WEAKLY SYMMETRIC SPACES AND BOUNDED SYMMETRIC DOMAINS

HIÊÚ NGUYÊÑ

Abstract. Let G be a connected, simply-connected, real semisimple Lie group and K a maximal compactly embedded subgroup of G such that $D = G/K$ is a hermitian symmetric space. Consider the principal fiber bundle $M = G/K_s \to G/K$, where K_s is the semisimple part of $K = K_s \cdot Z_K^0$ and Z_K^0 is the connected center of K. The natural action of G on M extends to an action of $G^1 = G \times Z_K^0$. We prove as the main result that M is weakly symmetric with respect to G^1 and complex conjugation. In the case where D is an irreducible classical bounded symmetric domain and G is a classical matrix Lie group under a suitable quotient, we provide an explicit construction of $M = D \times S^1$ and determine a one-parameter family of Riemannian metrics Ω on M invariant under G^1. Furthermore, M is irreducible with respect to Ω. As a result, this provides new examples of weakly symmetric spaces that are non-symmetric, including those already discovered by Selberg (cf. [M]) for the symplectic case and Berndt and Vanhecke [BV1] for the rank-one case.

Introduction

In his seminal 1956 paper [Se], Atle Selberg introduced the notion of a weakly symmetric space in his investigation of automorphic forms. Naturally, this notion generalizes that of a symmetric space. Let M be a Riemannian manifold with a transitive group of isometries G and a fixed isometry μ satisfying $\mu G \mu^{-1} = G$, $\mu^2 \in G$. Then M is called weakly symmetric with respect to G and μ if it has the property that given any two points $x, y \in M$, there exists an element $g \in G$ such that $gx = \mu y$ and $gy = \mu x$. According to J. Berndt and L. Vanhecke [BV1], if M is connected, then Selberg's definition is equivalent to that of a ray-symmetric space, a notion due to Z.I. Szabo [Sz]. This means that M has the following geometric property: given any point $x \in M$ and any geodesic γ passing through x, there exists an isometry $g \in G \cdot \mu$ of M which fixes x and reverses the direction of γ. Equivalently, W. Ziller [Z] characterizes this property at the infinitesimal level as g reversing the tangent vector of γ at x. Furthermore, if this reversal can be achieved for all geodesics passing through x by a single element g, then M is in fact a symmetric space and g is the geodesic symmetry at x.

It is classical result that the space of invariant differential operators on a symmetric space is commutative. Selberg showed in [Se] that this property is retained for weakly symmetric spaces. More precisely, if M is weakly symmetric with respect

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to G and μ, then the space of G-invariant differential operators on M is commutative[1]. His analysis of these spaces resulted in many applications, the most notable being the celebrated Selberg trace formula.

The first example of a weakly symmetric space that is non-symmetric was given by Selberg [Se] as the total space of a principal S^1-bundle over the upper-half plane. He extended the proof to principal S^1-bundles over the Siegel half-space (cf. [M]). Unfortunately, lack of any further examples forced the topic into hibernation. But recently, there has a been a reawakened interest in the topic, starting in 1994 with Berndt and Vanhecke [BV1], who used an equivalent geometric characterization of weakly symmetric spaces to obtain new examples that are non-symmetric, including horospheres in complex projective space, tubes in complex hyperbolic space, and spheres of Hopf fibrations. Since then, many papers have appeared on the subject with discoveries of many new examples, including ones that are not naturally reductive as homogeneous spaces (see [BV1], [BV2], [BKV], [BRV], [BTV], [GV1], [GV2]).

The aim of this paper is to demonstrate that Selberg's construction of weakly symmetric spaces as total spaces of principal S^1-bundles over the Siegel half-space extends to principal fiber bundles over hermitian symmetric spaces. This is achieved by translating Selberg's results into the context of Lie groups. Let G be a connected, simply-connected, real semisimple Lie group and K a maximal compactly embedded subgroup of G such that $D = G/K$ be a hermitian symmetric space. This forces D to contain only noncompact irreducible factors, but our arguments are valid for the compact case as well (see discussion after Corollary 6). Then K can be chosen to be the fixed point set of a Cartan involution σ of G. If s is the geodesic symmetry of D induced from σ, then D is obviously symmetric, hence weakly symmetric, with respect to s. The help of the group G is not needed here in order to reverse tangent vectors on D at the fixed point of s. Furthermore, by transitivity of G, this symmetry holds at any point of D.

As a next step, it is natural to look at principal fiber bundles $M \to D$ and determine whether M is weakly symmetric in order to possibly discover new examples which are not symmetric. This will require that M has a Riemannian metric which is not obtained as a product of lower-dimensional symmetric spaces. Under this constraint, our investigation leads us to consider weak symmetry of D with respect to an involution different from s. Namely, this involution will be complex conjugation on D and is known to exist for all hermitian symmetric spaces.

More precisely, the principal bundle $M = G/K_s \to G/K$, where $K = K_s Z_K^0$, K_s is the semisimple part of K and Z_K^0 is the connected center of K isomorphic to a vector group. Define μ to be complex conjugation of D induced from an involution θ of G. The main result of this paper (Theorem 5) is that M is weakly symmetric with respect to extensions G^1 of G and $\tilde{\mu}$ of μ. The extension of the isometry group G is necessary in order to ensure that the isotropy subgroup of M is large enough to reverse its tangent vectors with respect to $\tilde{\mu}$. This is done by defining $G^1 = G \times Z_K^0$. As Z_K^0 normalizes K_s, there is a standard construction of extending the natural action of G on M to G^1. Hence, $M \cong G^1/K^1$.

The proof of our main result involves first demonstrating that D is weakly symmetric with respect to G and μ (Prop. 3). Here is a sketch of the argument: let

[1] In general, a homogeneous space G/K having this commutative property with respect to G is called a G-commutative or G-spherical manifold.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the the Cartan decomposition with respect to σ. We identify $T_o(D)$, the tangent space of D at the origin o, with $\mathfrak{p}$ and the isotropy representation of K on $T_o(D)$ with the adjoint representation of K on $\mathfrak{p}$. The reversal of elements in $\mathfrak{p}$ with respect to $Ad(K)$ and θ will then follow from the property that every $Ad(K)$-orbit of $\mathfrak{p}$ intersects a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$ on which θ reverses all elements. We now extend this argument of weakly symmetry to M. This requires observing that the adjoint action of K^1 on $\mathfrak{p}^1 = \mathfrak{p} \oplus \mathfrak{z}$ agrees with $Ad(K)$ when restricted to $\mathfrak{p}$ and is trivial on $\mathfrak{z}$, the Lie algebra of Z_K^0. Hence, it follows that M is weakly symmetric with respect to G^1 and $\tilde{\mu}$.

Let Z_G be the center of G and $Q \subset Z_K^0 \cap Z_G$ be a cyclic subgroup. We obtain as a corollary that $M_Q = G/QK_s$ is weakly symmetric with respect to $G_Q^1 = G^1/(Q \times Q^{-1})$ and $\tilde{\mu}_Q$. In case D is a classical irreducible bounded symmetric domain and $G_Q = G/Q$ is isomorphic to $G_{\mathbb{R}}$, a simple matrix Lie group, we go on further to determine explicitly a one-parameter family of Riemannian metrics Ω on M_Q. This construction involves using the determinant factor of automorphy j, a concept inherently defined for bounded symmetric domains, to define a twisted group action Φ of $G_Q^1 \cong G_Q \times S^1$ on $M_Q = D \times S^1$:

$$(g,s)(z,t) \stackrel{\Phi}{\longmapsto} (gz, s+t+\arg j(g,z)), \quad (g,s) \in G_Q^1,\ (z,t) \in M_Q.$$

We prove in Prop. 14 that Φ induces a G_Q^1-invariant metric Ω on M_Q. Furthemore, the explicit description of Ω will show that M_Q is irreducible as a Riemannian manifold. Hence, this result provides new examples of weakly symmetric spaces that are not symmetric and include those already discovered by Selberg for the symplectic case, where $G_Q = \mathbf{Sp}(n,\mathbb{R})$, and Berndt and Vanhecke [BV1] for the rank-one case, where $G_Q = \mathbf{SU}(n,1)$. We also refer the reader to the two papers [Na] and [Z].

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1. **Weakly symmetric spaces.** Let $M = G/K$ be a homogeneous Riemannian manifold and μ a fixed isometry of M (not necessarily in G) satisfying $\mu G \mu^{-1} = G$ and $\mu^2 \in G$.

Definition 1. ([Se]) M is called **weakly symmetric** with respect to G and μ if given any $x, y \in M$, there exists an element $g \in G$ such that $gx = \mu y$ and $gy = \mu x$.

Motivated by a characterization of weakly symmetric spaces in [Z] in terms of the full isotropy subgroup of M, we give a lemma that characterizes a weakly symmetric space when μ fixes at least one point of M. This characterization will be especially useful later for proving weak symmetry of hermitian symmetric spaces. Its proof uses partly the same argument as that in [BPV], where weak symmetry is shown to be equivalent to ray-symmetry.

Lemma 1. *Assume that M is connected and μ has a fixed point z_o. Define K to be the isotropy subgroup of G at z_o and $T_{z_o}(M)$ to be the tangent space of M at z_o. Then M is weakly symmetric with respect to G and μ if and only if given any tangent vector $v \in T_{z_o}(M)$, there exists an element $k \in K$ such that $d(k \circ \mu)_{z_o}(v) = -v$.*

Proof. Let M be weakly symmetric with respect to G and μ. According to [BPV], Proposition 4.4, this is equivalent to M being ray-symmetric and implies that if γ is a geodesic passing through z_o with tangent vector v at z_o, then there exists $k \in G$ such that $s = k \circ \mu$ is the nontrivial involution on γ which fixes z_o and reverses the direction of γ, i.e. $d(k \circ \mu)_{z_o}(v) = -v$. It follows that k fixes z_o since μ does and hence, $k \in K$.

In the other direction, let $x, y \in M$. Define γ to be a geodesic connecting x and y with midpoint m so that $x = \gamma(t)$ and $y = \gamma(-t)$ for some real value t. We next choose an element $h \in G$ which maps m to z_o and γ, x and y to γ_h, x_h and y_h, respectively. If we set $v = \dot{\gamma}(0)$, then there exists by assumption an element $k \in K$ such that $d(k \circ \mu)_{z_o}(v) = -v$. As an isometry maps geodesics to geodesics, this implies that $k \circ \mu(\gamma_h) = \gamma_h$. In other words, $k \circ \mu$ reverses γ so that $(k \circ \mu)(x_h) = y_h$ and $(k \circ \mu)(y_h) = x_h$. As μ normalizes G and $\mu^2 \in G$, the element $g = \mu h^{-1} k \mu h = \mu h^{-1} k \mu^{-1} \mu^2 h$ lies inside G. It can now be verified that the conditions $gx = \mu y$ and $gy = \mu x$ are satisfied. □ □

2. **Hermitian symmetric spaces.** The following treatment, including most of our notation, is taken from [HW] and [He]. Let G be a connected, simply-connected, real semisimple Lie group of Hermitian type. Denote by Z_G the center of G. Fix a Cartan involution σ of G. The fixed point set $K = G^\sigma$ contains the center Z_G of G and K/Z_G is a maximal compact subgroup of G/Z_G. The space $D = G/K$ is a hermitian symmetric space with noncompact irreducible factors. If $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebras of G and K, respectively, then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition under σ. Now, $\mathfrak{k} = \mathfrak{k}_s \oplus \mathfrak{z}_{\mathfrak{k}}$ where $\mathfrak{k}_s = [\mathfrak{k}, \mathfrak{k}]$ is the semisimple part of $\mathfrak{k}$, and $\mathfrak{z}_{\mathfrak{k}}$ is its center. Let K_s denote the connected closed subgroup of K with Lie algebra $\mathfrak{k}_s$ and Z_K denote the center of K. Then K_s is compact, simply-connected, and normal in K. The subgroup $Z_K \cong E \times Z_K^0$, where E is a finite abelian subgroup and the vector subgroup Z_K^0 denotes the identity component of Z_K, and $K = K_s \cdot Z_K^0$.

Let $\pi : G \to G/K$ denote the natural projection map and view the action of an element $g \in G$ on $x = hK \in D$ as left translation on G, denoted by $\tau(g)$:

$$gx = \tau(g)hK. \tag{1}$$

Then under π, we can identify $\mathfrak{p}$ with $T_o(D)$, the tangent space of D at the origin $o = eK$ (e is the identity element in G) and the adjoint action of K on $\mathfrak{p}$ becomes the isotropy representation of K on $T_o(D)$.

Choose $\mathfrak{a}$ to be a maximal abelian subalgebra of $\mathfrak{p}$ and lift this picture to the level of Lie groups so that A is the subgroup of P with Lie algebra $\mathfrak{a}$. Then the following result of M. Flensted-Jensen gives a special involution corresponding to complex conjugation of D that will be crucial in constructing new examples of weakly symmetric spaces. We refer the reader to [F] for its proof.

Proposition 2. *(*[F]*, Prop. 2.1) There exists an involution θ of G such that*

(i) $\sigma\theta = \theta\sigma$,
(ii) $\theta(a) = a^{-1}$ *for all* $a \in A$,
(iii) $\theta(K_s) = K_s$ *and* $\theta(c) = c^{-1}$ *for all* $c \in Z_K^0$,

or equivalently,

(i′) $d\sigma d\theta = d\theta d\sigma$,
(ii′) $d\theta(H) = -H$ *for all* $H \in \mathfrak{a}$,
(iii′) $d\theta(\mathfrak{k}_s) = \mathfrak{k}_s$ *and* $d\theta(X) = -X$ *for all* $X \in \mathfrak{z}_{\mathfrak{k}}$.

Next, through π, we can push θ down to an involution μ of D defined by

(2) $$\mu(gK) = \theta(g)K, \qquad gK \in D.$$

It is clear that μ is an isometry with respect to any metric on D that is invariant under G (acting by left translation).

Proposition 3. *D is weakly symmetric with respect to G and μ.*

Proof. Under π, we identify $T_o(D)$ with $\mathfrak{p}$ so that if $v \in T_o(D)$, then $X \in \mathfrak{p}$ is the corresponding element satisfying $d\pi_e(X) = v$. Also under this correspondence, the isotropy action of K on $T_o(D)$ becomes the adjoint action of K on $\mathfrak{p}$. It is well-known (cf. [He], Ch. V, Lemma 6.3) that this adjoint action satisfies

$$Ad(K)(\mathfrak{a}) = \mathfrak{p}.$$

Furthermore, the Cartan involution σ satisfies $d\sigma_e(X) = -X$ for all $X \in \mathfrak{p}$ and it commutes with $Ad(K)$.

To prove the lemma, write X as $Ad(h)(H)$ for some $H \in \mathfrak{a}$ and $h \in K$. By definition of the adjoint action, we have $d\theta_e(X) = Ad(\theta(h))(-H)$. Define $k = h\theta(h^{-1}) \in K$. It then follows that

$$(Ad(k) \circ d\theta_e)(X) = Ad(h\theta(h^{-1})\theta(h))(-H) = -Ad(h)(H) = -X.$$

Expressing this result now in terms of the isotropy representation gives $d(k \circ \mu)_o(v) = -v$. Lastly, as D is connected and $\mu^2 = 1$ and $\mu G \mu^{-1} = G$, we appeal to Lemma 1 to conclude that D is weakly symmetric with respect to G and μ. □ □

Principal fiber bundles. Consider the principal fiber bundle $G/K_s \to G/K$ with fiber K/K_s. We intend to prove that $M = G/K_s$ is weakly symmetric with respect to certain extensions G^1 of G and $\tilde{\mu}$ of μ. The reason for the need to enlarge the isometry group G is that its isotropy subgroups in general do not contain enough elements to reverse tangent vectors on M. As Z_K^0 normalizes K_s (in fact Z_K^0 commutes with K_s), there is a standard construction for extending the left action of G on M to an action of $G^1 = G \times Z_K^0$, the direct product of G and Z_K^0, where Z_K^0 acts on the right (cf. [O], Sect. 4.4):

$$(g_0, c)(gK_s) = g_0 g c K_s, \quad (g_0, c) \in G^1,\ gK_s \in M.$$

It can be easily checked that the isotropy subgroup of G^1 at the base point $\tilde{o} = eK_s$ under this action is given by $K^1 = \{\tilde{k} = (k, c^{-1}) : k \in K\}$, where we have the decomposition $k = k'c$ with $k' \in K_s$ and $c \in Z_K^0$.

Lemma 4. $M \cong G^1/K^1$.

Next, since $\theta(K_s) = K_s$, this induces an isometry $\tilde{\mu}$ of M defined as follows:

$$\tilde{\mu}(gK_s) = \theta(g)K_s, \quad gK_s \in M.$$

We are now ready to state our main result.

Theorem 5. *(Main result) M is weakly symmetric with respect to G^1 and $\tilde{\mu}$.*

Proof. The argument is a straightforward extension of that used in the proof of Theorem 3. Again, consider the natural projection $\tilde{\pi} : G \to G/K_s$. As usual, we identify $T_{\tilde{o}}(M)$ with $\mathfrak{p}^1 = \mathfrak{p} + \mathfrak{z}$ through $\tilde{\pi}$. Correspondingly, the isotropy action of K^1 on $T_{\tilde{o}}(M)$ becomes the action of $Ad_{G^1}(K^1)$ on $\mathfrak{p}^1$. Furthermore, $Ad_{G^1}(K^1)$ agrees precisely with the action of $Ad_G(K)$ on $\mathfrak{p}$ and is trivial on $\mathfrak{z}$.

Let $\tilde{v} \in T_{\tilde{o}}(M)$ and write $\tilde{X} = X + Z$ as the corresponding element in $\mathfrak{p}^1$, where $X \in \mathfrak{p}$ and $Z \in \mathfrak{z}$. It is clear that $d\theta_e(\tilde{X}) = d\theta_e(X) - Z$. Now, as D is weakly symmetric with respect to G and μ, there exists an element $k \in K$ such that $Ad(k)(-X) = d\theta_e(X)$. Write $k = k'c$ with $k' \in K_s$ and $c \in Z_K^0$ and define $\tilde{k} = (k^{-1}, c)$ to be an element of K^1. It follows that

$$(Ad_{G^1}(\tilde{k}) \circ d\theta_e)(\tilde{X}) = Ad_{G^1}(\tilde{k})(Ad_G(k)(-X) - W) = -X - W = -\tilde{X}.$$

In terms of the isotropy representation, this translates to $d(\tilde{k} \circ \tilde{\mu})_{\tilde{o}}(\tilde{v}) = -\tilde{v}$. Of course, it can be checked that $\tilde{\mu} G^1 \tilde{\mu}^{-1} = G^1$, $\tilde{\mu}^2 \in G^1$ and $\tilde{\mu}$ fixes $\tilde{o}$. As M is connected, this proves that it is weakly symmetric with respect to G^1 and $\tilde{\mu}$ by Lemma 1. □ □

Let $Q \subset Z_K^0$ be a cyclic subgroup and consider the quotient $M_Q = G/QK_s$. Define $G_Q^1 = G^1/(Q \times Q^{-1})$ and $K_Q^1 = K^1/(Q \times Q^{-1})$, where $Q \times Q^{-1} = \{(q, q^{-1}) \in G^1 : q \in Q\}$. Then $M_Q \cong G_Q^1/K_Q^1$. Furthermore, it is well-known that D is contractible. Therefore, $M_Q \to D$ is a trivial bundle and hence, $M_Q = D \times S^1$ if D is irreducible. Now, pushing $\tilde{\mu}$ down to an involution $\tilde{\mu}_Q$ of M_Q, we get the following result as an easy consequence.

Corollary 6. *M_Q is weakly symmetric with respect to G^1 and $\tilde{\mu}_Q$. Furthermore, if $Q \subset Z_K^0 \cap Z_G$, then M_Q is weakly symmetric with respect to G_Q^1 and $\tilde{\mu}_Q$.*

Compact case. We mention that our arguments work for the compact case as well. Let $D = G/K$ be a compact hermitian symmetric space. Then G can be chosen to be a connected, simply-connected, simple Lie group and K to be the centralizer of its connected center Z_K^0. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}^*$ be the Cartan decomposition of the Lie algebra of G and fix $\mathfrak{a}^*$ to be a maximal abelian subspace of $\mathfrak{p}$. Now replace $\mathfrak{a}$ and $\mathfrak{p}$ by $\mathfrak{a}^*$ and $\mathfrak{p}^*$, respectively, in the discussion of this section to obtain dual results for the compact case, namely weak symmetry of D and M as presented in Prop. 3 and Theorem 5. This is because Prop. 2 again provides an involution θ of G whose differential preserves $\mathfrak{k}_s$ and reverses elements in $\mathfrak{a}^*$ and $\mathfrak{z}_{\mathfrak{k}}$. Furthermore, the property $Ad(K)(\mathfrak{a}^*) = \mathfrak{p}^*$ continues to hold. Hence, the arguments in the compact case now proceed along the same lines, in fact word for word, as in the noncompact case.

3. **Bounded symmetric domains.** In this section, we shall assume that D is a classical irreducible bounded symmetric domain and Q chosen so that $G_Q \cong G_{\mathbb{R}}$, a simple matrix Lie group. We present results about factors of automorphy, determinant factors of automorphy and kernel functions that will be needed in the following sections to explicitly construct M_Q and describe a certain one-parameter family of Riemannian metrics on M_Q invariant under G_Q.

Preliminaries. Let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of $\mathfrak{g}$. Extend σ to $\mathfrak{g}_{\mathbb{C}}$ so that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_{\mathbb{C}}$. If $\mathfrak{h}$ is a Cartan of subalgebra of $\mathfrak{k}$, then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and its complexification $\mathfrak{h}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Let Φ be root system for $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Then $\mathfrak{g}_{\mathbb{C}}$ has the root decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} + \sum_{\alpha\in\Phi} \mathfrak{g}_\alpha$. Decompose $\Phi = \Phi_c \cup \Phi_n$ into subsets of compact and noncompact roots. Choose a root ordering for Φ so that Φ^+ and Φ^- are the set of positive and negative roots, respectively. This allows us to write $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ + \mathfrak{p}_-$, where $\mathfrak{p}_+$ (resp. $\mathfrak{p}_-$) is the holomorphic (resp. antiholomorphic) tangent space. Furthermore, let Z_0 be the element in the center of $\mathfrak{k}$ which defines this complex structure. Set $\Phi_c^\pm = \Phi^\pm \cap \Phi_c$ and $\Phi_n^\pm = \Phi^\pm \cap \Phi_n$.

Let $G_{\mathbb{C}}$ be the connected simply connected Lie group for $\mathfrak{g}_{\mathbb{C}}$. Denote by $G_{\mathbb{R}}$, $K_{\mathbb{R}}$, $K_{\mathbb{C}}$, P_+ and P_- to be analytic subgroups of $G_{\mathbb{C}}$ corresponding to $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{k}_{\mathbb{C}}$, $\mathfrak{p}_+$ and $\mathfrak{p}_-$. Let $q : G \to G_{\mathbb{R}}$ be the projection map. Then $K = q^{-1}(K_{\mathbb{R}})$. If $x \in P_+K_{\mathbb{R}}P_-$, then we write its $P_+K_{\mathbb{R}}P_-$ decomposition as $x = p_+(x) \cdot \kappa_o(x) \cdot p_-(x)$. This allows us to define the map $\zeta : P_+K_{\mathbb{C}}P_- \to \mathfrak{p}_+$ by requiring that $p_\pm(x) = \exp\zeta(x)$, where $\exp : \mathfrak{p}_+ \to P_+$ is the exponential map. Since $G_{\mathbb{R}} \subset P_+K_{\mathbb{C}}P_- \subset G_{\mathbb{C}}$, we write $g \in \exp z \cdot K_{\mathbb{C}}P_-$ so that the restriction map $\zeta : G_{\mathbb{R}} \to \mathfrak{p}_+$ given by $g \mapsto z$ gives the Harish-Chandra embedding of $G_{\mathbb{R}}/K_{\mathbb{R}}$ onto a bounded domain D in the complex vector space $\mathfrak{p}_+$.

We now lift this picture up to G. The embedding $G_{\mathbb{R}} \subset P_+K_{\mathbb{C}}P_-$ lifts to a corresponding decomposition map $G \to P_+\widetilde{K}_{\mathbb{C}}P_-$ by lifting $p_\pm$ and κ_o to G via the universal covering $q : G \to G_{\mathbb{R}}$. This gives $G/K \cong G_{\mathbb{R}}/K_{\mathbb{R}}$. Now, let $q_K : \widetilde{K}_{\mathbb{C}} \to K_{\mathbb{C}}$ be the universal covering group. Then $\widetilde{K}_{\mathbb{C}}$ can be thought of as the complexification of K and $q_K|_K = q|_K$. As a result, $\kappa_o : G \to K_{\mathbb{C}}$ lifts to $\tilde{\kappa}_o : G \to \widetilde{K}_{\mathbb{C}}$ such that $\tilde{\kappa}_o|_K : K \hookrightarrow \widetilde{K}_{\mathbb{C}}$. This gives the embedding $G \to P_+\widetilde{K}_{\mathbb{C}}P_-$ as defined in [HW].

The picture can be made explicit when we choose $G_{\mathbb{R}}$ to be a matrix Lie group. Since our results in this section pertain only to the case where $G_{\mathbb{R}}$ is of classical type, we shall always make this assumption when referring to $G_{\mathbb{R}}$. There are four families of classical bounded symmetric domains and we list them according to type [He]:

Type	$G_{\mathbb{R}}/K_{\mathbb{R}}$	$D \subset \mathfrak{p}_+$
I.	$\mathbf{SU}(m,n)/S(\mathbf{U}(m) \times \mathbf{U}(n))$	$\{Z \in \mathbf{M}_{mn}(\mathbb{C}) : I - Z^*Z \gg 0\}$
II.	$\mathbf{Sp}(n,\mathbb{C}) \cap \mathbf{U}(n,n)/\mathbf{U}(n)$	$\{Z \in \mathbf{M}_{nn}(\mathbb{C}) : Z^t = Z$ and $I - Z^*Z \gg 0\}$
III.	$\mathbf{SO}^*(2n,\mathbb{C})/\mathbf{U}(n)$	$\{Z \in \mathbf{M}_{nn}(\mathbb{C}) : Z^t = -Z$ and $I - Z^*Z \gg 0\}$
IV.	$\mathbf{SO}_o(n,2)/(\mathbf{SO}(n) \times \mathbf{SO}(2))$	$\{Z \in \mathbf{M}_{n1}(\mathbb{C}) : 1 - Z^*Z > 0$ and $1 + \lvert{}^tZZ\rvert^2 - 2Z^*Z > 0\}$

Factors of automorphy. Recall that the action of G and $G_{\mathbb{R}}$ on D agree (via q). If $g \in G$, then we write its projection to $G_{\mathbb{R}}$ via q as $q(g) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. The function κ_o defined for G and $G_{\mathbb{R}}$ above is called the factor of automorphy and $\tilde{\kappa}_o$ the universal factor of automorphy. We extend κ_o to a map $\kappa : G_{\mathbb{R}} \times D \to K_{\mathbb{C}}$ satisfying the following relation [2]:

$$g \cdot \exp z \in \exp gz \cdot \kappa(g, z) \cdot P_-, \qquad g \in G_{\mathbb{R}}, z \in D. \tag{3}$$

Let $o \in D$ be the coset element $eK_{\mathbb{R}}$. Then κ has the following properties:

[2]The universal factor of automorphy can be extended similarly to $\tilde{\kappa} : G \times D \to \widetilde{K}_{\mathbb{C}}$ (due to Tirao [Ti], p. 64).

Lemma 7. *([Sa], p. 68)*
i) $\kappa(g,o)=\kappa_o(g)$, *for all* $g\in G_{\mathbb{R}}$.
ii) $\kappa(k,z)=k$, *for all* $k\in K_{\mathbb{C}}$ *and* $z\in D$.
iii) $\kappa(g_1g_2,z)=\kappa(g_1,g_2z)\kappa(g_2,z)$, *for all* $g_1,g_2\in G_{\mathbb{R}}$ *and* $z\in D$.

Proposition 8. *The factor of automorphy has the expression*

Types I-III: $\kappa(g,Z)=\begin{pmatrix} A-(gZ)C & 0 \\ 0 & CZ+D\end{pmatrix}$, $gZ=\dfrac{AZ+B}{CZ+D}$

Type IV: $\kappa(g,Z)=\begin{pmatrix} U & 0 \\ 0 & V\end{pmatrix}$, *with* U *and* V *given as follows:*

Express

$$gZ=\frac{1}{(1,i)(CZ_1+DZ_2)}(AZ_1+BZ_2)$$

with $Z_1=2iZ$ *and* $Z_2=\begin{pmatrix} 1+\ {}^tZZ \\ i-i{}^tZZ\end{pmatrix}$. *Then*

$$\begin{aligned} U&=A-B{}^tZ'_+ +(gZ)[CZ'_+ +D(I+\tfrac12 Z''_+)]{}^tW,\\ V&=(I+\tfrac12(gZ)''_+)\{{}^t(gZ)'_+[AZ'_+ +B(I+\tfrac12 Z''_+)]\\ &\quad+[CZ'_+ +D(I+\tfrac12 Z''_+)]\}(I-\tfrac12 W''_-), \end{aligned}$$

where $W=-\dfrac{1}{2iv}\,{}^t(C-D{}^tZ'_+)\begin{pmatrix} i \\ 1\end{pmatrix}$, *as given below in (5), (6).*

The proof really boils down to finding the $K_{\mathbb{C}}$ part of the $P_+K_{\mathbb{C}}P_-$-decomposition of $G_{\mathbb{R}}$. For Types I-III, this is rather easy and has been done (cf. [HW], p. 5). On the other hand, the Type IV case requires quite a bit more wrangling. Since the author has failed to find a reference for this decomposition (and probably for good reason because the calculations get very messy), details will be given below.

Lemma 9. *Let* $G_{\mathbb{R}}=\mathbf{SO}_o(n,2)$. *Then the* $P_+K_{\mathbb{C}}P_-$*-decomposition of* $g\in G_{\mathbb{R}}$ *is*

$$g=\begin{pmatrix} A & B \\ C & D\end{pmatrix}=\begin{pmatrix} I & Z'_+ \\ -{}^tZ'_+ & I+\frac12 Z''_+\end{pmatrix}\begin{pmatrix} U & 0 \\ 0 & V\end{pmatrix}\begin{pmatrix} I & W'_- \\ -{}^tW'_- & I+\frac12 W''_-\end{pmatrix},$$

where

$$\begin{aligned} &Z'_+=(iZ,Z),\quad Z=\tfrac{1}{2id}B\begin{pmatrix} 1 \\ i\end{pmatrix}\quad \text{with}\quad d=\tfrac{1}{2i}\begin{pmatrix} i & 1\end{pmatrix}D\begin{pmatrix} 1 \\ i\end{pmatrix},\\ &W'_-=(iW,-W),\quad W=-\tfrac{1}{2id}{}^tC\begin{pmatrix} i \\ 1\end{pmatrix},\\ &\begin{pmatrix} U & 0 \\ 0 & V\end{pmatrix}=\begin{pmatrix} A+Z'_+D{}^tW'_- & 0 \\ 0 & (I+\frac12 Z''_+)({}^tZ'_+B+D)(I-\frac12 W''_-)\end{pmatrix}. \end{aligned}$$

Before proving Lemma 9, we shall first need some preliminaries. Let $G=\mathbf{SO}_o(n,2)$. Then according to [W],

$$\mathfrak{p}_+=\left\{\widetilde{Z}=\begin{pmatrix} 0 & Z'_+ \\ -{}^tZ'_+ & 0\end{pmatrix}: Z'_+=(iZ,Z) \text{ where } Z \text{ is } n\times 1\right\},$$

and the embedding of $Z\in D\subset\mathfrak{p}_+$ is given naturally as $Z\mapsto\widetilde{Z}$. Similarly,

$$\mathfrak{p}_-=\widetilde{W}=\left\{\begin{pmatrix} 0 & W'_- \\ -{}^tW'_- & 0\end{pmatrix}: W'_-=(iW,-W) \text{ where } W \text{ is } n\times 1\right\}.$$

To obtain P_+ and P_- now requires exponentiating $\mathfrak{p}_+$ and $\mathfrak{p}_-$, respectively. Use the fact that

$$\widetilde{Z}\widetilde{Z} = \begin{pmatrix} 0 & 0 \\ 0 & Z''_+ \end{pmatrix} \text{ where } Z''_+ = \begin{pmatrix} {}^tZZ & -i{}^tZZ \\ -i{}^tZZ & -{}^tZZ \end{pmatrix},$$

and $(\widetilde{Z})^k = 0$ for $k \geq 3$ to compute $\exp \widetilde{Z} = \sum_{k=0}^{\infty} \frac{1}{k!}(\widetilde{Z})^k$. This gives

$$P_+ = \left\{ \exp \widetilde{Z} = \begin{pmatrix} I & Z'_+ \\ -{}^tZ'_+ & I + \frac{1}{2}Z''_+ \end{pmatrix} : \widetilde{Z} \in \mathfrak{p}_+ \right\}$$

A similar calculation shows that

$$P_- = \left\{ \exp \widetilde{W} = \begin{pmatrix} I & W'_- \\ -{}^tW'_- & I + \frac{1}{2}W''_- \end{pmatrix} : \widetilde{W} \in \mathfrak{p}_- \right\},$$

where W''_- now takes the form $W''_- = \begin{pmatrix} {}^tWW & i{}^tWW \\ i{}^tWW & -{}^tWW \end{pmatrix}$. Furthermore, $K_{\mathbb{C}}$ has the form

$$K_{\mathbb{C}} = \left\{ \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} : U \in \mathbf{SO}(n), V \in \mathbf{SO}(2) \right\}.$$

Proof of Lemma 9. Multiplying the three matrices on the right hand side above together and comparing entries leads to the set of equations

$$\begin{aligned} U - Z'_+ V {}^tW'_- &= A, \\ -{}^tZ'_+ U - (I + \tfrac{1}{2}Z''_+) V {}^tW'_- &= C, \\ UW'_- + Z'_+ V (I + \tfrac{1}{2}W''_-) &= B, \\ -{}^tZUW'_- + (I + \tfrac{1}{2}Z''_+) V (I + \tfrac{1}{2}W''_+) &= D. \end{aligned} \tag{4}$$

The last two equations above give $V = (I - \frac{1}{2}Z''_+)^{-1}({}^tZ'_+B + D)(I + \frac{1}{2}W''_-)^{-1}$. It can be checked that $\det(I - \frac{1}{2}Z''_+) = \det(I - \frac{1}{2}W''_-) = 1$ so that the inverses above makes sense. In fact, we easily see that

$$(I - \tfrac{1}{2}Z''_+)^{-1} = I + \tfrac{1}{2}Z''_+, \quad (I + \tfrac{1}{2}W''_-)^{-1} = I - \tfrac{1}{2}W''_-,$$

and so

$$V = (I + \tfrac{1}{2}Z''_+)({}^tZ'_+B + D)(I - \tfrac{1}{2}W''_-).$$

Now use this expression of V to obtain $U = A + Z'_+ D {}^tW'_-$, which follows from recognizing that $Z'_+ {}^tZ'_+ = W'_- {}^tW'_- = 0$.

It remains to compute what Z'_+ is in terms of g. This requires noticing that the map

$$\Theta : V \mapsto v = \frac{1}{2i} \begin{pmatrix} i & 1 \end{pmatrix} V \begin{pmatrix} 1 \\ i \end{pmatrix}$$

identifies $\mathbf{SO}(2,\mathbb{C})$ with $\mathbf{GL}(1) = \mathbb{C}^*$, and so v is never 0. This is because $\mathbf{SO}(2,\mathbb{C})$ consists of the matrices

$$\mathbf{SO}(2,\mathbb{C}) = \left\{ V = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{C} \right\},$$

and it is easy to check that $\Theta(V) = e^{\theta}$. We then use our expression for V above to obtain

$$\Theta(V) = v = \frac{1}{2i} \begin{pmatrix} i & 1 \end{pmatrix} V \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2i} \begin{pmatrix} i & 1 \end{pmatrix} D \begin{pmatrix} 1 \\ i \end{pmatrix} = d,$$

where we have used the relations $\begin{pmatrix} i & 1 \end{pmatrix} Z''_+ = 0$, $W''_- \begin{pmatrix} 1 \\ i \end{pmatrix} = 0$. Now, the equations in (4) involving B give

$$d \cdot Z = \frac{1}{2i} Z \begin{pmatrix} i & 1 \end{pmatrix} V \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2i} Z'_+ V \begin{pmatrix} 1 & i \end{pmatrix} = \frac{1}{2i} B \begin{pmatrix} 1 & i \end{pmatrix},$$

from which we obtain the desired expression for Z after dividing by d. A similar argument can be used to prove that

$$W = -\frac{1}{2id}\, {}^tC \begin{pmatrix} i \\ 1 \end{pmatrix}$$

and will be left for the reader. This completes the lemma. □ □

Proof of Proposition 8 for Type IV. We have

$$\begin{aligned} g \exp \widetilde{Z} &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & Z'_+ \\ -{}^tZ'_+ & I + \frac{1}{2} Z''_+ \end{pmatrix} \\ &= \begin{pmatrix} A - B{}^tZ'_+ & AZ'_+ + B(I + \frac{1}{2} Z''_+) \\ C - D{}^tZ'_+ & CZ'_+ + D(I + \frac{1}{2} Z''_+) \end{pmatrix}. \end{aligned}$$

We use our formula for the $P_+ K_{\mathbb{C}} P_-$ decomposition of $G_{\mathbb{R}} = \mathbf{SO}(n, 2)$ given in Lemma 9 to get the $K_{\mathbb{C}}$-component of $g \exp \widetilde{Z}$, which is precisely the factor of automorphy $\kappa(g, Z) = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$. First, by rewriting

$$CZ'_+ + D(I + \frac{1}{2} Z''_+) = \frac{1}{2} \begin{pmatrix} CZ_1 + CZ_2 \end{pmatrix} \begin{pmatrix} 1 & -i \end{pmatrix} + \frac{1}{2} D \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix},$$

we get

$$v = \Theta(V) = \frac{1}{2i} \begin{pmatrix} i & 1 \end{pmatrix} [CZ'_+ + D(I + \frac{1}{2} Z''_+)] \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2i} \begin{pmatrix} i & 1 \end{pmatrix} (CZ_1 + DZ_2).$$

Then a short calculation shows that we get the correct answer for g acting on Z:

$$gZ = \frac{1}{2iv} [AZ'_+ + B(I + \frac{1}{2} Z''_+] \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\begin{pmatrix} i & 1 \end{pmatrix} (CZ_1 + DZ_2)} (AZ_1 + BZ_2).$$

Denoting by $W = -\frac{1}{2iv} {}^t(C - D{}^tZ'_+) \begin{pmatrix} i \\ 1 \end{pmatrix}$, we can finally write the expression for V as

$$\begin{aligned} V = (I + \tfrac{1}{2}(gZ)''_+)\{{}^t(gZ)'[AZ'_+ + B(I + \tfrac{1}{2} Z''_+)] \\ + [CZ'_+ + D(I + \tfrac{1}{2} Z''_+)]\}(I - \tfrac{1}{2} W''_-), \end{aligned} \tag{5}$$

and the relation $(gZ)''_+ {}^t(gZ)' = 0$ is used to obtain a simplified answer for U:

$$U = A - B{}^tZ'_+ + (gZ)[CZ'_+ + D(I + \tfrac{1}{2} Z''_+)]{}^tW. \tag{6}$$

This completes the proof. □ □

Kernel functions. Our next goal will be to define a certain kernel function K for the classical domains. Now, we can view $K_{\mathbb{C}}$ as consisting of diagonal matrices $k = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$ with U and V submatrices of appropriate sizes and $\det(U)\det(V) = 1$, where we label $k^+ = U$ to designate the upper left diagonal part of k and $k^- = V$ for the lower right diagonal part. Then, following [Sa], we define the matrix kernel function $\mathcal{K} : D \times D \to K_{\mathbb{C}}$:

$$\mathcal{K}(z, w) = \kappa_o((\exp \overline{z})^{-1}(\exp w))^{-1},$$

where $\overline{z}$ represents conjugation with respect to the complex structure on $\mathfrak{p}$, so that elements of $\mathfrak{p}_+$ are sent to $\mathfrak{p}_-$ (and vice versa). For the classical domains, this means that if the element $z \in D \subset \mathfrak{p}_+$ is viewed as a matrix, then the complex structure is complex conjugate transpose of matrices:

$$z \leftrightarrow \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix}, \quad \overline{z} \leftrightarrow \begin{pmatrix} 0 & 0 \\ Z^* & 0 \end{pmatrix}.$$

Then the kernel function $K : D \times D \to \mathbb{C}^*$ is defined as:

$$K(z, w) = \begin{cases} \det(\mathcal{K}(z, w)^-), & \text{Types I} - \text{III} \\ \Theta(\mathcal{K}(z, w)^-), & \text{Type IV} \end{cases}$$

Proposition 10. *The kernel function K for the classical domains is given as*

Type I: $K(Z, W) = \det(I - Z^*W)^{-1}$.
Type II: $K(Z, W) = \det(I - \overline{Z}W)^{-1}$.
Type III: $K(Z, W) = \det(I + \overline{Z}W)^{-1}$.
Type IV: $K(Z, W) = (1 + {}^t\overline{Z}\overline{Z}\,{}^tWW - 2Z^*W)^{-1}$.

Proof. For Types I, II, III, just multiply the following matrices

$$(\exp Z^*)^{-1} \exp W = \begin{pmatrix} I & 0 \\ -Z^* & 0 \end{pmatrix} \begin{pmatrix} I & W \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & W \\ -Z^* & I - Z^*W \end{pmatrix}$$

to easily see that

$$\mathcal{K}(Z, W) = \begin{pmatrix} (I + W(I - Z^*W)^{-1}Z^*)^{-1} & 0 \\ 0 & (I - Z^*W)^{-1} \end{pmatrix},$$

and so $K(Z, W) = \det(I - Z^*W)^{-1}$. Now use the fact that ${}^tZ = Z$ for Type II and ${}^tZ = -Z$ for Type III to get the right expressions for K in these cases.

For Type IV, it can be checked that

$$\begin{aligned}(\exp (\widetilde{Z}')^*)^{-1} \exp \widetilde{W} &= \\ &= \begin{pmatrix} I & -\overline{Z}'_- \\ {}^t\overline{Z}'_- & I + \frac{1}{2}\overline{Z}''_- \end{pmatrix} \begin{pmatrix} I & W'_+ \\ -{}^tW'_+ & I + \frac{1}{2}W''_+ \end{pmatrix} \\ &= \begin{pmatrix} I + \overline{Z}'_-\,{}^tW'_+ & W'_+ - \overline{Z}'_-(I + \frac{1}{2}W''_+) \\ {}^t\overline{Z}'_- - (I + \frac{1}{2}\overline{Z}''_-)\,{}^tW'_+ & {}^t\overline{Z}'W'_+ + (I + \frac{1}{2}\overline{Z}'')(I + \frac{1}{2}W''_+) \end{pmatrix}.\end{aligned}$$

Now, notice that

$$\kappa_o \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \Longrightarrow \Theta(V) = \frac{1}{2i} \begin{pmatrix} i & 1 \end{pmatrix} D \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

It follows from the two identities

$$\overline{Z}''_- \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} i & 1 \end{pmatrix} W''_+ = 0, \quad \begin{pmatrix} i & 1 \end{pmatrix} \overline{Z}''_- W''_+ \begin{pmatrix} 1 \\ i \end{pmatrix} = 8i\,{}^t\overline{Z}\overline{Z}\,{}^tWW,$$

that

$$\begin{aligned}\Theta(V) &= \tfrac{1}{2i}\begin{pmatrix} i & 1\end{pmatrix}[{}^t\overline{Z}'_-W'_+ + (I+\tfrac{1}{2}\overline{Z}''_-)(I+\tfrac{1}{2}W''_+)]\begin{pmatrix}1\\ i\end{pmatrix}\\ &= 1+\overline{{}^tZZ}{}^tWW-2Z^*W.\end{aligned}$$

Now use the definition of $K(Z,W)=\Theta(V^{-1})=\Theta(V)^{-1}$ to obtained the desired expression for K. □ □

Determinant factors of automorphy. We will also need to define the determinant factor of automorphy $j: G_{\mathbb{R}}\times D\to S^1$ associated to the factor of automorphy:

$$j(g,z)=\begin{cases}\det(\kappa(g,z)^-)^{-1}, & \text{Types I -- III}\\ \Theta(\kappa(g,z)^-)^{-1}, & \text{Type IV}\end{cases} \tag{7}$$

Lemma 11. *The kernel function K enjoys the following properties:*
i) $K(z,w)=\overline{K(w,z)}$ and $K(z,z)>0$.
ii) $K(gz,gw)=\overline{j(g,z)}^{-1}K(z,w)j(g,w)^{-1}$

Proof. Property i) is obvious. Property ii) for Types I-III follows first from the equivalence of the two equations

$$\begin{gathered}(gZ)^*I_{m,n}(gW)=Z^*I_{m,n}W,\\ (AZ+B)^*(AW+B)-(CZ+D)^*(CW+D)=Z^*W,\end{gathered}$$

and the relations

$$\begin{gathered}I-(gZ)^*(gW)=(CZ+D)^{*-1}(I-Z^*W)(CW+D)^{-1},\\ j(g,Z)=(CZ+D)^{-1}.\end{gathered}$$

The identity for Type IV requires a much more complicated expression. To simplify the notation, recall the definition of Z_1 and Z_2 earlier in Proposition 8 and denote

$$\hat{Z}=\begin{pmatrix}Z_1\\ Z_2\end{pmatrix},\quad U=AZ_1+BZ_2,\quad V=\begin{pmatrix}CZ_1+DZ_2\end{pmatrix}.$$

We then define $g\hat{Z}$ as ordinary matrix multiplication to obtain

$$g\hat{Z}=\begin{pmatrix}U\\ V\end{pmatrix},\quad gZ=Uv^{-1},\quad v=\begin{pmatrix}i & 1\end{pmatrix}V.$$

In what follows, we use the letters z and w as subscripts to distinguish the terms U and V associated to each of the two different elements Z and W as defined immediately above. First, some preliminary identities will be needed:

$$\begin{aligned}\hat{Z}^*g^*I_{n,2}\,g\hat{W}=\hat{Z}^*\hat{W} &\iff U_z^*U_w-V_z^*V_w=Z_1^*W_1-Z_2^*W_2,\\ {}^t\hat{Z}\,{}^tgg\hat{Z}={}^t\hat{Z}\hat{Z}=0 &\iff {}^tUU+{}^tVV=0.\end{aligned}$$

Also, the following three equalities can be easily verified:

$$\begin{aligned}&Z_1W_1+Z_2W_2=4(1+\overline{{}^tZZ}\,{}^tWW-2Z^*W),\\ &(v_z^*v_w)^2-2v_z^*V_z^*V_wv_w=-v_z^*V_z^*\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}V_wv_w,\\ &\overline{{}^tV_zV_z}\,{}^tV_wV_w=v_z^*V_z^*\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}V_wv_w.\end{aligned}$$

This gives $(v_z^* v_w)^2 - 2v_z^* V_z^* V_w v_w + \overline{{}^tV_z V_z}\,{}^tV_w V_w = 0$. Using $j(g,Z) = (v_z/2i)^{-1}$ and $j(g,W) = (q_w/2i)^{-1}$, we get

$$
\begin{aligned}
1 + &\overline{{}^t(gZ)(gZ)}\,{}^t(gW)(gW) - 2(gZ)^*(gW) = \\
&= \frac{1}{v_z^* v_w}[v_z^* v_w + \frac{1}{v_z^* v_w}(\overline{{}^tU_z U_z}\,{}^tU_w U_w) - 2U_z^* U_w] \\
&= \frac{1}{v_z^* v_w}[v_z^* v_w + \frac{1}{v_z^* v_w}(\overline{{}^tV_z V_z}\,{}^tV_w V_w) - 2(Z_1^* W_1 - Z_2^* W_2 + V_z^* V_w)] \\
&= \frac{1}{(v_z^* v_w)^2}((v_z^* v_w)^2 - 2v_z^* V_z^* V_w v_w + \overline{{}^tV_z V_z}\,{}^tV_w V_w) \\
&\quad + \tfrac{4}{v_z^* v_w}(1 + \overline{{}^tZZ}\,{}^tWW - 2Z^*W) \\
&= 0 + j(g,Z)^*(1 + \overline{{}^tZZ}\,{}^tWW - 2Z^*W)j(g,W).
\end{aligned}
$$

The transformation property of $K(gZ, gW)$ for Type IV now follows. □ □

Let $K_{\mathbb{C}}$ act on $\mathfrak{p}_+$ by the adjoint action and consider the linear map $J(g,z) = \mathrm{Ad}_{\mathfrak{p}_+}\kappa(g,z)$. It is well-known that $d(gz) = J(g,z)dz$ (cf. [Sa], Ch. II, §5), Then the jacobian of $d(gz)$ is $\det(J(g,z)) = j(g,z)^p$, where the exponent p depends on the bounded symmetric domain (see the lemma below).

Let $B(Z,W)$ be the Bergman kernel on D. Then B can be written as a constant factor (namely the volume of D) of our kernel function K raised to an appropriate exponent p (depending on D). Furthermore, the Bergman kernel gives rise to the hermitian (or Bergman) metric Ψ on D defined by the fundamental 2-form $i\partial\overline{\partial}\log B(Z,Z)$ and the Riemannian metric ω is obtained by taking the imaginary part of Ψ (cf. [Sa], II. §6 and [TW], §1):

Lemma 12. *The Bergman metric Ψ on D is given by:*

Type I: $\Psi = ip\,\mathrm{Tr}\{(I - Z^*Z)^{-1}dZ^*(I - ZZ^*)^{-1}dZ\}$.
Type II: $\Psi = ip\,\mathrm{Tr}\{(I - \overline{Z}Z)^{-1}d\overline{Z}(I - Z\overline{Z})^{-1}dZ\}$.
Type III: $\Psi = ip\,\mathrm{Tr}\{(I + \overline{Z}Z)^{-1}d\overline{Z}(I + Z\overline{Z})^{-1}dZ\}$.
Type IV: $\Psi = ip\,(1 + \overline{{}^tZZ}\,tZZ - 2Z^*Z)^{-2}(2dZ^*dZ - d(\overline{{}^tZZ})d({}^tZZ))$.

4. **An explicit construction.** In this section, we follow [M] and demonstrate how M_Q can be constructed more explicitly. Assume that D and G/Q are the same as in the previous section. Then $M_Q = D \times S^1$. Because of this fact, we begin by describing a twisted group action Φ of $G_{\mathbb{R}}^1 = G_{\mathbb{R}} \times S^1$ on $D^1 = D \times S^1$ which makes use of the determinant factor of automorphy. We then prove that M_Q is diffeomorphic to D^1 under this action. Furthermore, we go on to describe a one-parameter family of Riemannian metrics Ω on D^1 invariant under G^1. Therefore, by using this identification in the next section, we shall obtain new examples of weakly symmetric spaces.

Let $D^1 = D \times S^1$ be the product manifold of D with the unit circle S^1 parametrized by the interval $[0, 2\pi)$. If $z \in D$ and $t \in S^1$, we shall write (z,t) or z_t to denote the corresponding element of D^1. Let $G_{\mathbb{R}}^1 = G_{\mathbb{R}} \times S^1$ be the direct product of $G_{\mathbb{R}}$ (listed in section 3) and S^1 as Lie groups. We write out its elements as $g_s = (g,s)$, $g \in G$ and $s \in [0,2\pi)$ and define the multiplication as $(g_1,s_1)(g_2,s_2) = (g_1g_2, s_2 + s_1 (\mathrm{mod}\ 2\pi))$. Let $j(g,z)$ be the determinant factor of automorphy map from $G_{\mathbb{R}} \times D \to S^1$ as defined in (7).

Lemma 13. *The automorphic group action $\Phi : G_{\mathbb{R}}^1 \times D^1 \to D^1$ defined by*

$$g_s z_t = (gz, \arg j(g,z) + t + s), \qquad g_s \in G_{\mathbb{R}}^1, \quad z_t \in D^1. \tag{8}$$

is transitive with isotropy subgroup $K^1_{\mathbb{R}} = \{(k, -\arg j(k,o)) : k \in K_{\mathbb{R}}\}$ *at the point* $(o,0)$ *so that* $D^1 \cong G^1_{\mathbb{R}}/K^1_{\mathbb{R}}$. *Hence,* D^1 *is diffeomorphic to* M_Q *under the action of* Φ, *where* $G/Q \cong G_{\mathbb{R}}$.

Proof. The lemma is trivial except for transitivity of Φ. In fact, we shall prove that $G_{\mathbb{R}}$, considered as a subgroup of $G^1_{\mathbb{R}}$, is transitive on D^1. Furthermore, as $G_{\mathbb{R}}$ is already transitive on D, it suffices to prove that $G_{\mathbb{R}}$ is transitive on any slice $\{z\} \times S^1 \subset D^1$. By identifying z with the coset gK so that $gK_{\mathbb{R}}g^{-1}$ is the isotropy subgroup of $G_{\mathbb{R}}$ at z, it is clear that the orbit of $gK_{\mathbb{R}}g^{-1}$ on the S^1 factor and described by $\arg j(k,o)$, $k \in K_{\mathbb{R}}$, is transitive.

Now, it follows from the isomorphism $G_Q \cong G_{\mathbb{R}}$ that $G^1_Q \cong G^1_{\mathbb{R}}$ and $K^1_Q \cong K^1_{\mathbb{R}}$. As M_Q is diffeomorphic to G^1_Q/K^1_Q, it is then clear that D^1 is diffeomorphic to M_Q under the action of Φ. □ □

The Riemannian metric. We now give explicit expressions for a one-parameter family of $G^1_{\mathbb{R}}$-invariant metrics on D^1 for the classical domains. If z is a complex number, let $\operatorname{Re} z$ and $\operatorname{Im} z$ denote its real and imaginary parts, respectively. Also, we write $\operatorname{Tr}\{A\}$ to denote the trace of the matrix A.

Proposition 14. *Let* ω *be the unique Riemannian metric on* D *invariant under* $G_{\mathbb{R}}$. *Then the following one-parameter family of Riemannian metrics* $\Omega = \Omega_\nu$ *on* D^1 *is invariant under* $G^1_{\mathbb{R}}$:

$$\Omega_\nu = \omega + \nu(dt - \delta)^2, \qquad \nu \in \mathbb{R}^+, \tag{9}$$

where the differential one-form $\delta = \delta(Z)$ *at* $(Z,t) \in D^1$ *is expressed as*

Type I: $\delta = \operatorname{Im}\operatorname{Tr}\{Z^* dZ(1 - Z^*Z)^{-1}\}$.
Type II: $\delta = \operatorname{Im}\operatorname{Tr}\{\overline{Z} dZ(1 - \overline{Z}Z)^{-1}\}$.
Type III: $\delta = \operatorname{Im}\operatorname{Tr}\{-\overline{Z} dZ(1 + \overline{Z}Z)^{-1}\}$.
Type IV: $\delta = \operatorname{Im}\left\{\dfrac{2Z^*dZ - 2q(Z)^*dq(Z)}{1 + {}^t\overline{Z}\overline{Z}\,{}^tZZ - 2Z^*Z}\right\}, \quad q(Z) = \dfrac{Z_2}{\begin{pmatrix} i & 1 \end{pmatrix} Z_2}.$

Furthermore, D^1 *is an irreducible Riemannian manifold with respect to* Ω.

The heart of the proof relies on the following lemma.

Lemma 15. *The differential one-form* δ *transforms under* $G^1_{\mathbb{R}}$ *as follows:*

$$d\arg j(g,Z) = \delta(Z) - \delta(gZ) \tag{10}$$

Proof. For Type I, we have $j(g,Z) = \det(CZ+D)^{-1}$ and so

$$\begin{aligned} d\arg j(g,Z) &= (1/2i)(d\log\det(CZ+D)^* - d\log\det(CZ+D)) \\ &= (1/2i)\operatorname{Tr}\{d(CZ+D)^*(CZ+D)^{*-1} \\ &\qquad - d(CZ+D)(CZ+D)^{-1}\}. \end{aligned}$$

By setting $U = (AZ+B)$ and $V = (CZ+D)$ to simplify our notation, it suffices to find an expression for $d(CZ+D)(CZ+D)^{-1} = dVV^{-1}$. First, notice that $gZ = UV^{-1}$ and check that the following two equations are equivalent:

$$\begin{aligned} \begin{pmatrix} Z^* & I \end{pmatrix} g^* I_{m,n} g d\begin{pmatrix} Z \\ I \end{pmatrix} &= \begin{pmatrix} Z^* & I \end{pmatrix} d\begin{pmatrix} Z \\ I \end{pmatrix}, \\ U^*dU - V^*dV &= Z^*dZ. \end{aligned}$$

Now,

$$\begin{aligned} dVV^{-1} &= V^{*-1}(V^*dV)V^{-1} \\ &= V^{*-1}(U^*dU - Z^*dZ)V^{-1} \\ &= V^{*-1}U^*(dU)V^{-1}) - V^{*-1}Z^*dZV^{-1} \\ &= V^{*-1}U^*(UV^{-1}dVV^{-1} + d(UV^{-1})) - V^{*-1}Z^*dZV^{-1} \\ &= (gZ)^*(gZ)dVV^{-1} + (gZ)^*d(gZ) - V^{*-1}Z^*dZV^{-1}, \end{aligned}$$

or equivalently

$$(I - (gZ)^*(gZ))dVV^{-1} = (gZ)^*d(gZ) - V^{*-1}Z^*dZV^{-1},$$

and so

$$dVV^{-1} = (I - (gZ)^*(gZ))^{-1}[(gZ)^*d(gZ) - V^{*-1}Z^*dZV^{-1}].$$

We multiply through and take the trace, simplifying the second term on the right hand side to

$$\begin{aligned} &\mathrm{Tr}\{(I - (gZ)^*(gZ))^{-1}V^{*-1}Z^*dZV^{-1}\} = \\ &\quad = \mathrm{Tr}\{V^{-1}(I - (gZ)^*(gZ))^{-1}V^{*-1}Z^*dZ\} \\ &\quad = \mathrm{Tr}\{(I - Z^*Z)^{-1}Z^*dZ\}, \end{aligned}$$

where the identity $(I - (gZ)^*(gZ))^{-1} = V^*(I - Z^*Z)^{-1}V$ has been used. We then get

$$\begin{aligned} \mathrm{Tr}\{dVV^{-1}\} = &\ \mathrm{Tr}\{(I - (gZ)^*(gZ))^{-1}(gZ)^*d(gZ)\} \\ &- \mathrm{Tr}\{(I - Z^*Z)^{-1}Z^*dZ\}. \end{aligned}$$

We now use this trace equation to compute

$$\begin{aligned} d\arg j(g,Z) &= d\arg\det(CZ + D)^{-1} \\ &= (1/2i)\mathrm{Tr}\{(dVV^{-1})^* - dVV^{-1}\} \\ &= -\mathrm{Im}\,\mathrm{Tr}\{(I - (gZ)^*(gZ))^{-1}(gZ)^*d(gZ)\} \\ &\qquad +\mathrm{Im}\,\mathrm{Tr}\{(I - Z^*Z)^{-1}Z^*dZ\} \\ &= \delta(Z) - \delta(gZ), \end{aligned}$$

as desired.

For Types II and III, we observe in these two cases that the argument parallels that used for Type I. This follows from recognizing that the expression for the determinant factor of automorphy $j(g, Z)$ doesn't change. Also, D lies inside $\{Z \in \mathbf{M}_{nn}(\mathbb{C}) \,:\, I - Z^*Z >> 0\}$ and $G_{\mathbb{R}}$ is a subgroup of $\mathbf{SU}(n, n)$, so $d\arg j(g, Z)$ produces the same identity formula as in the Type I case.

Again, the Type IV argument is complicated and must be derived separately. First, we recall some of the notation used earlier in showing the transformation property of the kernel function K for the Type IV case:

$$U = AZ_1 + BZ_2, \quad V = CZ_1 + DZ_2, \quad v = \begin{pmatrix} i & 1 \end{pmatrix} V.$$

We shall abuse notation and often write v^* to mean $\bar{v}$ even though v is not a matrix. Secondly, define $q(Z) = Z_2 z_2^{-1}$, where $z_2 = \begin{pmatrix} i & 1 \end{pmatrix} Z_2$. Then $q(gZ) = Vv^{-1}$ and $2q(Z)^*dq(Z) = \overline{{}^tZZ}d({}^tZZ)$. Now, $j(g, Z) = (v/2i)^{-1}$ and so

$$\begin{aligned} d\arg j(g,Z) &= \tfrac{1}{2i}(\log\det(v/2i)^* - \log\det(v/2i)) \\ &= \tfrac{1}{2i}\mathrm{Tr}\{(dvv^{-1})^* - dvv^{-1}\}. \end{aligned}$$

Therefore, to find an expression for dvv^{-1}, note that $gZ = Uv^{-1}$ and check the following equalities:

$$\begin{array}{lll} \hat{Z}^* g^* I_{n,2}\, d(g\hat{Z}) = {}^t\hat{Z} I_{n,2} \hat{Z} & \Longleftrightarrow & U^*dU - V^*dV = Z_1^*dZ_1 - Z_2^*dZ_2, \\ {}^t\hat{Z}\, {}^tgId(g\hat{Z}) = {}^t\hat{Z}d\hat{Z} = 0 & \Longleftrightarrow & {}^tUdU + {}^tVdV = 0. \end{array}$$

Then compute

$$\begin{array}{rcl} dvv^{-1} & = & v^{*-1}v^*dvv^{-1} \\ & = & v^{*-1}V^* \begin{pmatrix} i & 1 \end{pmatrix}^* \begin{pmatrix} i & 1 \end{pmatrix} dVv^{-1} \\ & = & v^{*-1}V^*dVv^{-1} + v^{*-1}V^* \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} dVv^{-1}. \end{array}$$

Add and subtract the term $2q(gZ)^*dq(gZ)$ to the right hand side above, but replace the addition by the equivalent expression obtained from the expansion below instead:

$$\begin{array}{rcl} 2q(gZ)^*d(gZ) & = & 2v^{*-1}V^*d(Vv^{-1}) \\ & = & 2v^{*-1}V^*(dVv^{-1} - Vdvv^{-2}) \\ & = & (2v^{*-1}V^*dVv^{-1} - dvv^{-1}) - (2v^{*-1}V^*Vv^{-1} - 1)dvv^{-1} \\ & = & v^{*-1}V^* \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} dVv^{-1} - \overline{{}^t(gZ)(gZ)}\,{}^t(gZ)(gZ)dvv^{-1} \\ & = & v^{*-1}V^*dVv^{-1} + v^{*-1}V^* \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} dVv^{-1} \\ & & \quad - \overline{{}^t(gZ)(gZ)}\,{}^t(gZ)(gZ)dvv^{-1}, \end{array}$$

where the identity $v^*v^*vv^{-1} - 2v^*V^*Vv^{-1} + \overline{{}^tVV}\,{}^tVV = 0$ (proved earlier in our discussion of the kernel function) is used below to justify the previous substitution:

$$\begin{array}{rcl} \overline{{}^t(gZ)(gZ)}\,{}^t(gZ)(gZ) & = & v^{*-2}\overline{{}^tUU}\,{}^tUUv^{-2} \\ & = & v^{*-2}(\overline{{}^tUU}\,{}^tUU)v^{-2} \\ & = & 2v^{*-1}V^*Vv^{-1} - 1. \end{array}$$

If our directions above are correctly put into practice, then our expression for dvv^{-1} simplifies (namely from cancellation of the terms involving the 2×2 matrices):

$$dvv^{-1} = 2v^{*-1}V^*dVv^{-1} - \overline{{}^t(gZ)(gZ)}\,{}^t(gZ)(gZ)dvv^{-1} - 2q(gZ)^*dq(gZ).$$

Now, expand the first term on the right hand side above as

$$\begin{array}{rcl} 2v^{*-1}V^*dVv^{-1} & = & 2v^{*-1}(U^*dU - Z_1^*dZ_1 + Z_2^*dZ_2)v^{-1} \\ & = & 2(gZ)^*dUv^{-1} - 2v^{*-1}(Z_1^*dZ_1 - Z_2^*dZ_2)v^{-1} \\ & = & 2(gZ)^*(d(Uv^{-1}) + 2Uv^{-1}dvv^{-1}) \\ & & \quad - 2v^{*-1}(4Z^*dZ - 2\overline{{}^tZZ}d({}^tZZ))z_2^{-1} \\ & = & 2(gZ)^*d(gZ) + 2(gZ)^*(gZ)dvv^{-1} \\ & & \quad - j(g,Z)^*(2Z^*dZ - 2q(Z)^*dq(Z))j(g,Z), \end{array}$$

and bring terms involving dvv^{-1} on the right to the left hand side to get:

$$\begin{array}{l} (1 + \overline{{}^t(gZ)(gZ)}\,{}^t(gZ)(gZ) - 2(gZ)^*(gZ))dvv^{-1} = \\ \quad = 2(gZ)^*d(gZ) - 2q(gZ)^*dq(gZ) \\ \qquad - j(g,Z)^*(2Z^*dZ - 2q(Z)^*dq(Z))j(g,Z). \end{array}$$

A division and application of the transformation property for the $K(gZ, gZ)$ term then brings our calculation for dvv^{-1} to an end:

$$dvv^{-1} = \frac{2(gZ)^*d(gZ) - 2q(gZ)^*d(gZ)}{1 + \overline{{}^t(gZ)(gZ)}\,{}^t(gZ)(gZ) - 2(gZ)^*(gZ)} - \frac{2Z^*dZ - 2q(Z)^*dq(Z)}{1 + \overline{{}^tZZ}\,{}^tZZ - 2Z^*Z}.$$

As a result,

$$\begin{aligned} d\arg j(g,Z) &= \tfrac{1}{2i}\mathrm{Tr}\{dv^* v^{*-1} - dv v^{-1}\} \\ &= -\delta(gZ) + \delta(Z). \end{aligned}$$

This completes the proof of the lemma. □ □

Remark 1 (Author's Remark). One might expect the expression for δ in the Type IV case to contain the term

$$\overline{{}^tZZ}d({}^tZZ) \tag{11}$$

instead of $q(Z)^* dq(Z)$. Oddly, it happens that

$$\overline{{}^t(gZ)(gZ)}d({}^t(gZ)(gZ)) = 4q(gZ)^*dq(gZ), \quad \text{but} \quad \overline{{}^tZZ}d({}^tZZ) = 2q(Z)^*dq(Z).$$

It is this difference by a factor of 2 between the two identities above that makes the $G^1_{\mathbb{R}}$-invariance of Ω unclear when the expression (11) is used.

Proof of Proposition 14. We first reduce the proof to showing that only the second term in Ω is invariant under $G^1_{\mathbb{R}} = G_{\mathbb{R}} \times S^1$. This follows from the fact that since ω is the Riemannian metric on D, it is $G_{\mathbb{R}}$-invariant, hence $G_{\mathbb{R}} \times S^1$-invariant, because the action of the circle S^1 on D is trivial.

Now, we have $G^1_{\mathbb{R}}$ acting on D^1 as follows: $(g,s)(Z,t) = (gZ,\check{t})$, where $\check{t} = t + s + \arg j(g,Z)$. Then $d\check{t} = dt + d\arg j(g,Z)$ and by Lemma 15, it follows that

$$d\arg j(g,Z) = \delta(gZ) - \delta(Z).$$

Hence,

$$d\check{t} - \delta(gZ) = dt - \delta(Z),$$

and shows the $G^1_{\mathbb{R}}$-invariance of Ω. Since nondegeneracy and positivity of Ω is clear, it follows that it is a Riemannian metric. It is also clear from the explicit expression of Ω that it does not agree with the metric induced from D^1 considered as a product of two symmetric spaces, namely D and S^1. Hence, D^1 is a non-symmetric irreducible Riemannian manifold. □ □

New examples. Again, we maintain the same assumptions about D and G_Q as before. Let μ be ordinary complex conjugation of D considered as a subset of $\mathbb{C}^n$. We claim that μ is induced from the involution θ of G described earlier in Prop. 2. Define $\theta_{\mathbb{R}}(g) = \mu g \mu^{-1}$ for any $g \in G_{\mathbb{R}}$. Then $\theta_{\mathbb{R}}(g)$ acts on D as follows:

$$\theta_{\mathbb{R}}(g)Z = \overline{\left(\frac{A\bar{Z}+B}{C\bar{Z}+D}\right)} = \bar{g}Z, \qquad Z \in D.$$

Therefore, $\theta_{\mathbb{R}}$ is just complex conjugation and hence a well-defined involutive map of $G_{\mathbb{R}}$. As $\theta_{\mathbb{R}}$ satisfies all the properties in Prop. 2, this proves our claim by the uniqueness of θ. By Prop. 3, it follows that

Proposition 16. *D is weakly symmetric with respect to $G_{\mathbb{R}}$ and μ.*

Corollary 17. *Let $Z_1, Z_2 \in D$. There exists an element $g \in G_{\mathbb{R}}$ such that*

$$\arg j(g,Z_1) = \arg j(g,Z_2).$$

Proof. Consider the kernel function $K(Z,W)$ defined in Proposition 10. We recall that $K(Z,W)$ satisfies $K(W,Z) = \overline{K(Z,W)}$ and has the following transformation property:

$$K(gZ,gW) = \overline{j(g,Z)}^{\,-1}K(Z,W)j(g,W)^{-1}.$$

To apply this to our situation, use the assumption that D is weakly symmetric with respect to $G_{\mathbb{R}}$ and μ. This provides us with an element $g \in G_{\mathbb{R}}$ such that $gZ_1 = \overline{Z_2}$ and $gZ_2 = \overline{Z_1}$. It follows that

$$K(gZ_1, gZ_2) = K(\overline{Z_2}, \overline{Z_1}) = \overline{K(Z_2, Z_1)} = K(Z_1, Z_2).$$

On the other hand,

$$K(gZ_1, gZ_2) = \overline{j(g, Z_1)}^{\,-1} K(Z_1, Z_2) j(g, Z_2)^{-1},$$

from which the identity $\overline{j(g, Z_1)} j(g, Z_2) = 1$ is clear. We conclude that $\arg j(g, Z_1) = \arg j(g, Z_2)$. □ □

Now, extend μ to an involution $\tilde{\mu}$ of D^1 defined by $\mu(Z, t) = (\mu(Z), -t) = (\overline{Z}, -t)$ and check that $\tilde{\mu} G^1_{\mathbb{R}} \tilde{\mu}^{-1} = G^1_{\mathbb{R}}$ and $\tilde{\mu}^2 \in G^1_{\mathbb{R}}$. According to Theorem 5, we can state that

Theorem 18. *D^1 is weakly symmetric with respect to $G^1_{\mathbb{R}}$ and $\tilde{\mu}$.*

This result provides new examples of weakly symmetric spaces that are not Riemannian products of symmetric spaces, as D^1 is irreducible with respect to Ω. In the case where $G_{\mathbb{R}} = \mathbf{Sp}(n, \mathbb{R})$, the manifolds D^1 correspond to those examples discovered by Selberg (cf. [M]), who used the unbounded realization of D as a generalized Siegel half-plane, and in the case where $G_{\mathbb{R}} = \mathbf{SU}(n, 1)$, D^1 was shown to be weakly symmetric with respect to its full isometry group in [BV1] by using its realization as a tube in complex hyperbolic space.

Unit disc. Let $G_{\mathbb{R}} = \mathbf{SU}(1, 1)$. We examine how weak symmetry of the unit disc D and its circle extension D^1 behaves at the group level. By definition, there exists an isometry $(g, s) \in G^1_{\mathbb{R}}$ that will exchange any two arbitrary distinct points (z, t) and (w, r) in D^1 with respect to complex conjugation $\tilde{\mu}$. By transitivity of $G_{\mathbb{R}}$, we can assume without loss of generality that one of the points, say (w, r), is the origin $(o, 0)$. The strategy is to first find an element $h \in G_{\mathbb{R}}$ that will switch o and z and then compose it with an element $k \in K_{\mathbb{R}}$ mapping z to $\bar{z}$. This will give the desired element g which will reverse the two points o and z with respect to μ. Define

$$k = \begin{pmatrix} e^{-i \arg z} & 0 \\ 0 & e^{i \arg z} \end{pmatrix}, \quad h = \frac{1}{\sqrt{\phi}} \begin{pmatrix} i & -iz \\ i\bar{z} & -i \end{pmatrix}, \quad \phi = 1 - \bar{z}z.$$

This leads to the element (g, s) as being

$$g = \frac{1}{\sqrt{\phi}} \begin{pmatrix} ie^{-i \arg z} & -i\sqrt{\bar{z}z} \\ i\sqrt{\bar{z}z} & -ie^{i \arg z} \end{pmatrix}, \quad s = -t + \arg \frac{z}{i}.$$

Now check that $\arg j(g, z) = \arg j(g, o) = -\arg \frac{z}{i}$ to obtain the desired property of weak symmetry: $(g, s)(o, 0) = (\overline{z}, -t)$ and $(g, s)(z, t) = (o, 0)$.

Lastly, as an application, we compute the Laplacian Δ on D^1. Write $z \in D$ as $z = x + iy$. By definition, any Riemannian manifold (M, Ω) has a Laplacian Δ defined by

$$\Delta f = \mathbf{div\ grad}\ f = \frac{1}{\sqrt{\bar{\Omega}}} \sum_l \frac{\partial}{\partial x_l} \left(\sum_k \Omega^{kl} \sqrt{\bar{\Omega}} \frac{\partial f}{\partial x_k} \right), \quad \bar{\Omega} = \det(\Omega_{ij}).$$

Lemma 19. *If $G = \mathbf{SU}(1,1)$, then the Laplacian of D^1 with respect to $\Omega = \Omega_1$ (described in Prop. 14) is given by*

$$\Delta = (1-x^2-y^2)^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) + 2(1-x^2-y^2)(x-y)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\frac{\partial}{\partial t} + (1+x^2+y^2)\frac{\partial^2}{\partial t^2}.$$

Proof. The metric $\Omega = \dfrac{dx^2+dy^2}{(1-x^2-y^2)^2} + \left(dt + \dfrac{ydx - xdy}{1-x^2-y^2}\right)^2$ on D^1 can be expressed in matrix form as

$$\Omega := (\Omega_{ij}) = \begin{pmatrix} (1+y^2)/\phi^2 & -xy/\phi^2 & y/\phi \\ -xy/\phi^2 & (1+x^2)/\phi^2 & -x/\phi \\ y/\phi & -x/\phi & 1 \end{pmatrix}, \qquad \phi = 1 - z\bar{z}.$$

It can be checked that

$$\sqrt{\Omega} = \frac{1}{\phi^2}, \quad (\Omega^{ij}) = \begin{pmatrix} \phi^2 & 0 & -y\phi \\ 0 & \phi^2 & x\phi \\ -y\phi & x\phi & (1+x^2+y^2) \end{pmatrix}. \tag{12}$$

It remains to just write out the definition of the Laplacian using the expressions in (12) above in order to obtain the desired formula in the lemma. We leave this tedious calculation for the interested reader. □ □

Compact case. It is natural to expect that a similar construction of M_Q should work in the compact case as well; however, a global description becomes more difficult to obtain when D is compact. One reason is that the determinant factor of automorphy is not well-defined on $G \times D$, at least not globally. Another reason is that the principal fiber bundle $M_Q \to D$ is no longer necessarily trivial. Of course, there are known explicit realizations of M_Q as a surface in complex projective space, but only when D has rank one (cf. [BV1], [Na]). Perhaps the duality that exists between compact and noncompact symmetric spaces could be employed. In any case, the author will conjecture that M_Q has a $G^1_{\mathbb{R}}$-invariant Riemannian metric locally given by the formula

$$\Omega = \omega + (dt - \delta)^2,$$

where for instance, if $G_{\mathbb{R}} = \mathbf{SU}(m+n)$, then ω is the unique $G_{\mathbb{R}}$-invariant Riemannian metric on $D = \mathbf{SU}(m+n)/S(\mathbf{U}(m)\times\mathbf{U}(n))$ and

$$\delta = \operatorname{Im}\operatorname{Tr}\{Z^* dZ (1 + Z^* Z)^{-1}\}.$$

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Department of Mathematics, Rowan University, Glassboro, NJ 08028, USA
E-mail address: `nguyen@rowan.edu`