following calculation

$$\begin{aligned} j(kg) &= (kg, 0) \\ &= (k, \arg\tau(k))(g, \arg\tau^{-1}(k)) \\ &= (k, \arg\tau(k))(gk, 0)(k^{-1}, \arg\tau^{-1}(k)) \\ &= (k, \arg\tau(k))j(gz)(k^{-1}, \arg\tau^{-1}(k)) \end{aligned}$$

implies that $f(j(kg)) = f(j(gk))$. Hence, $f \circ j \in C_c^K(G//K_s)$.

Conversely, let $h \in C_c^K(G//K_s)$. We wish to show that there exists an $f \in C_c^\infty(\mathbb{G}//\mathbb{K}_\nu)$ such that $f \circ j = h$. Define

$$f(g_s) = h(\exp(-\frac{s}{\nu}Z_0)g) \tag{4.4}$$

To see that f is indeed in $C_c^\infty(\mathbb{G}//\mathbb{K}_\nu)$, let $k_\nu \in \mathbb{K}_\nu$. Then

$$\begin{aligned}
f(k_\nu g_s) &= f(kg, s+t\nu) \\
&= h(\exp(-\tfrac{s+t\nu}{\nu}Z_0)kg) \\
&= h(\exp(-\tfrac{s}{\nu}\eta)\exp(-tZ_0)k^s k^t g) \\
&= h(k^s \exp(-tZ_0)\exp(tZ_0)\exp(-\tfrac{s}{\nu}Z_0)g) \\
&= h(\exp(-\tfrac{s}{\nu}Z_0)g) \\
&= f(g_s)
\end{aligned}$$

As a result, f is left $\mathbb{K}_\nu$-invariant. Similarly, reminding ourselves that h is invariant under inner automorphisms of K,

$$\begin{aligned}
f(g_s k_\nu) &= f(gk, s+t\nu)]) \\
&= h(\exp(-\tfrac{s+t\nu}{\nu}Z_0)gk) \\
&= h((k^s)^{-1}\exp(-tZ_0)\exp(-\tfrac{s}{\nu}Z_0)gk) \\
&= h(k^{-1}\exp(-\tfrac{s}{\nu}Z_0)gk) \\
&= h(\exp(-\tfrac{s}{\nu}Z_0)g) \\
&= f(g_s)
\end{aligned}$$

and so f is also right $\mathbb{K}_\nu$-invariant. It is clear that f must be unique and hence the mapping $f \mapsto f \circ j$ establishes a bijection between $C_c^K(G//K_s)$ and $C_c^\infty(\mathbb{G}//\mathbb{K}_\nu)$.

Let dx, dw, dy, dz be the Haar measures on G, $\mathbb{G}$, $\mathbb{G}/\mathbb{Z}$, $\mathbb{Z}$, respectively, and set $\int_{\mathbb{Z}} dz = 1$. Since $dw = dydz$, we use the diffeomorphism $j : G \to \mathbb{G}/\mathbb{Z}$ to normalize dy so that the bijection $f \mapsto f \circ j$ preserves the L^1-norm between our algebras. It remains to check that the convolution product is preserved. Let $f_1, f_2 \in C_c^\infty(\mathbb{G}//\mathbb{K}_\nu)$.

$$\begin{aligned}
((f_1 * f_2)\circ j)(g) &= \int_{\mathbb{G}^1} f_1(j(g)w^{-1})f_2(w)dw \\
&= \int_{\mathbb{G}^1/\mathbb{Z}}\int_{\mathbb{Z}} f_1(j(g)y^{-1}z^{-1})f_2(zy)dxdz \\
&= \int_G f_1(j(gx^{-1}))f_2(j(x))dx \\
&= ((f_1\circ j)*(f_2\circ j))(g)
\end{aligned}$$

Hence, $C_c^K(G//K_s)$ and $C_c^\infty(\mathbb{G}//\mathbb{K}_\nu)$ are isomorphic as algebras.

We have also proven that $C_c^\infty(\mathbb{G}//\mathbb{K}_\nu)$ is commutative. To see that we can also get Flensted-Jensen's result that $C_c^\infty(\mathbb{G}^1/\!\!/\mathbb{K}^1)$ is commutative, it suffices to prove that our definition of f in (4.4) is well-defined on the coset space $\mathbb{G}^1$ because $\mathbb{G}^1$ and $\mathbb{K}^1$ are quotients of $\mathbb{G}$ and $\mathbb{K}_\nu$, respectively. This requires the following expression for elements of $\mathbb{K}^1$. Decompose elements of K as $k = k^s k^t$ where $k^s \in K_s$ and $k^t \in Z_K^0$. Now write $k^t = \exp(tZ_0)$ for some real number t

so that $k_\nu = k^{t\nu}$. If k_ν is an element of Q_ν, then $k_\nu = k^{t\nu}$ and

$$\begin{aligned} f(g_s k_\nu) &= f(gk^t, s+t\nu) \\ &= h(\exp(-\tfrac{s+t\nu}{\nu} Z_0) g k^t) \\ &= h(\exp(-\tfrac{s}{\nu} Z_0) \exp(-tZ_0) g k^t) \\ &= h(\exp(-\tfrac{s}{\nu} Z_0) (k^t)^{-1} g k^t) \\ &= f(g, s) \end{aligned}$$

implies f is well-defined on $\mathbb{G}^1$. □

Remarks. Note that the proposition above does not hold when $\nu = 0$ because our definition in Equation (4.4) is absurd. The mapping $f \mapsto f \circ j$ fails to be one-to-one in this case because $\mathbb{G}/\mathbb{K}_\nu$ reduces to a direct product of G/K and $\mathbb{R}$, and so the strict containment $C_c^K(G//K_s) \subset C_c^\infty(\mathbb{G}//\mathbb{K}_\nu)$ becomes clear. However, commutativity of $C^\infty(\mathbb{G}/\mathbb{K}_\nu)$ still holds because now $\mathbb{G}/\mathbb{K}_\nu = G/K \times S^1$ is a direct product of two weakly symmetric spaces.

It is also interesting to observe that even though G/K_s and $\mathbb{G}/\mathbb{K}_\nu$ are diffeomorphic as Riemannian manifolds, (G, K_s) and $(\mathbb{G}, \mathbb{K}_\nu)$ are not equivalent as Gelfand pairs in general, since the space of double cosets $K_s \backslash G/K_s$ and $\mathbb{K}_\nu \backslash \mathbb{G}/\mathbb{K}_\nu$ may not be isomorphic. However, they do coincide when $\nu \neq 0$ and G is a bounded symmetric domain not of tube type.

Lemma 4.1.4 *(cf. [2], Theorem 3.3)* $C_c^\infty(G//K_s) = C_c^\infty(\mathbb{G}//\mathbb{K}_\nu)$ *if and only if G is not of tube type.*

4.2 Spherical Representations

Assume G and K as in the previous section so that (G, K) is a Gelfand pair. Then $\mathbf{E}(G/K)$ is commutative. This leads to us the notion of spherical kernels and their corresponding spherical functions. Let I be an integral operator on $C_c^\infty(G/K)$ of the form

$$I(f)(z) = \int_D p(z,w) f(w) dw, \quad f \in C^\infty(G/K), \quad z, w \in D.$$

Denote by $\mathbf{I}(G/K)$ the space of such operators I that are invariant under G, i.e. $p(gz, gw) = p(z, w)$ for all $g \in G$. Following Godement [6], p is then called an **invariant kernel** of I. Furthermore, p is said to be a **spherical kernel** if it satisfies the property

$$\int_D p(z,u) q(u,w) du = \mu_q p(z,w), \quad \mu_q \ constant,$$

for every invariant kernel $q \in \mathbf{I}(G/K)$ with compact support (as a function of one variable).

Lemma 4.2.1 *(Godement [6], p. 144-03)* $p(z, w)$ *is a spherical kernel if and only if*

$$D_z p(z,w) = \lambda_p p(z,w) \quad \text{for every} D \in \mathbf{D}(G/K),$$

where λ_p *is a constant depending only on* p.

Let φ be a complex-valued continuous function on a homogeneous space $D = G/K$. We say that φ is a spherical function on D if (i) $\varphi(o) = 1$, $o = eK$, (ii) φ is invariant under the left action of K, (iii) $D\varphi = \lambda_D \varphi$ for each $D \in \mathbf{D}(G/K)$.

Translating these properties to functions on G, we say that φ is a spherical function on G with respect to K if (i) $\varphi(e) = 1$, (ii) φ is bi-K-invariant, (iii) $D\varphi = \lambda_D \varphi$ for each $D \in \mathbf{D}_K(G)$.

Let $\pi : G \to G/K$ is a quotient map. Then $\tilde{\varphi} = \varphi \circ \pi$ is a spherical function on G with respect to K if and only if φ is a spherical function on G.

Lemma 4.2.2 *(Godement) If p is a spherical kernel on D, then $\varphi(gK) = p(gK, K)$ is a spherical function on D. Conversely, if φ is a spherical function on D, then $p(gK, hK) = \varphi(h^{-1}g)$ is a spherical kernel on D.*

Proof. Follows immediately from Lemma 7.1. □

Let $\mathbb{D}^1$ be a weakly symmetric space so that $(\mathbb{G}^1, \mathbb{K}^1)$ is a Gelfand pair. Flensted-Jensen [2] has characterized the spherical functions on $\mathbb{G}^1$ (with respect to $\mathbb{K}^1$). We summarize his results below, which are just generalizations of Harish-Chandra's results for spherical functions on G with respect to K. Let $\mathbb{G}^1 = \mathbb{K}^1 A_\nu N$ be the "Iwasawa decomposition" of $\mathbb{G}^1$. For $x \in \mathbb{G}^1$, let $H(x) \in \mathfrak{a}^1$ be the unique element such that $x \in \mathbb{K}^1 \exp(H(x))N$.

Lemma 4.2.3 *(Flensted-Jensen) Let dk, da, dn be Haar measures on $\mathbb{K}^1$, A_ν and N, respectively. The Haar measure dg on $\mathbb{G}^1$ can be normalized so that*

$$\int_{\mathbb{G}^1} f(g)dg = \int_{\mathbb{K}^1 A_\nu N} f(kan)e^{2\rho(H(g))}dkdadn \tag{4.5}$$

for each $f \in C_c(\mathbb{G}^1)$.

Theorem 4.2.4 *(Flensted-Jensen) Every spherical function on $\mathbb{G}^1$ w.r.t. $\mathbb{K}^1$ is of the form*

$$\varphi_\lambda(g) = \int_{\mathbb{K}^1} e^{(i\lambda - \rho)(H(gk))}dk$$

where $\lambda \in (\mathfrak{a}^1)^*_{\mathbb{C}}$. Moreover, $\varphi_\lambda \equiv \varphi_\mu$ if and only if $\lambda = w(\mu)$ for some $w \in W_\nu$.

A joint eigenfunction f on $\mathbb{D}^1$ is an eigenfunction of each of the operators $D \in \mathbf{D}(\mathbb{G}^1/\mathbb{K}^1)$, and so determines a homomorphism $\mu : \mathbf{D}(\mathbb{G}^1/\mathbb{K}^1) \to \mathbb{C}$ defined by $Df = \mu(D)f$. Any such homomorphism μ has a joint eigenspace given by

$$\mathcal{E}_\mu = \{f \in C^\infty(\mathbb{G}^1/\mathbb{K}^1) : Df = \mu(D)f \text{ for each } D \in \mathbf{D}(\mathbb{G}^1/\mathbb{K}^1)\} \tag{4.6}$$

Lemma 4.2.5 *(cf. [10] IV, Prop. 2.4) The joint eigenfunctions on $\mathbb{G}^1/\mathbb{K}^1$ are characterized by the following integral equation: Each joint eigenspace $\mathcal{E}_\mu \neq 0$ contains exactly one spherical function φ. The members f of $\mathcal{E}_\mu$ are characterized by*

$$\int_{\mathbb{K}^1} f(xky\mathbb{K}^1)dk = f(x\mathbb{K}^1)\varphi(y\mathbb{K}^1), \quad x, y \in \mathbb{G}^1 \tag{4.7}$$

Proof. The proof mirrors that given by Helgason [9] in the case of symmetric space. □

Lemma 4.2.6 *Every square-integrable joint eigenfunction of* $\mathbf{D}(\mathbb{G}^1/\mathbb{K}^1)$ *is a joint eigenfunction of* $\mathbf{I}(\mathbb{G}^1/\mathbb{K}^1)$.

Proof. Let $f \in \mathcal{E}_\mu$ with φ the corresponding spherical function. Denote by $\tilde{f} = f \circ \pi$ and $\tilde{\varphi} = \varphi \circ \pi$ to be their pullback to $\mathbb{G}^1$. If $I_{p_\psi} \in \mathbf{I}(\mathbb{G}^1/\mathbb{K}^1)$, then the following reordering of integration is permissible by Fubini's theorem, since ψ and f are square-integrable (and so is φ):

$$\begin{aligned}
I_{p_\psi} f(x\mathbb{K}^1) &= \int_{\mathbb{G}^1/\mathbb{K}^1} p_\psi(x\mathbb{K}^1, y\mathbb{K}^1) f(y\mathbb{K}^1) d(y\mathbb{K}^1) \\
&= \int_{\mathbb{K}^1} \int_{\mathbb{G}^1/\mathbb{K}^1} \psi(k^{-1}y^{-1}x) \tilde{f}(yk) d(y\mathbb{K}^1) dk \\
&= \int_{\mathbb{G}^1} \tilde{f}(g)\psi(g^{-1}x) dg \\
&= \int_{\mathbb{G}^1} \tilde{f}(xg)\psi(g^{-1}) dg \\
&= \int_{\mathbb{K}^1 A_\nu N} \tilde{f}(xkan)\psi(n^{-1}a^{-1}k^{-1}) e^{2\rho(H(a))} dkdadn \\
&= \int_{A_\nu N} \tilde{f}(x)\tilde{\varphi}(an)\psi(n^{-1}a^{-1}) e^{2\rho(H(a))} dadn \\
&= \tilde{f}(x) \int_{\mathbb{K}^1 A_\nu N} \tilde{\varphi}(kan)\psi(n^{-1}a^{-1}k^{-1}) e^{2\rho(H(a))} dkdadn \\
&= f(x\mathbb{K}^1) \int_{\mathbb{G}^1} \tilde{\varphi}(g)\psi(g^{-1}) dg \\
&= f(x\mathbb{K}^1) \int_{\mathbb{G}^1/\mathbb{K}^1} p(\mathbb{K}^1, g\mathbb{K}^1) q(g\mathbb{K}^1, \mathbb{K}^1) dg \\
&= f(x\mathbb{K}^1) \mu_q p(\mathbb{K}^1, \mathbb{K}^1)
\end{aligned}$$

where μ_q is a constant depending only on q since p is a spherical kernel. Hence, $I_{p_\psi} f = \mu_p f$. □

We next investigate the connection between spherical representations of $\mathbb{G}^1$ and discrete series representations of G. This will make use of some deep results of Langlands [14]. For any integer k, let $L^2(\mathbb{D}^1, k)$ denote the space of square-integrable functions on D that transform under G according to the one-dimensional K-type $\tau^k = j^k$, where j is the jacobian factor of automorphy defined in chapter I. Then $H^2(\mathbb{D}^1, k)$ will be the subspace of holomorphic functions. Write elements of D as $w = gK$. Now, consider the reproducing kernel $K_k(z, w)$ defined as follows for $f \in H^2(\mathbb{D}^1, k)$:

$$\int_D \frac{K_k(z,w)}{j(g,o)^k \overline{j(g,o)}^k} f(w) dw = f(z).$$

Then Langlands has proven the following:

Proposition 4.2.7 *(Langlands)*

$$P_k(f)(z) = \int_D \frac{K_k(z,w)}{j(g,o)^k \overline{j(g,o)}^k} f(w) dw$$

defines the orthogonal projection of $L^2(\mathbb{D}^1, k)$ *onto* $H^2(\mathbb{D}^1, k)$.

This proposition now implies that $K(z, w)$ has the following properties:

Corollary 4.2.8 *(Langlands) i)* $K(gz, gw) = j(g,z)^k K(z,w)\overline{j(g,w)}^{\,k}$.
ii) $\overline{K(z,w)} = K(w,z)$ *and* $K(z,z) > 0$.
iii) $\displaystyle\int_D \frac{K_k(z,w)K_k(w,u)}{j(g,o)^k\overline{j(g,o)}^{\,k}}dw = K(z,u)$.

Properties i) and ii) now tell us that $K_k(z,w)$ has the same transformation properties as our kernel function $K(z,w)$ raised to the exponent $-k$ and hence, must agree (see Satake [21]).

Lemma 4.2.9

$$K_k(z,w) = K(z,w)^{-k}.$$

Let $\mathcal{E}_\phi$ be an eigenspace of $\mathbf{D}(\mathbb{G}^1/\mathbb{K}^1)$. Since $\partial/\partial t$ is a fundamental operator of $\mathbf{D}(\mathbb{G}^1/\mathbb{K}^1)$, every $F \in \mathcal{E}$ can be written as $F(z,t) = K_k(z,z)^{-\frac{1}{2}}f(z)e^{ikt}$ for some integer $k \in \mathbb{Z}$ and function f on D. Define the operator $T : F \mapsto f$. Assume that g_sF represents the action of $\mathbb{G}^1$ on F by left translation. Then using the above transformation property of F and writing $j(g,z) = |j(g,z)|e^{i\arg j(g,z)}$, we can compute the image $T(g_sF)$:

Lemma 4.2.10

$$T(g_sF)(z) = j(g,z)^k f(z)e^{iks}$$

Proof.

$$\begin{aligned}
F(g_s(z,t)) &= F(gz, t+s+\arg\tau^{-1}(\kappa(g,z))) \\
&= K_k(gz,gz)^{-\frac{1}{2}}f(gz)e^{ik(t+s-\arg j(g,z))} \\
&= K_k(z,z)^{-\frac{1}{2}}|j(g,z)|^{-k}e^{-ik\arg j(g,z))}e^{iks}e^{ikt} \\
&= K_k(z,z)^{-\frac{1}{2}}j(g,z)^{-k}f(z)e^{iks}e^{ikt}
\end{aligned}$$

and so $T(g_sF)(z) = j(g,z)^{-k}f(z)e^{iks}$. □

We hope to show in a later section that $T : \mathcal{E} \to \mathrm{L}^2(D,k)$ is an intertwining operator for π_ϕ restricted to $G \subset \mathbb{G}^1$ and the discrete series representation π_λ.

Lemma 4.2.11

$$p_k(z_t, w_r) = \frac{K_k(z,w)e^{ik(t-r)}}{K_k(z,z)^{\frac{1}{2}}K_k(w,w)^{\frac{1}{2}}} \tag{4.8}$$

is an $\mathbb{G}^1$ *invariant kernel on* $\mathbb{D}^1 \times \mathbb{D}^1$ *for all* $k \in \mathbb{Z}$.

Proof. Again, using the transformation property of $K_k(z,w)$ described above and the fact that $j(g,z) = |j(g,z)|e^{i\arg j(g,z)}$, it becomes straightforward to check that $p_k(g_sz_t, g_sw_r) = p_k(z_t, w_r)$. Let I be the corresponding integral operator with kernel p_k. Then

$$\begin{aligned}
I(F)(z_t) &= \int_{\mathbb{D}^1} p_k(z_t,w_r)F(w_r)dw_r \\
&= \frac{e^{ikt}}{K_k(z,z)^{\frac{1}{2}}}\int_D\int_{S^1}\frac{K(z,w)}{K(w,w)}f(w)dwdr \\
&= K(o,o)^{-1}K(z,z)^{-\frac{1}{2}}e^{ikt}P_k(f)(z)
\end{aligned}$$

and so I is a projection of $\mathcal{E}$ such that $T(I(\mathcal{E})) = H^2(D,\tau^k)$. □

4.3 Holomorphic Discrete Series

Let π be an irreducible representation of G on a Hilbert space E. If τ is a representation of K on V, then we say that $v \in E$ transforms according to τ if $\pi(K)v$ is isomorphic to a finite number of copies of V. Let E_τ be the subspace of E consisting of vectors which transform according to τ and denote by $P_\tau : E \to E_\tau$ the corresponding projection. Consider the operator $P_\tau \pi(x) P_\tau$ and define $\phi(x) = \mathrm{Tr}(P_\tau \pi(x) P_\tau)$ to be its trace. If m denotes the multiplicity of τ in π_K, then we say that ϕ is a τ-spherical function of height m. In the case ϕ has height one, we have the following characterization of τ-spherical functions:

Theorem 4.3.1 *(Godement) Let ϕ be a continuous function on G. Then ϕ is a τ-spherical function of height one if and only if*

$$\int_{K/Z} \phi(k^{-1}xky)dk = \phi(x)\phi(y) \tag{4.9}$$

In this paper, we shall only be concerned with τ-spherical functions of height one. Therefore, ϕ will now simply be called a τ-spherical function.

Lemma 4.3.2 *If φ_λ is a spherical function on $\mathbb{G}^1$ with respect to $\mathbb{K}^1$, then its restriction to G is a τ-spherical function.*

Proof. The fact that φ_λ is a spherical function means that

$$\int_{\mathbb{K}^1} \varphi_\lambda(xk_\nu y)dk_\nu = \phi(x)\phi(y). \tag{4.10}$$

Define ϕ to be the restriction of φ_λ to G. If $k \in Z_K^0$ and $g \in G$, then $\varphi(kx) = \varphi(k)\varphi(g)$ and so the restriction of ϕ to Z_K^0 determines a character of Z_K^0. Denote by $\tau = \phi|_{Z_K^0}$. Extend τ to a one-dimensional representation of K by allowing τ to be trivial on K_s. Then

$$\phi(k^{-1}xky) = \varphi_\lambda(k^{-1}xky) = \tau^{-1}(k)\varphi_\lambda(xk_\nu y) \tag{4.11}$$

for $k \in K$. The transformation $k \mapsto k_\nu$ has Jacobian $\tau(k)$ and so

$$\int_{K/Z} \phi(k^{-1}xky)dk = \int_{\mathbb{K}^1} \phi(xk_\nu y)dk_\nu = \phi(x)\phi(y). \tag{4.12}$$

Hence, ϕ is a τ-spherical function. $\square$

We wish to show next that if $\varphi_\lambda \in L^2(\mathbb{G}^1/\mathbb{K}^1)$, then the restriction ϕ can be realized as a square-integrable matrix-coefficient of a holomorphic discrete series representation π. Denote by V the representation space of τ, Λ the highest weight of τ, and v_Λ the corresponding highest weight unit vector. Consider the associated line bundle $\mathbb{V}_\tau \to G/K$, where $\mathbb{V}_\tau = G \times_K V$, induced from τ. If $\mathcal{H}$ denotes the space of L^2-holomorphic section of $\mathbb{V}_\tau \to G/K$, then we may view

$$\mathcal{H} \cong \left\{ f : G \mapsto V \quad \begin{array}{l} 1.\ f \text{ holomorphic} \\ 2.\ f(gk) = \tau^{-1}(k)f(g),\ k \in K,\ g \in G \\ 3.\ \int_{G/Z} \|f(g)\|_\tau^2 \, dg < \infty \end{array} \right\},$$

where the norm of $f(g)$ is given by the inner product $(\cdot\,,\cdot)_\tau$ on V.

Let π denote the unitary representation of G on the Hilbert space $\mathcal{H}$ acting by left translation. Denote by E_τ to be the subspace of $\mathcal{H}$ consisting of elements which transform according to τ, i.e. if $v \in E_\tau$, then $\pi(K)v$ is isomorphic to a finite number of copies of V. The projection of $\mathcal{H}$ onto E_τ we will write as P_τ. Now define the operator $\Phi^\pi_\tau(x) = P_\tau \pi(x) P_\tau$ on $\mathcal{H}$ and set $\phi^\pi_\tau(x)$ to be its trace.

Extend τ to a representation of $\widetilde{K}_{\mathbb{C}}$ so that Λ becomes a linear functional on $\mathfrak{h}_{\mathbb{C}}$. Let μ be the maximal root of Φ^+_n.

Theorem 4.3.3 *(Harish-Chandra) If* $\langle \lambda + \rho, \mu \rangle < 0$*, then* π *is an irreducible unitary representation of* G *and* $\pi|_K$ *has* τ *as the unique minimal* K*-type.*

Lemma 4.3.4 *If* $\langle \lambda + \rho, \mu \rangle < 0$*, then* $\phi = \phi^\pi_\tau$ *is a matrix coefficient of* π*.*

Before proving the lemma, we shall need to describe Harish-Chandra's construction of the holomorphic discrete series (extended to our situation for G simply-connected). More precisely, we define a space of functions on $W = P_- \widetilde{K}_{\mathbb{C}} G$ as follows:

$$H := \left\{ F : W \mapsto \mathbb{C} \;\; \begin{array}{l} 1.\ F \text{ holomorphic} \\ 2.\ F(pkx) = \tau(k)F(x),\ p \in P_-,\ k \in \widetilde{K}_{\mathbb{C}},\ x \in W \\ 3.\ \int_{G/Z} |F(g)|^2 \, dg < \infty \end{array} \right\},$$

Let $f \in \mathcal{H}$ and define a function F by $F(g) = (f(g^{-1}), v_\tau)_\tau$. Then under the correspondence $f \leftrightarrow F$, we have that f is holomorphic section if and only if F is holomorphic function. It is easily shown that $\mathcal{H} \cong H$ as Hilbert spaces and the action of G on H by right translation is equivalent to π (see [14], p.102). We keep this equivalence in mind and define

$$\psi(x) = (\tau(\kappa(x))v_\Lambda, v_\Lambda)_\tau$$

Theorem 4.3.5 *(Harish-Chandra) If* $\langle \Lambda + \rho, \mu \rangle < 0$*, then*
(i) $\|\psi\|^2 < \infty$ *and* $\psi \in H$*.*
(ii) $(\pi(x)\psi, \psi) = \psi(x)\|\psi\|^{-2}$*.*
(iii) $\|\psi\|^{-2} = \Pi_{\beta \in \Phi^+_n} |(\Lambda(H_\beta) + \rho(H_\beta))/\rho(H_\beta)|$*.*

Proof. For part (i), see [7], Lemma 9. For parts (ii), see [7], corollary to Theorem 2. For part (iii), see [8], Theorem 4. □

Proof. (of the lemma) By definition, Harish-Chandra's theorem says that $\psi \in E_\tau$ and π irreducible means that $\{\psi/\|\psi\|\}$ is an orthonormal basis for E_τ since E_τ has dimension one. Then the Schur orthogonality relations imply

$$\phi^\pi_\tau(x) = \mathrm{tr}(P_\tau \pi(x) P_\tau) = (\pi(x)\frac{\psi}{\|\psi\|}, \frac{\psi}{\|\psi\|})$$

and so part (ii) of Harish-Chandra's results now gives us $\phi = \phi^\pi_\tau = \psi$. We conclude that ϕ is a square-integrable matrix coefficient of the holomorphic discrete series representation π. □

4.4 Invariant Eigenfunctions

Let Γ be a discrete subgroup of G which acts discontinously on D. Then $\mathcal{F} = \Gamma\backslash D$ is a fundamental domain of D. If we also view Γ as a subgroup of $\mathbb{G}^1$, then $\mathcal{F}_\nu = \Gamma\backslash\mathbb{D}^1$ will be called a fundamental domain of $\mathbb{D}^1$.

Lemma 4.4.1 *Let Γ be a discrete subgroup of G. Then*
(i) $\mathcal{F}$ is a fundamental domain of D if and only if $\mathcal{F}_\nu$ is a fundamental domain for $\mathbb{D}^1$.
(ii) $\mathcal{F}$ is compact if and only if $\mathcal{F}_\nu$ is compact.
(iii) $\mathcal{F}$ has finite area if and only if $\mathcal{F}_\nu$ has finite area.

Proof. (i) Γ acts discontinuously on D if and only if $\mathcal{F}_\nu$ acts discontinuously on $\mathbb{D}^1$. □

Let χ be a representation of Γ on V_χ and τ_ν a one-dimensional K-type. Extend χ to Γ_ν and define

$C^\infty(\mathbb{D}^1, \chi)$ to be the space of infinitely differentiable functions on $\mathbb{D}^1$ with values in V_χ and which transform as follows:

$$F(m(z,t)) = \chi(m)F(z,t) \tag{4.13}$$

Similarly, let $L^2(\mathbb{D}^1, \chi)$ be the space of square integrable functions F on $\mathbb{D}^1$ with values in V_χ and transforms as in (4.13):

$$\int_{\mathbb{D}^1} F(z,t)\overline{F(z,t)}d(z,t) = \int_{\mathcal{F}_\nu} F(z,t)\overline{F(z,t)}d(z,t) < \infty, \tag{4.14}$$

where $d(z,t)$ is a $\mathbb{G}^1$-invariant measure on $\mathbb{D}^1$. We define an inner product on $L^2(\mathbb{D}^1, \chi)$:

$$(F_1, F_2) = \int_{\mathcal{F}_\nu} F_1(z,t)\overline{F_2(z,t)}d(z,t). \tag{4.15}$$

Then $L^2(\mathbb{D}^1, \chi)$ becomes a Hilbert space under this inner product.

Let us assume χ to be one-dimensional and k an integer. We define

$$\mathcal{E}(\chi, k) = \left\{ F \in L^2(\mathbb{D}^1, \chi) : \begin{array}{l} 1.\ DF = \mu(D)F \text{ for all } D \in \mathbf{D}(\mathbb{G}^1/\mathbb{K}^1) \\ 2.\ \frac{\partial}{\partial t}F = kF \end{array} \right. \tag{4.16}$$

to be the eigenspace of $\mathbf{D}(\mathbb{G}^1/\mathbb{K}^1)$ corresponding to μ. Also, define $\mathcal{A}(\chi, k)$ to be the space of square-integrable functions on D which transform as follows:

$$f(z) = \chi(m)\tau^k(\kappa(m,z))f(z),\ m \in \Gamma. \tag{4.17}$$

The following lemma now gives us the first connection between eigenfunctions of $\mathbf{D}(\mathbb{G}^1/\mathbb{K}^1)$ and functions on D that transform automorphically.

Lemma 4.4.2 *Let $F(z,t) \in \mathcal{E}(\chi, k)$. Then*

$$F(z,t) = K(z,z)^{\frac{k}{2}}f(z)e^{ikt}, \tag{4.18}$$

where $f(z) \in \mathcal{A}(\chi, k)$.

Proof. Since $F \in \mathcal{E}(\chi, k)$ has eigenvalue k under the differential operator $\frac{\partial}{\partial t}$, it must be of the form

$$F(z,t) = g(z)e^{ikt}$$

for some integer k. Now $K(z,z)$ is nonvanishing so we can write $g(z) = K(z,z)^{\frac{k}{2}}f(z)$. Then using the transformation property of F, we have

$$\begin{aligned} F(m(z,t)) &= \chi(m)F(z,t) \\ F(mz, t+\arg\tau^{-1}(\kappa(m,z))) &= \chi(m)K(z,z)^{\frac{k}{2}}f(z)e^{ikt} \\ K(z,z)^{\frac{k}{2}}|\tau(\kappa(m,z))|^k e^{i\arg\tau^{-k}(\kappa(m,z))}f(mz)e^{-ikt} &= \chi(m)K(z,z)^{\frac{k}{2}}f(z)e^{ikt} \end{aligned}$$

and so $f(mz) = \chi(m)\tau^k(\kappa(m,z))f(z)$, which means $f \in \mathcal{A}(\chi, k)$. $\square$

4.5 Selberg Trace Formula

Let $I_{p_\psi} \in \mathbf{I}(\mathbb{G}^1/\mathbb{K}^1)$ act on $F \in L^2(\mathbb{D}^1, \chi)$ as follows:

$$\int_{\mathbb{D}^1} p_\psi(z,w)F(w)dw = \int_{\mathcal{F}_\nu} \sum_{\Gamma} \chi(m)p_\psi(z,mw)F(w)dw. \tag{4.19}$$

We define

$$P_\psi(z,w,\chi) = \sum_{m\in\Gamma} \chi(m)p_\psi(z,mw) \tag{4.20}$$

and view P_ψ as the kernel of the integral operator I_{p_ψ} on $L^2(\mathbb{D}^1, \chi)$. To insure that the summation on the right side of (4.20) converges, we assume that P_ψ has the following properties (see Selberg [23], p. 60): there exists a majorant Ξ of p_ψ (meaning $|p_\psi(z,w)| \le \Xi(z,w)$ for all $z, w \in \mathbb{D}^1$) such that

(i) $\int_{\mathbb{D}^1} \Xi(z,w)dw < \infty$

(ii) Ξ has regular growth, i.e. there exists positive constants A and δ such that

$$\Xi(z,w) < A\int_{d(w,\tilde{w})<\delta} k_\psi(z,\tilde{w})d\tilde{w}, \tag{4.21}$$

where $d(w,\tilde{w})$ denotes the shortest geodesic distance between w and $\tilde{w}$. Selberg has proven that under these assumptions, the series for $p_\psi(z,w,\chi)$ converges absolutely for z, w in $\mathbb{D}^1$.

Since the set of fundamental differential operators $\{D_1, ..., D_l, \frac{\partial}{\partial t}\}$ may be chosen to be self-adjoint, there exists an orthonormal system of eigenfunctions $\{F_i\}$ which forms a basis for $C^\infty(\mathbb{D}^1, \chi)$ and satisfying

$$D_jF_i = \lambda_j^i F_i, \quad j = 1, ..., l \tag{4.22}$$

such that the l-tuples $\lambda^i = (\lambda_1^i, ..., \lambda_l^i)$ have no finite point of accumulation. We expand P_ψ as follows:

$$P_\psi(z_t, w_u, \chi) = \sum_i h(\lambda^i)F_i(z_t)F_i^*(w_u). \tag{4.23}$$

Let us formally compute the trace of I_{p_ψ}:

$$\mathrm{Tr}(I_{p_\psi}) = \sum_i h(\lambda^i). \tag{4.24}$$

On the other hand, we have $\mathrm{Tr}(I_{p_\psi}) = \int_{\mathcal{F}_\nu} \mathrm{Tr}(P_\psi(z_t, z_t, \chi))dz_t$, and so

$$\mathrm{Tr}(I_{p_\psi}) = \sum_{m\in\Gamma} \mathrm{Tr}(\chi(m)) \int_{\mathcal{F}_\nu} p_\psi(z_t, mz_t)dz_t \tag{4.25}$$

Following Selberg [23], we divide Γ into its conjugacy classes $\{m\}_\Gamma$ inside $\mathbb{G}^1$, where m is a chosen representative. Let Γ_m denote the centralizer of m in Γ and write $[\tilde{m}]$ as the coset element in $\Gamma_m\backslash\Gamma$ represented by $\tilde{m}$. The right hand side of (4.25) now becomes

$$\sum_{\{m\}_\Gamma} \sum_{\Gamma_m\backslash\Gamma} \mathrm{Tr}(\chi(\tilde{m}^{-1}m\tilde{m}) \int_{\mathcal{F}_\nu} p_\psi(z_t, \tilde{m}^{-1}m\tilde{m}z_t)dz_t \tag{4.26}$$

Before simplifying this expression any further, we first observe that $\mathrm{Tr}(\chi(\tilde{m}^{-1}m\tilde{m})) = \mathrm{Tr}(\chi(m))$. Also, for any coset $[\tilde{m}] \in \Gamma_m\backslash\Gamma$, we have

$$\int_{\mathcal{F}_\nu} p_\psi(z_t, \tilde{m}^{-1}m\tilde{m}z_t)dz_t = \int_{\tilde{m}\mathcal{F}_\nu} p_\psi(z_t, mz_t)dz_t, \tag{4.27}$$

where $\tilde{m}\mathcal{F}_\nu$ is the translation of $\mathcal{F}_\nu$ by $\tilde{m}$. As a result, (4.26) can be rewritten as

$$\sum_{\{m\}_\Gamma} \sum_{\Gamma_m\backslash\Gamma} \mathrm{Tr}(\chi(m) \int_{\mathcal{F}_\nu} p_\psi(z_t, mz_t)dz_t, \tag{4.28}$$

where $\mathcal{F}_\nu^m$ is the fundamental domain for Γ_m in $\mathbb{D}^1$. We transform the integral on the right hand side to integration over $\mathbb{G}^1/\mathbb{K}^1$ as

$$\int_{\mathcal{F}_\nu^m} p_\psi(x\mathbb{K}^1, mx\mathbb{K}^1)d(x\mathbb{K}^1). \tag{4.29}$$

According to Selberg ([23], p. 65), this integral becomes

$$\int_{\mathbb{G}^1/G_m} \psi(g_s^{-1}m^{-1}g_s)dg_s, \tag{4.30}$$

where G_m is the centralizer of m in $\mathbb{G}^1$. Hence, the final equation for the Selberg trace formula becomes

$$\mathrm{Tr}(I_{p_\psi}) = \sum_{\{m\}_\Gamma} \mathrm{Tr}(\chi(m))\mathrm{vol}(G_m/\Gamma_m)\Upsilon_\psi(m), \tag{4.31}$$

where

$$\Upsilon_\psi(m) = \int_{\mathbb{G}^1/G_m} \psi_(g_s^{-1}m^{-1}g_s)dg_s \tag{4.32}$$

is a function which depends only on the conjugacy class of m in $\mathbb{G}^1$.

Let I_{p_ψ} be a integral operator on $\mathbb{D}^1$ with spherical kernel p and corresponding spherical function ψ. Assume that p is square-integrable. Then I_{p_φ} is in fact a Hilbert-Schmidt operator, which implies that it has a discrete spectrum and that its eigenspaces are finite-dimensional. More precisely, the spectrum contains only one nontrivial eigenvalue $h(I_{p_\varphi})$ with eigenfunctions consisting of those elements $F \in \mathcal{E}(\chi, k)$ such that $F(z_t) = K(z,z)^{\frac{k}{2}} f(z) e^{ikt}$ with $f(z) \in \mathcal{A}(\chi, k)$. To compute $h(I_\varphi)$, let $F \in \mathcal{E}(\chi, k)$. Then

$$I_{p_\varphi} F = \mu_p F \tag{4.33}$$

and so $h(I_{p_\varphi}) = \mu_p$. If we introduce a basis $F_1, ..., F_N$ of $\mathcal{E}(\chi, k)$, then the trace of I_{p_φ} is $\mathrm{Tr}(I_{p_\varphi}) = \mu_p N$. On the other hand, the Selberg trace formula tells us that

$$\mu_p N = \sum_{\{m\}_\Gamma} \mathrm{Tr}(\chi(m)) \mathrm{vol}(G_m/\Gamma_m) \Upsilon_\varphi(m). \tag{4.34}$$

Since the dimension of $\mathcal{A}(\chi, k)$ equals N, we now have a formula for computing this dimension.

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